Conditional Expectation and Calderón-Zygmund Operators

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The world is full of inequalities

There are countess books on mathematical inequalities, among them

- Hardy, Littlewood and Pólya (1934) *"Inequalities"*
  "It is often really difficult to trace the origins of a familiar inequality. It is quite likely to occur first as an auxiliary proposition, often without explicit statement, in a memoir on geometry or astronomy; it may have been rediscovered, many years later, by half a dozen different authors; and no accessible statement may be quite complete."

- Mitrinovic, Dragoslav S., Pecaric, J., Fink, A.M (1993), "Classical and New Inequalities in Analysis"
  Math Reviews: "This is an excellent book that seems to prove that there is no possibility of a last word on equalities ..."

  Over 700 pages, with about 1,200 references for about $700

Given an inequality, many questions arise

Questions

1. Is the inequality sharp? If not, what is the sharp inequality?
2. Once a sharp version is proved, is equality ever attained or is the inequality always strict?
3. If the sharp inequality is attained, can the set of extremal (the quantities that give equality) be fully described/characterized?
4. If the sharp inequality is attained, and an admissible quantity gives strict inequality, how far is it from the set of externals and how to make the "how far" precise? How to measure the "deficit"? These go by the name of "Quantitive sharp inequalities."

RESEARCH ANNOUNCEMENTS

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 9, Number 1, July 1983

SOME RESULTS IN HARMONIC ANALYSIS IN $\mathbf{R}^n$, FOR $n \to \infty$

BY E. M. STEIN

1. Introduction. The purpose of this note is to bring to light some further results whose thrust is that certain fundamental estimates in harmonic analysis in $\mathbf{R}^n$ have formulations with bounds independent of n, as $n \to \infty$.

Stein then lists several examples of his new estimates

1. The classical Hardy-Littlewood maximal function
2. The "spherical" maximal function
3. Various Fourier Multipliers (imaginary powers of the Laplacian, etc)

2. The theorem. In $\mathbf{R}^n$ we define the familiar Riesz transforms by $(R_j f)\hat{}\,(\xi) = i(\xi_j/|\xi|)\hat{f}(\xi)$, $j = 1, \ldots, n$, and write $R = (R_1, \ldots, R_n)$; also $|R(f)(x)|$ will stand for $(\sum_{j=1}^n |R_j(f)(x)|^2)^{1/2}$.

THEOREM.

$$||R(f)||_p \leq A_p ||f||_p, \qquad 1 < p < \infty,$$

with A_p independent of n.

From Calderón-Zygmund theory: For fix p, A_p grows exponentially with n. For fix n,

$$A_p = \mathcal{O}(p), \quad p \to \infty, \quad \text{and} \quad \mathcal{O}(\frac{1}{p-1}), \quad p \to 1$$

Best possible order in on p. Stein's gets: $\mathcal{O}(p^2)$ and $\mathcal{O}(\frac{1}{(p-1)^2})$

Stein concludes his paper with

(ii) The above results raise the following general question. Can one find an appropriate infinite-dimensional formulation of (that part of) harmonic analysis in $\mathbf{R}^n$, which displays in a natural way the above uniformity in n? A related question is to study the "limit as $n \to \infty$" of the above results, insofar as such limits may have a meaning. One might guess that a further understanding of these questions would involve, among other things, notions from probability theory: i.e. Brownian motion and possibly some variant of the central limit theorem.

Stein was right:

For many classical operators in harmonic analysis $T : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$, probability theory (**and more precisely Brownian motion and martingale theory**) not only gives bounds for their L^p-norms that are independent of dimension but in several cases gives sharp estimates on operator norms:

$$\|T\|_{p\to p} = \sup_{f\in L^p(\mathbb{R}^n)} \frac{\|Tf\|_p}{\|f\|_p}$$

These applies to many operators, including: Riesz transforms, Beurling-Ahlfors operators, various types of Fourier multipliers, Littlewood-Paley square functions, etc, on different geometric setting

Large literature on this topic–recent paper with lots of references)

Fabrice Baudoin, Chen Li, & R.B. (Math Annalen (2019)) "Gundy–Varopoulos martingale transforms and their projection operators on manifolds and vector bundles."

We illustrate these techniques with a very simple to state problem that has been of interest for a while

First a little pre Calderón-Zygmund history

A question that excited the interest of many analysts during the early part of the 20th century:

How does the size of a periodic function control the size of its conjugate?

$$f(\theta) = \sum_{k=1}^{N} (a_k \cos k\theta + b_k \sin k\theta), \quad a_k, b_k \text{ real numbers}, \ \theta \in [-\pi, \pi]$$

Conjugate function (switch placement of $\sin$ and $\cos$ and "+" by "-" in f)

$$\tilde{f}(\theta) = \sum_{k=1}^{N} (a_k \sin k\theta - b_k \cos k\theta)$$

"Size" of f is measured by:

$$\|f\|_p = \left(\int_{-\pi}^{\pi} |f(\theta)|^p d\theta \right)^{1/p}, \ \ 1 \leqslant p < \infty$$

Question of interest was: Is there a constant C_p, depending only on p, s.t.

$$\|\tilde{f}\|_p \leqslant C_p \|f\|_p ?$$

If $p = 1$ or $p = \infty$, NO. Problem was $1 < p < \infty$.

Since $\tilde{\tilde{f}} = -f$, question equivalent to

If one of these series is in L^p is the other also in L^p with comparable norms?

Orthogonality of the trigonometric functions gives:

$$\|\tilde{f}\|_2 = \|f\|_2.$$

For general $1 < p < \infty$, the problem was solved by M. Riesz 1923.

“Conjugate function”?

Set $c_k = a_k - ib_k$. Consider the analytic function (polynomial) in the unit disc $\{z \in \mathbb{C} : |z| < 1\}$, $z = x + iy = re^{i\theta}$,

$$F(z) = \sum_{k=1}^{N} c_k z^k = \sum_{k=1}^{N} c_k r^k e^{k\theta} = U(z) + iV(z)$$

U and V are Conjugate Harmonic functions satisfying the Cauchy-Riemann equations:

$$\frac{\partial U}{\partial x} = \frac{\partial V}{\partial y}, \quad \frac{\partial V}{\partial x} = -\frac{\partial U}{\partial y}$$

Furthere

$$\lim_{r \to 1} U(re^{i\theta}) = \lim_{r \to 1} \sum_{k=1}^{N} \left(a_k r^k \cos k\theta + b_k r^k \sin k\theta \right) = f(\theta)$$

$$\lim_{r \to 1} V(re^{i\theta}) = \lim_{r \to 1} \sum_{k=1}^{N} \left(a_k r^k \sin k\theta - b_k r^k \cos k\theta \right) = \tilde{f}(\theta)$$

In general: For a periodic function on $[-\pi, \pi]$, the conjugate function can be defined by

$$\tilde{f}(\theta) = p.v. \frac{1}{2\pi} \int_{-\pi}^{\pi} cot \left(\frac{\theta - \varphi}{2} \right) f(\varphi) d\varphi$$

Theorem (M. Riesz–Published 1927, Proved 1923)

$$\|\tilde{f}\|_p \leqslant C_p \|f\|_p, \;\; 1 < p < \infty,$$

C_p depends only on p.

Non-periodic Hilbert transform on the real line

$$Hf(x) = p.v. \frac{1}{\pi} \int_{\mathbb{R}} \frac{f(x-y)}{y} \, dy = \lim_{\varepsilon \to 0} \frac{1}{\pi} \int_{|y| > \varepsilon} \frac{f(x-y)}{y} \, dy$$

Riesz theorem equivalent to

$$\|Hf\|_p \leqslant C_p \|f\|_p, \;\; 1 < p < \infty,$$

Discrete Hilbert transform, D. Hilbert early 1900's. $\{a_n, n \in \mathbb{Z}\}$ a sequence

$$\mathcal{H}a_n = \frac{1}{\pi} \sum_{m \in \mathbb{Z} \setminus \{0\}} \frac{a_{n-m}}{m}$$

In his paper, M. Riesz: L^p boundedness of H implies l^p boundedness for $\mathcal{H}$.

$$\|\mathcal{H}a_n\|_p \leqslant C_p' \|a_n\|_p, \quad 1 < p < \infty, \quad \|a_n\|_p = \left(\sum_{n \in \mathbb{Z}} |a_n|^p \right)^{1/p}$$

and operator norms satisfy:

$$\|\mathcal{H}\|_{p \to p} \leqslant C \|H\|_{p \to p}, \quad \text{and} \quad \|H\|_{p \to p} \leqslant \|\mathcal{H}\|_{p \to p}$$

E. C. Titchmarsh (Published 1926, Proved 1924)

1. $\|\mathcal{H}a_n\|_p \leqslant C_p \|a_n\|_p, \quad 1 < p < \infty$
2. $\|\mathcal{H}\|_{p \to p} = \|H\|_{p \to p}, \quad 1 < p < \infty,$
3. That is, the best constants in the l^p and L^p inequalities for the discrete and continuous Hilbert transforms are the same!

Correction.

Von

E. C. Titchmarsh.

I. In paragraph 4 of my paper on 'Reciprocal formulae involving series and integrals' (Math. Zeitschr. 25 (1926), pp. 321—347), the proof that $N_p \leq N'_p$ is incorrect, and should be deleted. This does not affect anything else in the paper.

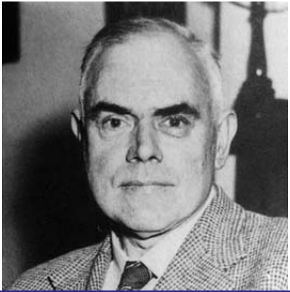
II. In obtaining the inequality which follows formula (2.32), we have assumed that (4a) as well as (3a) holds for the particular value of p taken. This merely involves a slight rearrangement of the proof.

III. The following references to the work of M. Riesz should have been given:

Comptes Rendus 178 (Apr. 28, 1924), pp. 1464—1467 and Proc. London Math. Soc. (2) 23 (1925), pp. XXIV—XXVI (Records for Jan. 17, 1924). I should have said that I was already familiar with Riesz's methods, and not merely his results, when I wrote my paper.

(Eingegangen am 10. November 1926.)

“Manuscripts of Hardy, Littlewood, Marcel Riesz and Titchmarsh,” Mary Lucy (M.L.) Cartwright, Bull London. Math. Soc, 14 (1982), 472-532



The following quotes are from Cartwright's LMS paper:

§6. I shall now try to tell the story leading to the long letter [Add. Ms. a.275[18]] on conjugate functions written by Riesz in December 1923 and received by Hardy early in the following January. I shall quote it in full because several references to Riesz's work are based on this letter long before he published the full account in *Math. Zeitschrift* in 1927, and it must have affected the thinking of Hardy and Littlewood, and also Titchmarsh.

At the end of Riesz's letter of 19 June 1923, he wrote

> J'ai démontré (on peut le formuler de beaucoup d'autres manières) que deux séries trigonométriques conjuguées sont toujours en même temps les séries de Fourier de fonctions de la classe L^p $(p > 1)$. C'est-à-dire, si l'une l'est, l'autre l'est aussi. Ma démonstration n'a rien à faire avec le théorème de Young–Hausdorff.

On 29 October Hardy wrote to Riesz because the printers said that Riesz had not returned the proofs of his equivalence note. No other topic was mentioned in this letter, but on 18/12/1923, Hardy wrote

> Some months ago you said 'j'ai démontré que 2 sér. trig. conjuguées sont toujours en même temps les séries de F. de fonctions de la classe L^p ($p > 1$) ...' *I want the proof.* Both I & my pupil Titchmarsh have tried in vain to prove it. Some attempts go close, and I feel that probably I could do it with sufficient determination. But it is not worth while if you have done it already. As T. is engaged on corresponding questions about integrals, it is of vital importance to him to have the proof of this rather fundamental theorem.
>
> So *please* send me (a) the grandes lignes de la dém (so that if need be we can reconstruct it without serious difficulty) or (b) a complete proof. I need hardly say that I should be delighted to get it published here.

Döbelnsgatan 5, Stockholm.
23 décembre 1923.

Mon cher ami,

Voici les grandes lignes de la démonstration désirée:

I. Deux séries trigonométriques conjuguées sont en même temps les séries de Fourier de fonctions de la classe L^p, $p > 1$.

Ce théorème est évidemment équivalent au théorème suivant:

II. Soit $F(z) = F(re^{ix}) = U(x,r) + iV(x,r)$ une fonction holomorphe pour $r < 1$. Si $\int_0^{2\pi} |U(x,r)|^p dx$ reste borné lorsque r tend vers 1, il en sera de même de $\int_0^{2\pi} |V(x,r)|^p dx$.

Ou plus précisément:

III. Posons $F(0) = u + iv$. Si l'on a

$$\text{(A)} \qquad \frac{1}{2\pi} \int_0^{2\pi} |U(x,r)|^p dx \leqslant 1\,, \qquad r < 1\,,$$

on aura

$$\frac{1}{2\pi} \int_0^{2\pi} |V(x,r)|^p dx \leqslant (M_p + |v|)^p$$

où M_p ne dépend que de p.

Pour $p = 2$, tout ce que je viens de dire est un corollaire immédiat du théorème de Parseval. Or pour arriver à la démonstration dans le cas général, le pas le plus important, pour ne pas dire le seul pas nécessaire, c'est d'oublier le théorème de Parseval. Il est en effet manifeste que le théorème de Young–Hausdorff n'étant pas un théorème reversible, on n'en pourra rien tirer pour la question actuelle. L'histoire de ma démonstration est des plus drôles. Au printemps j'avais à préparer des problèmes d'examen pour un élève très peu doué qui voulait passer son examen de "licenciat" (examen plus haut que la licence française). Parmi les 8 problèmes se trouvait aussi la question actuelle pour $p = 2$ (sous la forme II). Or il était à peu près évident que mon candidat ne connaissait pas le théorème de Parseval. Avant de lui proposer le problème, il me fallait donc réfléchir s'il y avait d'autres voies par lesquelles il pouvait arriver à la solution. Je constatai tout de suite que c'est le théorème de Cauchy qui est la vraie source du théorème et cette observation me fournissait pour ainsi dire toute prête la solution de la question générale, question qui m'avait si longtemps occupé. (Tous mes essais antérieurs se basaient sur la

§8. This evoked a reply [Add. Ms. a.275[33]] from Hardy dated 5/1/24

> Very many thanks—you supply all that is essential, & I have sent on your letter to Titchmarsh.
>
> *Most* elegant & beautiful. Of course p. 2 is the real point. It is amazing that none of us should have seen it before (even for $p = 4$!). Your student's life is not entirely without value (though I suppose he will never understand why).
>
> I make a few miscellaneous & ill assorted remarks (though I have nothing to say about the main theorems).

The page 2 in question is most of the proof of III for p an even integer. There is a later remark

> (À propos of your remark about forgetting Parseval's theorem, do you know the puzzle (which defeated Einstein, Jeans, J. J. Thomson etc).
>
> Two cyclists A, B, 20 miles apart, ride towards one another at 10 miles an hour. A fly C can fly 15 miles an hour. It leaves A and flies to meet B, then back to A, & so on. *How far does the fly fly*?
>
> One thing only is necessary: you must not know the formula for the sum of a geometrical progression. If you do, you will take 15–20 minutes: if not, 2 seconds.)

Theorem (S. Pichorides 1972 (independent & unpublished by B.Cole))

$$\|Hf\|_p \leqslant cot\left(\frac{\pi}{2p^*}\right)\|f\|_p, \quad 1 < p < \infty,$$

where $p^* = \max\{p, q\}$, $\frac{1}{p} + \frac{1}{q} = 1$. *Same as*

$$cot\left(\frac{\pi}{2p^*}\right) = \begin{cases} tan(\frac{\pi}{2p}) & 1 < p \leqslant 2, \\ cot(\frac{\pi}{2p}) & 2 \leqslant p < \infty \end{cases}$$

Further, $\|H\|_{p\to p} = cot\left(\frac{\pi}{2p^*}\right)$. *That is, the constant* $cot\left(\frac{\pi}{2p^*}\right)$ *is best possible.*

The inequality is strict unless $p = 2$. *(No extremal otherwise)*

Sharpness with truncations of $|x|^{-1/p}$:

$$f_\epsilon(x) = (4\log(1/\epsilon))^{-1/p}\,\chi_{\{x:\epsilon<|x|<\frac{1}{\epsilon}\}}|x|^{-1/p}, \quad 0 < \epsilon < 1$$

Theorem (M. Kwaśnicki* & R.B. 2018)

$$\|\mathcal{H}a_n\|_{\ell^p} \leqslant cot\left(\frac{\pi}{2p^*}\right)\|a_n\|_{\ell^p}, \quad 1 < p < \infty,$$

In particular,

$$\|\mathcal{H}\|_{p\to p} \leqslant \|H\|_{p\to p}.$$

Together with Riesz/Titchmarsh:

$$\|\mathcal{H}\|_{p\to p} = \|H\|_{p\to p}$$

*With Mateusz Kwaśnicki, Wrocław University of Science &Technolgy, Poland

1) T. Gokhberg, N.Y Krupnik (1968): $\|H\|_p = cot\left(\frac{\pi}{2p^*}\right)$ for $p = 2^n$ or $p = \frac{2^n}{2^n-1}$, $n = 1, 2, \ldots$ Proof by induction and the identity

$$|Hf|^2 = 2H(fHf) + f^2$$

which leads to

$$\|H\|_{2p} \leqslant \|H\|_p + \sqrt{\|H\|_p^2 + 1}$$

Now use $cot(\frac{\alpha}{2}) = cot(\alpha) + \sqrt{cot^2(\alpha) + 1}$

---

2) Argument extended to $\|\mathcal{H}\|_p$ by I.E. Verbitisky (1984), E.Laeng (2007).

$$|\mathcal{H}f|^2 = 2\mathcal{H}(f \cdot \mathcal{H}f) + J(f^2) + 2f \cdot Jf$$

Many other proofs of

$$\|\mathcal{H}a_n\|_p \leqslant C_p\|a_n\|_p$$

exist, including via Littlewood-Paley theory for discrete Laplacian (random walks). None Sharp.

$$N_t = \int_0^t K_s \cdot d\,B_s, \quad M_t = \int_0^t H_s \cdot d\,B_s, \quad 0 \leqslant t \leqslant T$$

1. N subordinate to M ($N << M$) if: $|K_s| \leqslant |H_s|$ a.s. for all s
2. N & M are orthogonal: ($N \perp M$) if: $K_s \cdot H_s = 0$ a.s. for all s

- D. Burkholder (1984). Under subordination (T any "time")

$$\|N_T\|_p \leqslant (p^* - 1)\|M_T\|_p, \;\; 1 < p < \infty,$$

$(p^* - 1) = (p - 1)$, if $2 \leqslant p < \infty$, and $\frac{1}{(p^*-1)}$, if $1 < p \leqslant 2$)

- G. Wang & R.B. (1995). Under subordination and orthogonality

$$\|N_T\|_p \leqslant cot\left(\frac{\pi}{2p^*}\right) \|M_T\|_p, \;\; 1 < p < \infty$$

Both sharp. (Also strict!)

Poisson kernel & Harmonic functions in upper half-space

$$p_w(x,y) = \frac{1}{\pi}\frac{y}{(|x-w|^2+y^2)}, \quad x, w \in \mathbb{R}, y \in (0,\infty)$$

$$U_f(x,y)) = \frac{1}{\pi}\int_{\mathbb{R}} p_w(x,y) f(w)dw, \quad \text{(nice } f)$$

is harmonic in $\mathbb{R}^2_+ = \{x,y) : x \in \mathbb{R}, y > 0\}$.

$$\lim_{y\downarrow 0} U_f(x,y) = f(x), \quad \lim_{y\uparrow\infty} U_f(x,y) = 0,$$

and

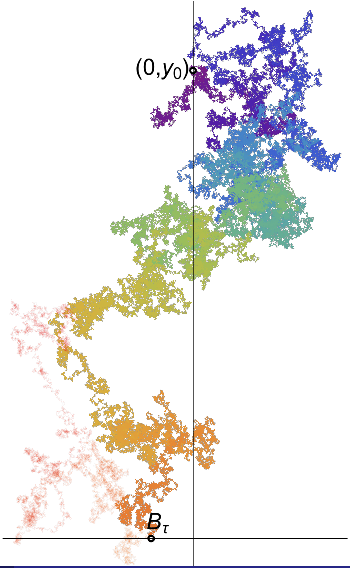
$$\lim_{y\uparrow\infty}(y\pi U_f(x,y)) = \int_{\mathbb{R}} f(w)dw$$

In particular,

$$\lim_{y\uparrow\infty}(y\pi U_f(0,y)) = \int_{\mathbb{R}} f(w)dw$$

Harmonic functions composed with Brownian motion

$B_t = (X_t, Y_t)$ Brownian motion in $\mathbb{R}^2_+$, $\tau = \inf\{t > 0 : Y_t = 0\}$ is exit time.



Then

$$U_f(0,y) = E_{(0,y)}\left(f(B_\tau)\right),$$

$$\lim_{y\to\infty}\Big(\pi y E_{(0,y)}\left(f(B_\tau)\right)\Big) = \int_{\mathbb{R}} f(w)dw$$

"Fundamental" Theorem of Stochastic Calculus (Itô's formula):

$$U_f(B_{t\wedge\tau}) = U_f(B_0) + \int_0^{t\wedge\tau} \nabla U_f(B_s)\cdot dB_s$$

Letting $t \to \tau$,

$$f(X_\tau) = U_f(0,y) + \int_0^{\tau} \nabla U_f(B_s)\cdot dB_s$$

Trivial observations

- V_f is the conjugate harmonic function of U_f. Cauchy-Riemann

$$|\nabla U_f(x,y)| = |\nabla V_f(x,y)|.$$

$$\nabla V_f = \mathbb{H}\nabla U_f, \;\; \mathbb{H} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

$$V_f(B_{t\wedge\tau}) = V_f(0,y) + \int_0^{t\wedge\tau} \mathbb{H}\nabla U_f(B_s)\cdot dB_s$$

and so

$$Hf(X_\tau) = V_f(0,y) + \int_0^{\tau} \mathbb{H}\nabla U_f(B_s)\cdot dB_s$$

$V_f(B_{t\wedge\tau}) << U_f(B_{t\wedge\tau})$, $V_f(B_{t\wedge\tau}) \perp U_f(B_{t\wedge\tau})$ Wang & R.B. $\Rightarrow$

$$\mathbb{E}_{(0,y)}|Hf(X_\tau) - V_f(0,y)|^p \leqslant \left(cot\left(\frac{\pi}{2p^*}\right)\right)^p \mathbb{E}_{(0,y)}|f(X_\tau) - U_f(0,y)|^p, \quad 1 < p < \infty.$$

Multiply both sides by πy and send y to infinity to get Pichorides.

For general (constant or even variable coefficient) 2×2 matrix A.

Fix $(0, y) \in \mathbb{R}^2_+$. Define:

$$T^y_A f(x) = E_{(0,y)} \left(\int_0^\tau A \nabla U_f(B_s) \cdot dB_s | B_\tau = x \right)$$

$$\textit{Any } A : \quad \left(E_{(0,y)} |T^y_A f(B_\tau)| \right)^{1/p} \leqslant (p^* - 1) \|A\| \left(E_{(0,y)} |f(B_\tau)|^p \right)^{1/p}$$

$$\|A\| = \| \sup_{|v| \leqslant 1} (|A(z,w)v|) \|_{L^\infty(\mathbb{R}^2 \times [0,\infty))} < \infty,$$

(From Burkholder's inequality)

$$\textit{Orthogonal } A : \quad \left(E_{(0,y)} |T^y_A f(B_\tau)| \right)^{1/p} \leqslant \textit{cot}\left(\frac{\pi}{2p^*} \right) \|A\| \left(E_{(0,y)} |f(B_\tau)|^p \right)^{1/p}$$

(From Wang and R.B. inequality)

As $y \to \infty$, we get for any A with $(n+1) \times (n+1)$ matrix we get on $\mathbb{R}^n$, any $n \geqslant 1$,

$$\|A\| = \| \sup_{|v| \leqslant 1} (|A(z,w)v|)\|_{L^\infty(\mathbb{R}^n \times [0,\infty))} < \infty,$$

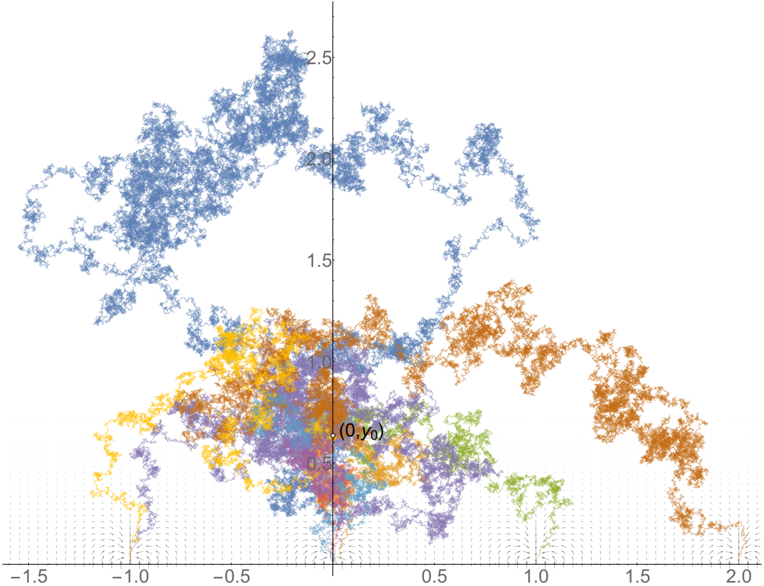
$$T_A f(x) = \int_{\mathbb{R}^n} K(x,\tilde{x}) f(\tilde{x}) d\tilde{x},$$

$$K_A(x,\tilde{x}) = \int_0^\infty \int_{\mathbb{R}^n} 2wA(z,w)\nabla p_{\tilde{x}}(z,w) \cdot \nabla p_x(z,w) dz\, dw,$$

Remark

- *Some are convolution, some are not, some are C-Z, some are not ... Not all C-Z arise this way*
- *Get Riesz transforms, $\mathbb{R}^n$, Lie groups, manifolds, Ornstein-Uhlenbeck (Wiener space), etc.*
- *Can replace Brownian motion by other processes– including Lévy processes*

Replace Brownian motion by "conditioned" Brownian motion (Doob h-processes) $Z_t = (X_t, Y_t)$. Only exists $\mathbb{R}^2_+$ on the lattice $\mathbb{Z}$. $\tau = \inf\{t \geqslant 0 : Y_t = 0\}$.



$L = \{2\pi n : n \in \mathbb{Z}\}$ (normalization of some constants)

$$p_n(x,y) = \frac{1}{\pi}\,\frac{y}{(x-2\pi n)^2 + y^2},$$

for $x \in \mathbb{R}$ and $y > 0$. We define (periodic Poisson kernel)

$$h(x,y) = \sum_{n\in\mathbb{Z}} p_n(x,y) = \left(\frac{1}{2\pi}\,\frac{\sinh y}{\cosh y - \cos x}\right).$$

h positive harmonic. For $f : L \to \mathbb{R}$ defined by $f(2\pi n) = a_n$ (compact support)

$$U_f(x,y) = \sum_{n\in\mathbb{Z}} f(2\pi n)\frac{p_n(x,y)}{h(x,y)}, \quad x \in \mathbb{R}, \quad y > 0$$

U_f is h-harmonic in $\mathbb{R}^2_+$:

$$\frac{1}{2}\Delta U_f(x,y) + \frac{\nabla h(x,y)\cdot\nabla U_f(x,y)}{h(x,y)} = 0.$$

$U_f(2\pi n, 0) = a_n$; U_f is the h-harmonic extension of $\{a_n\}$.

$$dZ_t = dB_t + \frac{\nabla h(Z_t)}{h(Z_t)} dt.$$

Consider two stochastic integrals (martingales) $M_t = U_f(Z_{t\wedge\tau})$. (A bit of stochastic calculus)

$$\begin{aligned} M_t = U_f(Z_{t\wedge\tau}) &= M_0 + \int_0^{t\wedge\tau} \nabla U_f(Z_s) \cdot dZ_s + \frac{1}{2}\int_0^{t\wedge\tau} \Delta U_f(Z_s) ds \\ &= M_0 + \int_0^{t\wedge\tau} \nabla U_f(Z_s) \cdot dZ_s - \int_0^{t\wedge\tau} \frac{\nabla h(Z_s) \cdot \nabla U_f(Z_s)}{h(Z_s)} ds \\ &= M_0 + \int_0^{t\wedge\tau} \nabla U_f(Z_s) \cdot dB_s. \end{aligned}$$

Any 2×2 matrix with A, $\|A\| \leqslant 1$, consider the new stochastic integral:

$$N_t \quad = \quad \int_0^{t\wedge\tau} A \nabla U_f(Z_s) \cdot dB_s$$

Then $N_t << M_t$. If in addition, A is orthogonal, then $N_t \perp M_t$.

For $y > 0$, we now define the conditional expectation operator $\mathcal{T}_A^y$:

$$\mathcal{T}_A^y(f)(n) = \mathbb{E}_{(0,y)}\Big[N_\tau \,\Big|\, X_\tau = 2\pi n\Big]$$

1. For any 2×2 matrix A and $1 < p < \infty$,

$$\mathbb{E}_{(0,y)}|\mathcal{T}_A^y f(X_\tau)|^p \leqslant \mathbb{E}_{(0,y)}|N_\tau|^p \leqslant (p^* - 1)^p \|A\|^p \mathbb{E}_{(0,y)}|f(X_\tau)|^p.$$

2. If A is orthogonal ($A\vec{v} \cdot \vec{v} = 0$, all $\vec{v} \in \mathbb{R}^2$)

$$\mathbb{E}_{(0,y)}|\mathcal{T}_A^y f(X_\tau)|^p \leqslant \mathbb{E}_{(0,y)}|(A\star M^f)_\tau|^p \leqslant \left(cot\left(\frac{\pi}{2p^*}\right)\right)^p \|A\|^p \mathbb{E}_{(0,y)}|f(X_\tau)|^p.$$

Theorem: As $y \to \infty$, $\mathcal{T}_A^y(f) \to \mathcal{J}_A(f)$

1. $\|\mathcal{J}_A(f)\|_{\ell^p(\mathbb{Z})} \leqslant (p^* - 1)\|A\|\|f\|_{\ell^p(\mathbb{Z})}$, any A
2. $\|\mathcal{J}_A(f)\|_{\ell^p(\mathbb{Z})} \leqslant cot(\frac{\pi}{2p^*})\|A\|\|f\|_{\ell^p(\mathbb{Z})}$, any orthogonal A

Question: What are these operators $\mathcal{J}_A(f)$?

Theorem (M. Kwaśnicki & R.B.)

For $\mathbb{H} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, *orthogonal matrix of norm 1:*

$$\mathcal{J}_{\mathbb{H}}(f)(n) = \sum_{m \in \mathbb{Z}} \mathcal{J}(n) f(n-m),$$

$$\begin{aligned} \mathcal{J}(n) &= \int_{\mathbb{R}} \int_0^\infty \frac{2y}{h(x,y)} \mathbb{H} \nabla p_n(x,y) \cdot \nabla p_0(x,y) \, dy dx \\ &+ \int_{\mathbb{R}} \int_0^\infty 4y p_0(x,y) \mathbb{H} \nabla p_n(x,y) \cdot \nabla \left(\frac{1}{h(x,y)}\right) dy dx \\ &= \frac{1}{\pi n} \left(1 + \int_0^\infty \frac{2y^3}{(y^2 + \pi^2 n^2) \sinh^2 y} \, dy \right) \mathbf{1}_{\mathbb{Z} \setminus \{0\}} \end{aligned}$$

Theorem

The discrete Hilbert transform is the convolution of $\mathcal{J}_{\mathbb{H}}$ *with a probability kernel:*

$$\mathcal{H}f(n) = \sum_{m\in\mathbb{Z}} \mathcal{K}(n-m)\mathcal{J}_{\mathbb{H}}(f)(m),$$

$\{\mathcal{K}(n); n \in \mathbb{Z}\}$ *nonnegative, mass* 1

Corollary

$$\|\mathcal{H}f(n)\|_{\ell^p(\mathbb{Z})} \leqslant \|\mathcal{J}_{\mathbb{H}}(f)\|_{\ell^p(\mathbb{Z})} \leqslant cot\left(\frac{\pi}{2p^*}\right)\|f\|_{\ell^p(\mathbb{Z})}, \quad 1 < p < \infty,$$

$p^* = \max(p, p/(p-1))$ *and hence*

$$\|H\|_{L^p(\mathbb{R})} = \|\mathcal{H}\|_{\ell^p(\mathbb{Z})} = \|\mathcal{J}_{\mathbb{H}}\|_{\ell^p(\mathbb{Z})}$$

Calderón-Zygmund (1952)

$$Tf(x) = p.v. \int_{\mathbb{R}^d} K(y)f(x-y)dy,$$

K a Calderón-Zygmund (C-Z) kernel.

$$\|T\|_p \leqslant C_{p,d}\|f\|_p, \quad 1 < p < \infty$$

Discrete versions: $f : \mathbb{Z}^d \to \mathbb{R}$, $n \in \mathbb{Z}^d$

$$\mathcal{T}f(n) = \sum_{m \in \mathbb{Z}^d \setminus \{0\}} K(m)f(n-m).$$

As Riesz and Titchmarsh:

$$\|T\|_p \approx \|\mathcal{T}\|_p$$

A. Magyar, E.M. Stein, S Waigner (2002):

$$\|T\|_{L^p(\mathbb{R}^d)} \leqslant \|\mathcal{T}\|_{\ell^p(\mathbb{Z}^n)} \leqslant C_d \|T\|_{L^p(\mathbb{R}^d)},$$

C_d depending only on dimension.

Magyar–Stein–Waigner: Remark

"It would be interesting to know if C_d can be taken to be independent of d or for that matter if $C = 1$."

For Riesz transforms, R_j, $j = 1, 2, \ldots d$,

$$K_j(x) = \frac{c_d x_j}{|x|^{d+1}}$$

Known (Iwaniec–Martin (1996), Wang-R.B(1995))

$$\|R_j\|_p = \|H\|_p = cot\left(\frac{\pi}{2p^*}\right)$$

For discrete Riesz Transforms on $\mathbb{R}^d$, $d > 1$. ± 1 in the $(j^{th} + 1)$ place:

$$H_j = \begin{bmatrix} 0 & 0 & \dots & -1 & 0 & \dots & 0 \\ 0 & 0 & \dots & \dots & \dots & \dots & 0 \\ \vdots & & & & & & \\ 1 & 0 & \dots & \dots & \dots & \dots & 0 \\ \vdots & & & & & & \\ 0 & \dots & \dots & \dots & \dots & \dots & 0 \\ \vdots & \vdots & & & & & \vdots \\ 0 & \dots & \dots & \dots & \dots & \dots & 0 \end{bmatrix} \quad |H_j v| \leqslant |v|, \quad \langle H_j v, v \rangle = 0, \ \ v \in \mathbb{R}^d$$

One computes:

$$\begin{aligned} \mathcal{J}_n = & \int_{\mathbb{R}^d} \int_0^\infty \frac{2y}{h(x,y)} H_j \nabla p_n(x,y) \cdot \nabla p_0(x,y) dy dx \\ & + \int_{\mathbb{R}^d} \int_0^\infty 4y p_0(x,y) H_j \nabla p_n(x,y) \cdot \nabla \Big(\frac{1}{h(x,y)}\Big) dy dx \end{aligned}$$

$$\mathcal{J}_n = c_d \mathcal{R}_j(n) + E_j(n)$$

with $E_j(n)$ "error" term. But ...

Thank You!