Probability and the Geometry of the Laplacian

Lecture 1: The isoperimetric property

Lecture 2: Heat and Weyl asymptotics

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The Laplace (Laplacian) Operator:

$$\Delta f(x,y) = \frac{\partial^2 f}{\partial x^2}(x,y) + \frac{\partial^2 f}{\partial y^2}(x,y) \quad (d=2)$$

Named after the Marquis Pierre–Simon De Laplace (1749–1827). It lies at the heart of the mathematical description of heat, light, sound, electricity, magnetism, gravitation, fluid motion, as well as at the heart of modern mathematical analysis. It has been extensively studied by mathematicians (of all walks of life) and physicists for more than 200 years, often focusing on the geometric properties of its solutions.

Purpose of these talks:

Explore probabilistic ideas to study quantities associated with Laplacian that have been of interest for many years, that remain of interest and that lead to new and questions when the stochastic process that "goes" with the Laplacian changes.

1. Probability is area of mathematics used for modeling in many disciplines–science, engineering, social sciences, business,. . . where **uncertainty/randomness** is present.

2. In these talks we look at problems that on the surface do not have randomness. But in fact, randomness is “very nearby."

3. $$\Delta \Longleftrightarrow \textbf{heat flow} \Longleftrightarrow \textbf{Brownian motion}$$

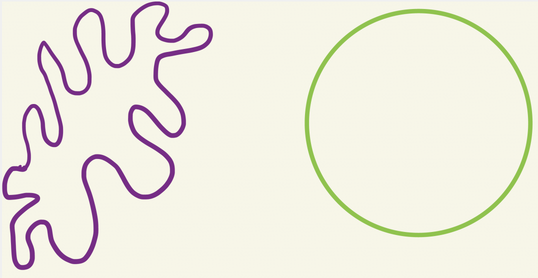
Illustrate this with three problems: “deeply rooted in our common experiences & intuition . . . .”

1. **Problem 1:** Amongst all simple closed planar curves of the same perimeter, determine the one that encloses the most area–discovered 800 or 900 B.C.
2. **Problem 2:** We know that big drums produce low tones and small drums produce high tones. Amongst all drums of fixed area, which one produces the lowest tone?
3. **Problem 3:** Amongst all regions of fixed area, which one will give the largest time for a “random walker” to reach the boundary of the region?

Problem 1: The isoperimetric inequality

Discovered by **Queen Dido** a Phoenician princess from the city of Tyre, shortly after her arrival in North Africa in 800 or 900 B.C.

Amongst all figures of equal perimeter ("iso-perimeter") the circle encloses the largest area. Equivalently (Euler 1744): Amongst all regions of fixed equal area ("iso-area"), the disk has the smallest perimeter.



When trading land, clever farmers used to cheat the not so clever ones by measuring the size of their fields by the time it took to walk around them.

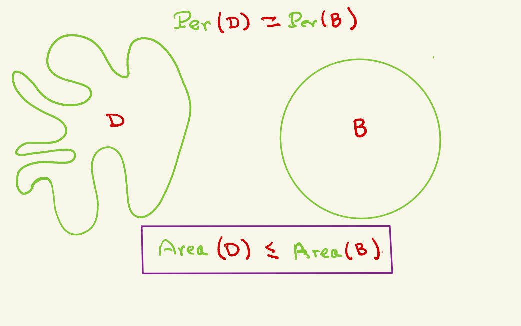
Dido Dividing the Ox-Hide, 1630 Engraving
Mathäus Merian The Elder

Theorem (The Isoperimetric Inequality)

- *D a domain (region) in the plane*
- *B = a circle of same perimeter*
- *Then*

$$Area(D) \leq Area(B)$$

Equality holds if and only if D is a circle.



It holds in higher dimensions (replacing area by volume and perimeter by surface area) and in many geometric settings...

Commonly used formulation

Let r=radius of B. What is r? $Per(B) = 2\pi r = Per(D) = L$. $r = \frac{L}{2\pi}$

$$Area(D) \leq Area(B) = \pi \left(\frac{L}{2\pi}\right)^2$$

Theorem: If $D \subset \mathbb{R}^2$ with perimeter L and area A

$$4\pi A \leq L^2$$

Equality iff $D = circle$

Countless proofs exists using different techniques: Fourier transform, Rearrangement inequalities, Brunn-Minkowski, Optimal mass transport,... heat kernel, . . . , Hundredths of papers written, dozens of books, hundredths of lectures given, . . .

- P. D. Lax, *"A Short Path to the Shortest Path,"* The American Mathematical Monthly, Vol. 102, No. 2 (Feb., 1995), pp. 158-159. ("Eminently suitable for presentation in an honors calculus course.")

- Jakob Steiner:
  - *"Einfache Beweise der Isoperimetische Hauptsatze,"* J. Reine Agnew (1838)
  - *"Gesammete, Werke"*, Vol 2, Berlin, 1882.

- G. Pólya and G. Szegö, *"Isoperimetric Inequalities in Mathematical Physics"* . Princeton University Press, 1951
- C. Bandle, *"Isoperimetric inequalities and applications,"* Pitmam, 1980.
- Robert Osserman, *"Isoperimetric Inequality,"* Bull, AMS , Vol 84, 1978.
- Fabrice Baudoin: 2021 AMS inited lecture: For a modern look at problems in "geometric measure theory" motivated by the isoperimetric inequality *"Heat Flow and sets of Finite Perimeter"*

Anonymous Venitian (16th century): Dido Founding Carthage.
Villa Giusti, Magnadola de Cessalto

Francesco Fontebasso's "Dido cuts the Oxhide" at the Albertina, Vienna

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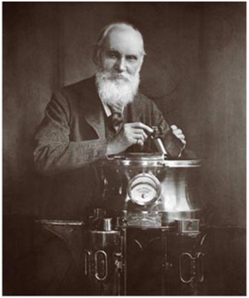


ISOPERIMETRICAL PROBLEMS.

[*Being a Friday evening Lecture delivered to the Royal Institution, May 12th,* 1893.]

Dido, B.C. 800 or 900.
Horatius Cocles, B.C. 508.
Pappus, Book V., A.D. 390.
John Bernoulli, A.D. 1700.
Euler, A.D. 1744.
Maupertuis (Least Action), b. 1698, d. 1759.
Lagrange (Calculus of Variations), 1759.
Hamilton (Actional Equations of Dynamics), 1834.
Liouville, 1840 to 1860.

``The isoperimetric theorem, deeply rooted in our experience and intuition, so easy to conjecture, but not so easy to prove, is an inexhaustible source of inspiration."

G. Polya

MATHEMATICS AND PLAUSIBLE REASONING

X

THE ISOPERIMETRIC PROBLEM

The circle is the first, the most simple, and the most perfect figure.—PROCLUS[1]
Lo cerchio è perfettissima figura.—DANTE[2]

1. Descartes' inductive reasons. In Descartes' unfinished work *Regulae ad Directionem Ingenii* (or *Rules for the Direction of the Mind*, which, by the way, must be regarded as one of the classical works on the logic of discovery) we find the following curious passage:[3] "In order to show by enumeration that the perimeter of a circle is less than that of any other figure of the same area, we do not need a complete survey of all the possible figures, but it suffices to prove this for a few particular figures whence we can conclude the same thing, by induction, for all the other figures."

THE ISOPERIMETRIC PROBLEM

Table I. Perimeters of Figures of Equal Area

Circle	3.55
Square	4.00
Quadrant	4.03
Rectangle 3 : 2	4.08
Semicircle	4.10
Sextant	4.21
Rectangle 2 : 1	4.24
Equilateral triangle	4.56
Rectangle 3 : 1	4.64
Isosceles right triangle	4.84

Problem 2: Can one hear the "size" of a drums?

People through out the world have known for centuries that "small drums" produce "high tones" and "big drums" produce "low tones."

Sacred Mountain Drums, Alan J. Willes, Drum Maker.

World's Largest Drum?

World's Lowest Tone?

Eigenvalues/Eigenfunctions

Given a (bounded) region D in the plane there is a sequence of numbers $\{\lambda_n\}$

$$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \ldots \to \infty$$

and a sequence of functions $\{\varphi_n\}$ which solve the boundary value problem (**"Dirichlet boundary" conditions**)

$$\begin{cases} \Delta\varphi_n = -\lambda_n\varphi_n & \text{in } D \\ \varphi_n = 0 & \text{on } \partial D. \end{cases}$$

The functions

$$u_n(t,z) = \varphi_n(z)e^{i\sqrt{\lambda_n}\,t} \qquad z = (x,y)$$

solve the wave equation:

$$\frac{\partial^2 u_n}{\partial t^2} = \Delta u_n = \frac{\partial^2 u_n}{\partial x^2} + \frac{\partial^2 u_n}{\partial y^2}.$$

1. Both $\{\lambda_n\}$ and $\{\varphi_n\}$ have been systematically studied for many (200+) years.
2. Special attention paid to the **"fundamental tone $\lambda_1(D)$"**.

"Theorem" (Deeply rooted in our intuition...)

The lower the **"one"** $\lambda_1(D)$, the larger the **"drum"**. The larger the **"drum"**, the lower the **"tone"** $\lambda_1(D)$.

Remark

At the age of eight and ten, my daughters told me: "We knew this since we were in kindergarten." My 4 and a half year-old granddaughter (preschooler) confirms this!

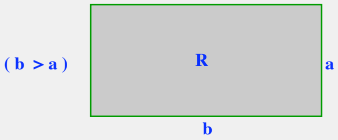
Variational formula

$$\lambda_1(D) = \inf\left\{ \int_D |\nabla f(z)|^2\, dz, \quad f \in C_0^\infty(D), \quad \int_D |f(z)|^2 dz = 1 \right\}$$

Explicitly computable only for very few regions:

Circles, rectangles, annuli, circular sectors, equilateral triangle, some right triangles.

Example ($D = rectangle$)



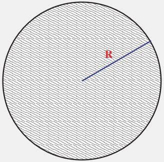
$$\varphi_1(x, y) = \sin(\frac{\pi x}{b}) \sin(\frac{\pi y}{a}), \qquad \lambda_1 = \frac{\pi^2}{a^2} + \frac{\pi^2}{b^2},$$

Note

As $b \to \infty$, the rectangle becomes an infinite strip (with infinite area) and eigenvalue $\frac{\pi^2}{a^2}$.

There are many examples of regions of infinite area with positive eigenvalue.

Example ($D =$ Circle of radius R)



Polar coordinates and separation of variables gives

$$\lambda_1 = \frac{j_0^2}{R^2} \approx \frac{5.7831}{R^2},$$

$j_0 \approx 2.405$ the smallest positive zero of the Bessel function J_0.

The isoperimetric property of $\lambda_1(D)$

Theorem: Faber-Krahn inequality, (1923) (Conjectured by Lord Rayleigh, 1877)

Amongst all D of finite area the circle minimizes $\lambda_1(D)$. $Area(D) = Area(D^*) \Longrightarrow$

$$\lambda_1(D^*) \leq \lambda_1(D)$$

Equality iff D is a circle.

Equivalent:

$$\pi r^2 = Area(D^*) = Area(D) \quad \Longrightarrow \quad \lambda_1(D^*) = \frac{j_0^2}{r^2} = \frac{j_0^2 \pi}{Area(D)}$$

$$\lambda_1(D^*) \leq \lambda_1(D) \quad \Longleftrightarrow \quad \frac{j_0 \pi}{Area(D)} \leq \lambda_1(D)$$

Equality iff $D = D^*$

"The lower the tone, the larger the area. Converse is false. Example: infinite strip."

A short detour on hearing the "size" of a drum

Theorem (R.B.& T. Carroll-1994)

Any $D \subset \mathbb{R}^2$ simply connected (no holes). R_D=radius of largest circle you can put inside D.

$$\frac{0.6197}{R_D^2} \leq \lambda_1(D) \leq \frac{j_0^2}{R_D^2}$$

The "drum" D produces an arbitrarily low tone if and only if it contains and arbitrarily large "circular" drum

R.B. and T. Carroll, *"Brownian motion and the fundamental frequency of a drum," Duke Math. J.*, (1994) pp 575–602

Open problem (70+ years)

What is the best constant for the lower bound and what is the D that gives it?

1. Previous best bound was $1/4$ given by Endre Makai (1966)
2. Robert Osserman Conjecture: $1/4$ is best possible (1977).

Problem 3: The survival time probability of a Brownian particle in D

Brownian motion:

- R. Brown (Scottish botanist) 1827: Brown was studying pollen particles floating in liquid under the microscope. He observed minute particles within the vacuoles of the pollen grains executing a jittery motion.
- A. Einstein 1905:
  1. Brownian motion should have independent increments (displacements should be independent).
  2. It should have no memory
  3. Its trajectories should be continuous with normal distribution
- N. Wiener 1923: There is a probability space and a family of random variables $\{B_t, t > 0\}$ with the stated properties and

$$P_z\{B_t \in A\} = \frac{1}{4\pi t}\int_A e^{\frac{-|w-z|^2}{4t}}\,dw, \quad \text{A subset in the plane.}$$

That is, the probability (density) that a Brownian particle moves from point z to point w in time t is given by the normal distribution.

$$p_t(z,w) = \frac{1}{4\pi}e^{\frac{-|w-z|^2}{4t}}$$

Heat Semigroup: Average of f along Brownian trajectories

$$u(t,z) = T_t f(z) = E_z\{f(B_t)\} = \frac{1}{4\pi t}\int_{\mathbb{R}^2} e^{\frac{-|z-w|^2}{4t}} f(w)dw$$

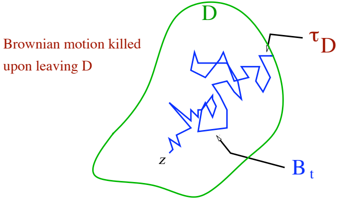
$$\Rightarrow \quad \frac{\partial u}{\partial t}(t,z) = \Delta u(t,z), \quad u(0,z) = f(z)$$

(Heat Equation)

Connection to Laplacian is built into the constructions of BM

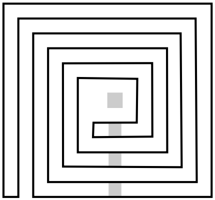
But, but...

We are interested in geometric information of regions D. Need to restrict Brownian motion to region. What Brownian motion quantity captures (some) geometric information of D?



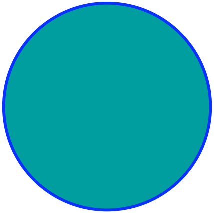
$\tau_D = \inf\{t : B_t \in \partial D\}$ = first time B_t reaches boundary

τ_D, called the "killing time" of BM in D. It is the time when B_t reaches the boundary of D and stops running (forever).



D

$D^* =$ circle same area as D



Theorem: Isoperimetric property of Brownian motion: Amongst all regions of same area the circle has the largest survival probability

1. Any $D \subset \mathbb{R}^2$ of finite area. For all $z \in D$ and all $t \geq 0$,

$$P_z\{\tau_D > t\} \leq P_0\{\tau_{D^*} > t\}$$

   Equality iff $D = \textit{circle}$ and $z = 0 =$ center of circle.

2. Also,

$$\int_D P_z\{\tau_D > t\}dz \leq \int_{D^*} P_z\{\tau_{D^*} > t\}dz \quad \text{(average probability)}$$

Easy Fact (M. Kac, early 50's) "Large deviation principle"

$$P_z\{\tau_D > t\} \sim e^{-t\lambda_1},$$

$$-\lambda_1(D) = \lim_{t\to\infty} \frac{1}{t} \log P_z\{\tau_D > t\}$$

#1 Gives Faber-Krahn

$$\lambda_1(D^*) \leq \lambda_1(D).$$

#2 $\iff$

$$\int_D P_z\{\tau_D > t\}dz \leq \int_{D^*} P_z\{\tau_{D^*} > t\}dz \iff$$

$$-\int_D^* P_z\{\tau_D^* > t\}dz \leq -\int_D P_z\{\tau_D > t\}dz \iff$$

$$\boxed{\int_{D^*}(1 - P_z\{\tau_{D^*} > t\})dz \leq \int_D (1 - P_z\{\tau_D > t\})dz}$$

Many authors, different settings (back to M. Kac): For $D \subset \mathbb{R}^2$

$$\lim_{t\to 0} \frac{1}{\sqrt{t}} \int_D (1 - P_z\{\tau_D > t\})\, dx = \frac{2}{\sqrt{\pi}} Per(D)$$

#2 Gives (iso-area same as iso-perimeter)

Amongst all regions of equal area the circular one has the smallest perimeter

$$Per(D^*) \leq Per(D).$$

The probability that the Brownian particle is alive by time t, starting from $z \in D$,

$$\boxed{P_z\{\tau_D > t\}}$$

holds many secrets.

"The Heart of the Matter" Show that the Finite Dimensional Distributions of BM

$$P_z\{B_{t_1} \in D_1, \dots B_{t_m} \in D_m\},$$

$$0 < t_1 < t_2 \cdots < t_m, \quad D_1, D_2, \dots, D_m \subset \mathbb{R}^2$$

for any $m = 1, 2, \dots$ have the isoperimetric properties.

- M.Aizenman and B. Simon (1982) (Brownian motion)
- R.B & P. J. Méndez-Hernández (2012) (General Lévy Processes)

Another 70+year old (intuitive and simple to state) conjecture

Conjecture (Pólya-Szgö, Isoperimetric Inequalities in Mathematical Physics, 1951, p.159)

Amongst all n-gons of equal area the regular one has the lowest fundamental tone (smallest eigenvalue)

Known only for triangles ($n = 3$) and quadrilaterals ($n = 4$). Both due to Pólya-Szgö 1951

Conjecture (Even more intuitive for a Brownian traveller (R.B. 2012))

Amongst all n-gons of equal area, the regular has the largest survival time probability for Brownian motion.

Remark

Second conjecture implies first

Topic of current interest: "Quantitative/deficit" Isoperimetric inequalities

1. $Per(D) - Per(D^*) \geq 0$
2. $\lambda_1(D) - \lambda_1(D^*) \geq 0$
3. $\mathbb{P}_0\left(\tau_{D^*} > t\right) - \mathbb{P}_z\left(\tau_D > t\right) \geq 0, \quad (z \neq 0\text{, center of circle})$

With equality if and only of *D* is a circle (similarly in higher dimensions)

Question (Initiated 1920's by T. Bonnesen)

Is there a non-negative function $\kappa(D)$ which is 0 only when $D =$ circle and measures in a "good" way the deviation of D from being a circle and

$$Per(D) - Per(D^*) \geq \kappa(D)?$$

Similarly for (2) and (3)?

- Yes: For (1) and (2) and various other similar "extremal" inequalities ...
- For (3) Only one partial results for the inequality $\mathbb{E}_0(\tau_{D^*}) - \mathbb{E}_z(\tau_D) > 0$.

R. B., P. Mariano and J. Wang, *“Bounds for exit times of Brownian motion and the first Dirichlet eigenvalue for the Laplacian,"* (2020 Preprint).

R. B. and P. J. Méndez-Hernández, “Symmetrization of Lévy processes and applications", J. Funct. Anal. **258** (2012), no. 12, 4026–4051.

A. Burchard and M. Schmuckenschläger, *"Comparison theorems for exit times"* Geometric And Functional Analysis, (2001) pp. 651–692

R. Osserman, *“Bonnesen-type isoperimetric inequalities"*, Amer. Math. Monthly (1979).

D. Kim, *“Quantitative inequalities for the expected lifetime of Brownian Motion,"* Michigan Math. J.,(2021) 1-22.

N. Fusco, F. Maggi, A. Pratelli, *“The sharp quantitative isoperimetric inequality,"* Ann. of Math. (2) 168 (3) (2008) 941–980

L. Brasco and G. De Philippis, Spectral inequalities in quantitative form, Shape optimization and spectral theory, De Gruyter Open, Warsaw, 2017, pp. 201–281.

Thank you/Muchas gracias