Probability and the Geometry of the Fractional Laplacian
Lecture 2: Heat and Weyl asymptotics

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Classical local PDE on $\mathbb{R}^d$, $d \geq 1$: The Laplacian

$$\Delta f(x) = \sum_{j=1}^{d} \frac{\partial^2 f}{\partial x_j^2}(x)$$

"It goes with" Brownian motion.

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"It goes with" Brownian motion.

The Fractional Laplacian: $0 < \alpha < 2$

$$\Delta^{\alpha/2} f(x) = A_{\alpha,d} \int_{\mathbb{R}^d} \frac{f(x+y) - f(x)}{|y|^{d+\alpha}} dy$$

$A_{\alpha,d}$ a normalizing constant. Equivalently

$$\boxed{\Delta^{\alpha/2} f(x) = \int_{\mathbb{R}^d} (f(x+y) - f(x))\, d\nu(y), \qquad d\,\nu(x) = \frac{A_{\alpha,d}}{|x|^{d+\alpha}} dx}$$

ν is called the "Lévy" measure.
"It goes with" stochastic processes known as "stable processes." Note that $\alpha/2 \in (0,1)$.

1. There has been intense activity in the last 25 years or so in the study of problems which arise when “local” PDE’s (the Laplacian for example) are replaced by “nonlocal" operators (the fractional Laplacian for example).

2. Many Nonlocal Operators arise as “(jump) diffusion operators” of Lévy processes which are “natural” extensions of Brownian motion. The study of these operators branches in different directions depending on motivation and point of view of problem-pure math, applied math,. . .

3. On the analysis, PDF, spectral theory side, many questions similar to those for the Laplacian have been raised, for jump diffusion operators, particularly for the Fractional Laplacian.

Rupert Frank (2018) *“Eigenvalue Bounds for the Fractional Laplacian: A Review”* (From the book Recent Developments in Nonlocal Theory.)

The problems for this talk:

(1) Weyl asymptotics and (2) Heat asymptotics for the fractional Laplacian

More precisely,

1. Geometric knowledge obtained by looking at all the eigenvalues and their growth as they "march up to infinity."
2. Geometric knowledge obtained by looking at certain probabilistic (heat) quantities as the stochastic processes begin to diffuse.

For the Laplacian

1. (1) Initiated by the a famous theorem of Hermann Weyl (1912)
2. (2) Largely influence by Mark Kac (1966) famous paper "Can one hear the shape of a drum?" (Monthly & video lecture "Archives of American Mathematics," UT, Austin)

Not a talk about "Can one hear the shape of a drum?"–answer to which is **"NO"**. C. Gordon-D.Webb-S. Wolpert (1992).

We start with the Laplacian

- $D \subset \mathbb{R}^d$, with finite volume and "nice smooth" boundary ∂D.

$$|D| = Vol(D), \quad |\partial D| = Surface\,Area(\partial D).$$

- The boundary value problem

$$\begin{cases} \Delta \varphi_k(x) = \sum_{j=1}^d \frac{\partial^2 \varphi_k}{\partial x_j^2}(x) = -\lambda_k \varphi_k(x), & x \in D \\ \\ \varphi_k(x) = 0, & x \in \partial D \end{cases}$$

has eigenvalues satisfying:

$$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \lambda_k \to \infty.$$

Example

The interval $I = [0, 1]$ has eigenfunctions

$$\varphi_k(x) = \sin(k\pi x)$$

and eigenvalues: $\lambda_k = k^2\pi^2$.

Example

For square $S = [0, 1] \times [0, 1]$, eigenfunctions are products of

$$sin(k\pi x)\sin(n\pi y),$$

and eigenvalues $\lambda_{n,k} = k^2\pi^2 + n^2\pi^2 = \pi^2(k^2 + n^2), \quad k, n = 1, 2, \dots$

Similarly for unit cute $Q = [0, 1] \times [0, 1] \times [0, 1]$.

Example

For the disc in the plane or ball in $\mathbb{R}^d$, eigenfunctions are Bessel functions and eigenvalues are their roots.

Hermann Weyl (1885–1955) Asymptotics

Counting function: $N_D(\lambda) = \#\{\lambda_k | \lambda_k < \lambda\}$**=number of eigenvalues less than** λ**.**

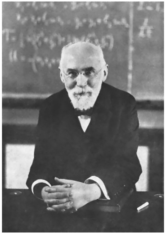
Example

For $S = [0,1] \times [0,1]$, $N_S(\lambda) = \#\left\{(k,n) \in \mathbb{Z}_+^2 : k^2 + n^2 < \left(\frac{\sqrt{\lambda}}{\pi}\right)^2\right\}$

Theorem (Weyl's Law, 1912: $N_D(\lambda) \sim |D|\lambda^{d/2}$**)**

$$\lim_{\lambda \to \infty} \lambda^{-d/2} N_D(\lambda) = \frac{|D|}{(4\pi)^{d/2}\Gamma(d/2+1)} = C_d|D|$$

(Weyl's proof was for $D \subset \mathbb{R}^2$*)*



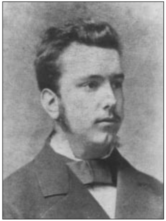
Hendrik Antoon Lorentz, 1953–1928
1902 Noble Prize in Physics
(magnetism upon radiation)

- In October 1910, he delivered a series of six lectures to the faculty of the University of Göttingen titled "Old and new problems in physics." The conjecture was stated at the end of the 4th lecture. (Conjecture also made around same time by Arnold Sommerfeld.)
- Göttingen had an endowed prize (the Paul Wolfskehl prize) for proving, or disproving, Fermat's last theorem. The donor stipulated that as long as the prize was not awarded, the proceeds from the principal should be used to invite an eminent scientists to deliver a series of lectures. Other Wolfskehl Lectures included Poincaré, Einstein, Planck and Bohr.

The prize was awarded to Andrew Wiles in 1997 (AMS Notices)

Paul Wolfskehl and the Wolfskehl Prize

Klaus Barner



Paul Wolfskehl, ca. 1880.

To the question often posed to Andrew Wiles in interviews—namely, what fascinated him so greatly in the Fermat conjecture—he seldom refrained from answering by emphasizing the long history of this problem. When I asked him the same question in Boston in 1995, he answered, "Because of its romantic history." When I then went further and asked him to explain to me in more detail what he meant by romantic, he answered merely, "Because Fermat said he had a proof, but none was found." That Wiles avoided answering in detail what is so romantic about the history of Fermat's Last Theorem reflects the fact that *he* also has a particularly romantic part to play in this story. The first time that I became aware of this was on October 28, 1995, the day after the awarding of the Prix Fermat to Wiles in the Salle des Illustres in the town hall in Toulouse. It was the last true day of autumn, with striking blue skies and temperatures worthy of summer, when Andrew Wiles visited the house in which Fermat was born in Beaumont-de-Lomagne. There he found the people in the highest of spirits on account of his mastering of this ancient enigma, and he was truly the man of the hour in this small relaxed town in the south of France, whose character had scarcely changed since the time of Fermat himself: Andrew had met Pierre.

Wiles also met the romance in the history of Fermat's Last Theorem on June 27, 1997, in Göttingen, where he was presented with the Wolfskehl Prize by the Academy of Science. Gerhard Frey gave the closing lecture, "On the Fermat problem, the conjecture of Taniyama and the theorem of Wiles". Since so much nonsense has been written about this prize and also about its donor Paul Wolfskehl, even by respected authors, and taken up blindly by other authors, I now see, through the presenting of this prize and the public awareness that goes with it, the last opportunity to do Paul Wolfskehl and his donation the justice they deserve.

Klaus Barner is professor of mathematics at the University of Kassel, Germany. His e-mail address is `klaus@mathematik.uni-kassel.de`.

"This article is an updated and revised translation of my paper 'Paul Wolfskehl und der Wolfskehlpreis', Mathematische Schriften Kassel, Vordruckreihe des Fachbereichs 17, Preprint Nr. 4/97, March 1997. I would like to thank Alex Reckless for his valuable help during the process of translating.

In my attempt to learn more about Wolfskehl and his family, I have been greatly assisted by Kurt-R. Biermann, Eckhart G. Franz, Erhard Heil, Paul Hoffmann, Charlotte Kühner-Wolfskehl, Sabine Rickmann, Heinz Georg Wagner, and Ingeborg Wolfskehl. I thank them all deeply. Special thanks go to Ingeborg Wolfskehl for letting me have a copy of the possibly last existing photo of Paul Wolfskehl."

—Klaus Barner

Andrew Wiles and Rudolph Smend (president of the Göttingen Academy) during the Wolfskehl Prize press conference, Göttingen, June 27, 1997.

M. Kac in his “Can one hear the shape of a drum?":

“Hilbert predicted that the theorem would not be proved in his life time. Well, he was wrong by many, many years [he passed away in 1943]. For less than two years later Hermann Weyl, who was present at the Lorentz’ lecture and whose interest was aroused by the problem, proved the theorem in question. Weyl used in a masterful way the theory of integral equations, which his teacher Hilbert developed only a few years before, and his proof was a crowning achievement of this beautiful theory."

Weyl’s Law was proved for cubes in $\mathbb{R}^3$ by L. Rayleigh and J.H. Jeans in 1905. Jean’s contribution to the paper was the observation: “It seems that Lord Rayleigh has introduced an unnecessary factor of 8 by counting negative as well as positive values of his integers.”

Weyl wrote (at least) three other papers on this topic (including other boundary value problems) around this time. He also made the following conjecture.

Conjecture (Weyl, 1913)

$$N_D(\lambda) = C_d|D|\lambda^{d/2} - C_d^{'}|\partial D|\lambda^{(d-1)/2} + o(\lambda^{(d-1)/2}), \quad \textit{as} \quad \lambda \to \infty,$$

Theorem (V. Ivrii, 1980))

Conjecture true provided domain is "nice".

In "Ramifications, old and new, of the eigenvalue problem" (Bulletin AMS 1950) Weyl writes:

"I feel that these informations about the proper oscillations of a membrane, valuable as they are, are still incomplete. I have certain conjectures on what a complete analysis of their asymptotic behavior should aim at; but since for more than 35 years I have made no serious attempt to prove them, I think I had better keep them to myself."

Many survey publications exist on this topic, including many historical accounts, ... Huge literature on the subject view from many different angles, . . .

1. For historical account: "Mathematical Analysis of Evolution, information and complexity."

   Chapter titled: "Weyl's Laws: Spectral Properties of the Laplacian in Mathematics and Physics" Wolfgang Arendt, Robin Nittka, Wolfgang Peter, Frank Steiner

2. V. Ivrii, "100 years of Weyl's law," (2016) Bull. Math. Sci. 6:379–452

Back to M. Kac

1. M. Kac early 50's: The "first order" asymptotic of the counting function

$$N_D(\lambda) = \#\{\lambda_k < \lambda\}, \text{ as } \lambda \to \infty$$

follows from "first order" asymptotic of its Laplace transform

$$Z_t(D) = \int_0^\infty e^{-t\lambda} dN_D(\lambda) = \sum_{k=1}^\infty e^{-t\lambda_k}, \text{ as } t \to 0$$

2. Main tools in Kac's proof:
   - (1) A beautiful observation on the behavior of Brownian particles for small time.
   - (2) The use of "Karamata tauberian theorem"

M. Kac (1951) Principle of not feeling the boundary. Let $x, y \in D$

For very small time t the probability density that Brownian motion moves from point x to point y in time t without leaving D is the same as if the boundary were not there.

In other words, for $x, y \in D$,

$$p_t^D(x,y) \sim p_t(x,y) = \frac{1}{(4\pi t)^{d/2}} e^{\frac{-|x-y|^2}{4t}}, \quad \text{as } t \to 0$$

In his 1966 "Can one hear the shape of a drum?"

“As the Brownian particles begin to diffuse they are not aware, so to speak, of the disaster that awaits them when they reach the boundary.”

In other words

Since Brownian particles have no memory (start afresh at any time) the same can be said for any “starting time” during their trajectory until they reach the boundary where they “face the music" and stop forever!

Island of Ikaria (Greece)
"The island where people forget to die"
New York Times, Oct 24, 2012

A little functional analysis (eigenfunction expansion of heat kernel)

$$p_t^D(x,x) \quad = \quad \sum_{k=1}^{\infty} e^{-t\lambda_k}\varphi_k^2(x)$$

Integrate with respect to x:

$$Z_D(t) = \sum_{k=1}^{\infty} e^{-t\lambda_k} = \int_D p_t^D(x,x)dx \sim \frac{1}{(4\pi t)^{d/2}}|D|, \;\; t \to 0$$

$$\lim_{t\downarrow 0} t^{d/2} Z_D(t) = \frac{|D|}{(4\pi)^{d/2}}$$

(Heat trace asymptotics!)

The Karamata tauberian theorem

Suppose μ is a measure on $[0,\infty)$ with

$$\lim_{t\to 0} t^{\gamma} \int_0^{\infty} e^{-t\lambda} d\mu(\lambda) = A, \quad \gamma > 0.$$

$$\Rightarrow \quad \lim_{a\to\infty} a^{-\gamma}\mu[0,a) = \frac{A}{\Gamma(\gamma+1)}$$

With $\mu = N_D$, $\gamma = d/2$, $A = \frac{|D|}{(4\pi)^{d/2}}$,

Corollary

$$\lim_{\lambda\to\infty} \lambda^{-d/2} N_D(\lambda) = \frac{|D|}{(4\pi)^{d/2}\Gamma(d/2+1)}$$

Since

$$p_t(x) = \frac{1}{(4\pi t)^{d/2}} e^{\frac{-|x|^2}{4t}}$$

we see that

$$\frac{1}{(4\pi)^{d/2}} = p_1(0)$$

Can write

$$\lim_{\lambda\to\infty} \lambda^{-d/2} N_D(\lambda) = \frac{p_1(0)\,|D|}{\Gamma(d/2+1)}.$$

In his 1966 paper Kac derived

1. For smooth drums ($d = 2$):

$$Z_t(D) \sim \frac{A}{4\pi t} - \frac{L}{4}\frac{1}{\sqrt{4\pi t}}, \qquad t \to 0.$$

2. For polygonal drums:

$$Z_t(D) \sim \frac{A}{4\pi t} - \frac{L}{4}\frac{1}{\sqrt{4\pi t}} + \frac{1}{6}, \quad t \to 0.$$

3. For polygonal drums with r polygonal holes:

$$Z_t(D) \sim \frac{A}{4\pi t} - \frac{L}{4}\frac{1}{\sqrt{4\pi t}} + (1-r)\frac{1}{6}, \quad t \to 0.$$

4. He conjectured that this should hold for “a smooth drum with r smooth holes."

(4) True: McKean-Singer (1967) “Curvature and eigenvalues of the Laplacian"

Question (Natural Given Kac's Polygonal Result)

Does the second order expansion hold for all domains in $\mathbb{R}^d$, $d \geq 2$ with Lipschitz boundaries? That is, for $D's$ with ∂D being locally the graph of a Lipschitz function?

Theorem (R. Brown 1993)

Yes:

$$Z_D(t) = (4\pi t)^{-d/2}\left(|D| - \frac{\sqrt{\pi t}}{2}|\partial D| + o(t^{1/2})\right), \quad t \downarrow 0$$

Questions (Natural Given Kac's Probabilistic Arguments)

1. Are there first order asymptotics when Brownian motion is replaced by other stochastic processes?
2. Are there second order asymptotics when Brownian motion is replaced by other stochastic processes and D is Lipschitz?

A Lévy Process: Constructed by **Paul Lévy** in the 30's (shortly after Wiener constructed Brownian motion). Other names: **de Finetti, Kolmogorov, Khintchine, Itô.**

1. **Stationary increments:** $0 < s < t < \infty$,

$$\mathbb{P}\{X_t - X_s \in A\} = \mathbb{P}\{X_{t-s} \in A\}$$

2. **Independent increments:** For any given sequence of ordered times

$$0 < t_1 < t_2 < \cdots < t_m < \infty,$$

the random variables

$$X_{t_1} - X_0,\ X_{t_2} - X_{t_1}, \ldots, X_{t_m} - X_{t_{m-1}}$$

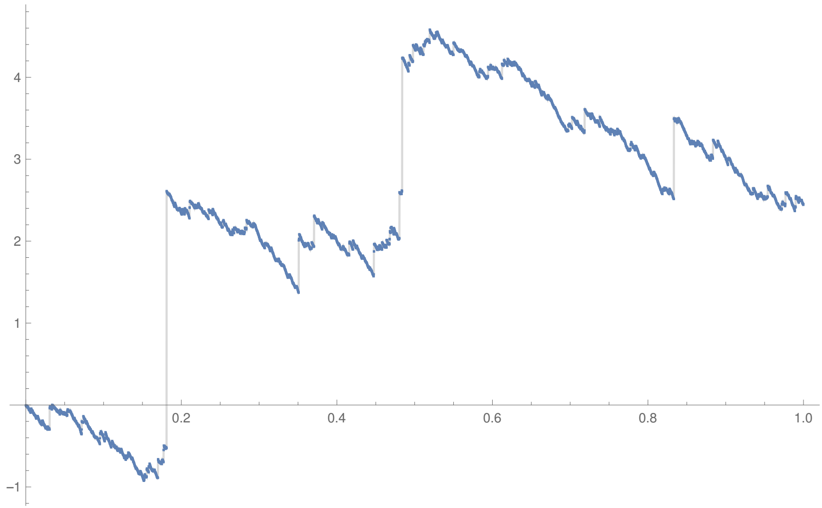
are independent.

3. X is *stochastically continuous*: For all $\varepsilon > 0$,

$$\lim_{t \to s} \mathbb{P}\{|X_t - X_s| > \varepsilon\} = 0$$

Continuity $\Rightarrow$ stochastically continuity.

(1)–(3) $\Rightarrow$ paths right continuous with left limits.


4
3
2
1
−1
0.2
0.4
0.6
0.8
1.0

The Lévy–Khintchine Formula (characterization): X_t is a Lévy process $\Longleftrightarrow$

$$\varphi_t(\xi) = \mathbb{E}\left(e^{i\xi\cdot X_t}\right) = e^{-t\rho(\xi)},$$

$$\rho(\xi) = -ib\cdot\xi + \mathbb{A}\xi\cdot\xi + \int_{\mathbb{R}^d}\left(1 - e^{i\xi\cdot x} + i\xi\cdot x 1_{\{|x|<1\}}(x)\right)\nu(dx)$$

1. $b \in \mathbb{R}^d$,
2. $\mathbb{A}$ $d \times d$ non–negative definite symmetric.
3. ν a measure on $\mathbb{R}^d$ with $\nu\{0\} = 0$ and

$$\int_{\mathbb{R}^d} \min\left(|x|^2, 1\right)\nu(dx) < \infty.$$

Bottom line:

Lévy processes X_t are "linear" combinations of t, of Brownian motion, and "pure" jump processes.

The (rotationally invariant) Stable processes: $\rho(\xi) = |\xi|^\alpha, \quad 0 < \alpha \leq 2$. That is,

$$\varphi_t(\xi) = \mathbb{E}\left(e^{i\xi \cdot X_t}\right) = e^{-t|\xi|^\alpha}$$

$\alpha = 2$, **Brownian motion,** $\alpha = 1$**, Cauchy processes.** Transition probabilities:

$$p_t^\alpha(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-i\xi \cdot x} e^{-t|\xi|^\alpha} d\xi$$

$$\mathbb{P}_x\{X_t \in A\} = \int_A p_t^\alpha(x-y)dy, \qquad A \subset \mathbb{R}^d$$

Lévy measure:

$$d\nu = \frac{A_{\alpha,d}}{|x|^{d+\alpha}} dx, \quad 0 < \alpha < 2$$

Heat Semigroup

$$T_t f(x) = \mathbb{E}_x[f(X_t)] = \int_{\mathbb{R}^d} f(y) p_t^\alpha(x-y)dy = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{-ix\cdot\xi} e^{-t|\xi|^\alpha}\widehat{f}(\xi)d\xi$$

$$\begin{aligned} Af(x) = \frac{\partial T_t f(x)}{\partial t}\Big|_{t=0} &= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{-ix\cdot\xi}|\xi|^\alpha \hat{f}(\xi)d\xi \\ &= A_{\alpha,d}\int_{\mathbb{R}^d} \frac{f(x+y)-f(x)}{|y|^{d+\alpha}}dy \\ &= \Delta^{\alpha/2} f(x) \end{aligned}$$

The eigenvalues for domains for stable processes $\{X_t\}$

Run the processes until the first time it exists D and construct the "killed" semigroup:

$$T_t^D f(x) = \mathbb{E}_x \left(f(X_t); \tau_D > t \right), \quad \tau_D = \text{exit time of X from D}$$

There exist number $\{\lambda_k\}$ and functions $\{\varphi_k\}$ with

$$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \lambda_n \to \infty,$$

$$\boxed{T_t^D \varphi_k(x) = e^{-\lambda_k t} \varphi_k(x), \quad x \in D.}$$

$$\boxed{\begin{cases} \Delta^{\alpha/2} \varphi_k = -\lambda_k \varphi_k & \text{in } D \\ \varphi_k = 0 & \text{on } D^c. \end{cases}}$$

Remark

λ_k and $\{\varphi_k\}$ not known explicitly even for the interval $(0,1)$. There are open problems/conjectures as to the "general shape" of φ_1.

Blumenthal and Getoor 1959

M. Kac's first order asymptotics for Brownian motion (the Laplacian) extend to Stable Processes (fractional Laplacian):

- $$\Rightarrow \quad \lim_{t\to 0} t^{d/\alpha} Z_D(t) = p_1^{\alpha}(0)|D|$$
- $$\Rightarrow \quad \lim_{\lambda\to\infty} \lambda^{-d/\alpha} N_D(\lambda) = \frac{p_1^{\alpha}(0)|D|}{\Gamma(d/\alpha+1)}$$

Sarah Bryant: "Path and spectral properties of certain Lévy process," (Purdue PhD thesis, 2009)

- Similar results hold for a wider collection of Lévy processes.
- Sarah also explored in more detail the relationship between second order asymptotics for the partition function $Z_t(D)$ for small t, and $N_D(\lambda)$ for large λ.

One big difference between stable and Brownian Motion. JUMPS

The first time they exit D they are really "outside of D": $P_x\{X_t \in \partial D\} = 0$

Conjecture (Blumenthal, Getoor 1959)

Second order trace term for stable processes should involve the Lévy measure of the complement (not surface area of ∂D).

- Since these processes "think" that the whole complement of D is "their" boundary, the conjecture seems natural.
- Unfortuantely, even if natural the conjecture is false and is the **wrong conjecture. A place where intuition fails.**

Theorem (R.B., T. Kulczycki, B. Siudeja (2009–2012): Stable of order $0 < \alpha < 2$)

Despite the fact that stable processes do not "see" the boundary, we have

$$t^{d/\alpha} Z_D(t) = C_1(\alpha, d)\,|D| - C_2(\alpha, d)\,|\partial D| t^{1/\alpha} + o\left(t^{1/\alpha}\right), \quad t \downarrow 0,$$

for domains with Lipschitz boundaries in $\mathbb{R}^d$. (In particular for polygonal domains in the plane!)

Second order asymptotic results for other Lévy processes:

R.B, J. Mijena, E. Nane (2014), H. Park, R. Song (2014), K. Bogdan, B. A. Siudeja (2015), L. Acuña-Valverde (2013), (2014), (2016), (2020).

Extensions to Schrödinger operators:

$(-\Delta)^{\alpha/2} + V$. L. Acuña-Valverde (2013-2014), R.B. & Acuña-Valverde (2015)...

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Thanks!

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Thank you/Muchas Gracias!