Finite Dimensional Distributions. What can you do with them?

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Want to study:

properties of the functions:

$$\Phi_m(x,D) = P_x\{B_{t_1} \in D, B_{t_2} \in D, \ldots, B_{t_m} \in D\}$$

B_t = Brownian motion (twice the speed) in $\mathbb{R}^d$, $D \subset \mathbb{R}^d$ open connected (referred to as "domains"), $x \in D$,

$$0 < t_1 < t_2 \cdots < t_m$$

Same as studying Multiple Integrals:

$$\Phi_m(x,D) = \int_D \cdots \int_D \prod_{j=1}^m p^{(2)}_{t_j - t_{j-1}}(x_j - x_{j-1}) dx_1 \ldots dx_m,$$

$$x_0 = x \quad \text{and} \quad p_t^{(2)}(y) = \frac{1}{(4\pi t)^{d/2}} e^{-|y|^2/4t}$$

More general, study for any times:

$$\Phi_m(x,D) = \int_D \cdots \int_D \prod_{j=1}^m p^{(2)}_{t_j}(x_j - x_{j-1}) dx_1 \ldots dx_m, \; 0 < t_j < \infty$$

Question

What is the smallest Dirichelt eigenvalue $\lambda_{1,\alpha}$ for the rotationally invariant stable processes of order $0 < \alpha < 2$ for the interval $(-1, 1)$?

Note: I learned this from Davar Khoshnevisan about 8 years ago.
Has been investigated by

- Investigated by: M.Kac-H. Pollar (1950). H. Widom (1961), J. Taylor (1967), B. Fristedt (1974), J. Bertoin (1996), Khosnevisan–Z. Shi (1998).

- I don't know the answer and, to be perfectly honest, don't care.
- In the process of investigating this "simple" question we "discovered" that little is known about the "fine" spectral theoretic properties of stables.
- More Exciting: The techniques give new Theorem for the Laplacian (BM).

R.B.and R. Latala and P. Méndez (2001) and R.B.and T. Kulczycki (2004)

$$C_{\alpha,d} = \frac{\Gamma(\frac{d}{2})}{2^{\alpha}\Gamma(1+\frac{d}{2})\Gamma(\frac{d+\alpha}{2})}$$

$B(0,1) =$ unit ball in $\mathbb{R}^d$.

$$\frac{1}{C_{\alpha,d}} \leq \lambda_{1,\alpha}(B(0,1)) \leq \frac{1}{C_{\alpha,d}} \frac{B(d/2, \alpha/2+1)}{B(\alpha/2, \alpha+1)}$$

For $\alpha = 1$ (Cauchy processes), $B(0,1) = (-1,1)$ (as in Davar's question)

$$1 \leq \lambda_{1,1} \leq \frac{3\pi}{8} \approx 1.178$$

Note:

$$\frac{3\pi}{8} < \frac{\pi}{2} = \sqrt{\frac{\pi^2}{4}}$$

That is, eigenvalue for Cauchy is not the square root of the one for Brownian motion!

Variational Formula: For any $D \subset \mathbb{R}^d$

The Dirichlet form, $(\mathcal{E}, \mathcal{F})$, for stables processes, $0 < \alpha < 2$, in $\mathbb{R}^d$ is:

$$\mathcal{E}(f,g) = A_{\alpha,d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{(f(x)-f(y))(g(x)-g(y))}{|x-y|^{\alpha+d}}\, dx\, dy$$

and

$$\mathcal{F} = \left\{ f \in L^2(\mathbb{R}^d) : \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|f(x)-f(y)|^2}{|x-y|^{\alpha+d}}\, dx\, dy < \infty \right\}$$

with

$$A_{\alpha,d} = \frac{\alpha \Gamma(\frac{d+\alpha}{2})}{2^{1-\alpha} \pi^{d/2} \Gamma(1-\frac{\alpha}{2})}$$

From this we have for any region $D \subset \mathbb{R}^d$:

$$\lambda_{1,\alpha}(D) = \inf\left\{A_{\alpha,d}\int_D\int_D \frac{|u(x)-u(y)|^2}{|x-y|^{\alpha+d}}\,dx\,dy + 2A_{\alpha,d}\int_D |u(x)|^2 k_D(x)dx\right\}$$

where "inf" is over all $u \in C_0^\infty$ with

$$\int_D |u(y)|^2 dy = 1.$$

$$K_D(x) = \int_{D^c} \frac{dy}{|x-y|^{\alpha+d}}$$

Theorem (Chungs's LIL. Set $B_t^* = \sup_{0 \le s \le t} |B_s|$)

$$\liminf_{t\to\infty} \left(\frac{\log\,\log t}{t}\right)^{1/2} B_t^* = \frac{\pi}{2}, \quad a.s. \tag{1}$$

But, is $\frac{\pi}{2}$ really just our "good-old-friend" $\frac{\pi}{2}$ or is it something else?
(1) comes from Borel–Cantelli arguments and the "small balls" probability estimate.

$$P_0\left\{B_1^* < \varepsilon\right\} \approx e^{-\frac{\pi^2}{4\varepsilon^2}}, \ \varepsilon \to 0$$

$$P_0\left\{B_1^* < \varepsilon\right\} = P_0\left\{\frac{1}{\varepsilon} B_t^* < 1\right\} = P_0\left\{B^*_{\frac{1}{\varepsilon^2}} < 1\right\} = P_0\left\{\tau_{(-1,1)} > \frac{1}{\epsilon^2}\right\}$$

$$\tau_{(-1,1)} = \inf\{t > 0 : B_t \notin (-1,1)\} = \text{ first exit time from the interval}$$

As we shall see,

$$P_0\left\{\tau_{(-1,1)} > t\right\} \approx e^{-\lambda_1 t}\varphi_1(0)\int_1^1 \varphi_1(y)\,dy, \quad t \to \infty,$$

where λ_1 is the smallest eigenvalue for one half of the Laplacian in the interval $(-1,-1)$ with Dirichlet boundary conditions and φ_1 is the corresponding eigenfunction. That is, $\pi^2/4$ and the "sin" function.

For any $0 < \alpha < 2$, let X_t^α be the rotationally invariant stable process of order α. A similar statement holds for the "small ball" probabilities and there is

Theorem (J. Taylor 1967)

$$\liminf_{t\to\infty}\left(\frac{\log\,\log t}{t}\right)^{1/\alpha} X_t^* = (\lambda_{1,\alpha})^{1/\alpha}, \quad a.s. \tag{2}$$

For several other occurrences of the eigenvalue in "sample path behavior," see **Erkan Nane:** "Higher order PDE's and iterated Processes" and "Iterated Brownian motion in bounded domains in $\mathbb{R}^n$ "

A Lévy Process is a stochastic process $X = (X_t), t \geq 0$ with

- X has independent and stationary increments
- $X_0 = 0$ (with probability 1)
- X is *stochastically continuous*: For all $\varepsilon > 0$,

$$\lim_{t \to s} P\{|X_t - X_s| > \varepsilon\} = 0$$

Note: Not the same as a.s. continuous paths. However, it gives "cadlag" paths: Right continuous with left limits.

- **Stationary increments:** $0 < s < t < \infty$, $A \in \mathbb{R}^d$ Borel

$$P\{X_t - X_s \in A\} = P\{X_{t-s} \in A\}$$

- **Independent increments:** For any given sequence of ordered times

$$0 < t_1 < t_2 < \cdots < t_m < \infty,$$

the random variables

$$X_{t_1} - X_0,\; X_{t_2} - X_{t_1}, \ldots, X_{t_m} - X_{t_{m-1}}$$

are independent.

The characteristic function of X_t is

$$\varphi_t(\xi) = E\left(e^{i\xi\cdot X_t}\right) = \int_{\mathbb{R}^d} e^{i\xi\cdot x} p_t(dx) = (2\pi)^{d/2}\widehat{p_t}(\xi)$$

where p_t is the distribution of X_t. Notation (same with measures)

$$\widehat{f}(\xi) = \frac{1}{(2\pi)^{d/2}}\int_{\mathbb{R}^d} e^{ix\cdot\xi} f(x)dx, \;\; f(x = \frac{1}{(2\pi)^{d/2}}\int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(\xi)d\xi$$

The characteristic function has the form $\varphi_t(\xi) = e^{t\rho(\xi)}$, where

$$\rho(\xi) = ib \cdot \xi - \frac{1}{2}\xi \cdot A\xi + \int_{\mathbb{R}^d} \left(e^{i\xi \cdot x} - 1 - i\xi \cdot x 1_{\{|x|<1\}}(x)\right)\nu(dx)$$

for some $b \in \mathbb{R}^d$, a non–negative definite symmetric $n \times n$ matrix A and a Borel measure ν on $\mathbb{R}^d$ with $\nu\{0\} = 0$ and

$$\int_{\mathbb{R}^d} \min\left(|x|^2, 1\right)\nu(dx) < \infty$$

$\rho(\xi)$ is called the **symbol** of the process or the **characteristic exponent.** The triple (b, A, ν) is called the **characteristics of the process.**

Converse also true. Given such a triple we can construct a Lévy process.

7. **The rotationally invariant stable processes:** These are self–similar processes, denoted by X_t^α, in $\mathbb{R}^d$ with symbol

$$\rho(\xi) = -|\xi|^\alpha, \qquad 0 < \alpha \leq 2.$$

That is,

$$\varphi_t(\xi) = E\left(e^{i\xi\cdot X_t^\alpha}\right) = e^{-t|\xi|^\alpha}$$

$\alpha = 2$ is **Brownian motion.** $\alpha = 1$ is the **Cauchy processes.**

Transition probabilities:

$$P_x\{X_t^\alpha \in A\} = \int_A p_t^\alpha(x-y)dy, \qquad \text{any Borel} \qquad A \subset \mathbb{R}^d$$

$$p_t^\alpha(x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{-i\xi\cdot x}e^{-t|\xi|^\alpha}d\xi$$

$$p_t^2(x) = \frac{1}{(4\pi t)^{d/2}}e^{-\frac{|x|^2}{4t}}, \quad \text{(log-concave)} \quad \alpha = 2, \quad \textbf{Brownian motion}$$

$$p_t^1(x) = \frac{C_d\, t}{(|x|^2+t^2)^{\frac{d+1}{2}}}, \quad \text{(NOT log-concave)} \quad \alpha = 1, \quad \textbf{Cauchy Process}$$

A subordinator is a one-dimensional Lévy process $\{T_t\}$ such that

(i) $T_t \geq 0$ a.s. for each $t > 0$, (ii) $T_{t_1} \leq T_{t_2}$ a.s. whenever $t_1 \leq t_2$

Theorem (Bertoin, p.73: Laplace transforms)

$$E(e^{-\lambda T_t}) = e^{-t\psi(\lambda)},\ \lambda > 0,$$

$$\psi(\lambda) = b\lambda + \int_0^\infty \left(1 - e^{-\lambda s}\right) \nu(ds)$$

$b \geq 0$ and the Lévy measure satisfies $\nu(-\infty, 0) = 0$ and $\int_0^\infty \min(s,1)\nu(ds) < \infty$. ψ is called the Laplace exponent of the subordinator.

Theorem (**Applebaum, p. 53**)

If X is an arbitrary Lévy process and T is a subordinator independent of X, then $Z_t = X_{T_t}$ is a Lévy process. For any Borel $A \subset \mathbb{R}^d$,

$$p_{Z_t}(A) = \int_0^\infty p_{X_s}(A) p_{T_t}(ds)$$

Lévy semigroup

For the Lévy process $\{X(t); t \geq 0\}$, define

$$T_t f(x) = E[f(X(t))|X_0 = x] = E_0[f(X(t)+x)], \qquad f \in \mathcal{S}(\mathbb{R}^d).$$

This is a Feller semigroup (takes $C_0(\mathbb{R}^d)$ into itself). Setting

$$p_t(A) = P_0\{X_t \in A\} \;\; \text{(the distribution of } X_t\text{)}$$

we see that (by Fourier inversion formula)

$$T_t f(x) = \int_{\mathbb{R}^d} f(x+y)p_t(dy) = p_t * f(x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{-ix\cdot\xi} e^{t\rho(\xi)}\widehat{f}(\xi)d\xi$$

with generator

$$\begin{aligned} Af(x) &= \left.\frac{\partial T_t f(x)}{\partial t}\right|_{t=0} = \lim_{t\to 0}\frac{1}{t}\Big(E_x[f(X(t)] - f(x)\Big) \\ &= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{-ix\cdot\xi}\rho(\xi)\hat{f}(\xi)d\xi = \text{a pseudo diff operator, in general} \end{aligned}$$

Many questions on the “fine” potential theoretic properties of solutions for $(-\Delta)^{\alpha/2}$ and more general Lévy processes, especially subordinations of Brownian motion, have been studied by many authors in recent years. **Examples:**

- Regularity of heat kernels, general solutions of “heat equation”, Sobolev, log-Sobolev inequalities, “intrinsic ultracontractivity,” . . .
- “Boundary” regularity of solutions, including boundary Harnack principle, “gauge theorems,” Fatou theorems, Martin boundary, . . .

I am interested in the “fine” spectral theoretic properties of these processes

- Estimates on eigenvalues, including the ground state $\lambda_{1,\alpha}$ and the spectral gap $\lambda_{2,\alpha} - \lambda_{1,\alpha}$, Number of “nodal” domains (Courant–Hilbert Nodal domain Theorem), geometric properties of eigenfunctions, including a “Brascamp–Lieb” log–concavity type theorem for $\varphi_{1,\alpha}$, . . .

From now on $X_t = X_t^\alpha$ is rotationally invariant stable with symbol

$$\rho(\xi) = -|\xi|^\alpha, \qquad 0 < \alpha \leq 2.$$

Let D be a bounded connected subset of $\mathbb{R}^d$. The first exit time of X_t^α from D is

$$\tau_D = \inf\{t > 0 : X_t^\alpha \notin D\}$$

Heat Semigroup in D is the self-adjoint operator

$$\begin{aligned} T_t^D f(x) &= E_x\Big[f(X_t^\alpha); \tau_D > t\Big], \qquad f \in L^2(D) \\ &= \int_D p_t^{D,\alpha}(x,y) f(y) dy, \end{aligned}$$

$$p_t^{D,\alpha}(x,y) = p_t^\alpha(x-y) - E^x(\tau_D < t; p_{t-\tau_D}^\alpha(X_{\tau_D}^\alpha, y)).$$

$p_t^{D,\alpha}(x,y)$ is called the **Heat Kernel for the stable process in D.**

$$\begin{aligned} p_t^{D,\alpha}(x,y) \leq p_t^{\alpha}(x-y) \leq p_1^{\alpha}(0)t^{-d/\alpha} &= \left(\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{-|\xi|^{\alpha}}d\xi\right)t^{-d/\alpha} \\ &= t^{-d/\alpha}\,\frac{\omega_d}{(2\pi)^d\alpha}\int_0^{\infty} e^{-s}s^{(\frac{n}{\alpha}-1)}ds \\ &= t^{-d/\alpha}\,\frac{\omega_d\Gamma(d/\alpha)}{(2\pi)^d\alpha} \end{aligned}$$

The general theory of heat semigroups gives an orthonormal basis of eigenfunctions

$$\{\varphi_{m,\alpha}\}_{m=1}^{\infty} \quad on \quad L^2(D)$$

with eigenvalues $\{\lambda_{m,\alpha}\}$ satisfying

$$0 < \lambda_{1,\alpha} < \lambda_{2,\alpha} \leq \lambda_{3,\alpha} \leq \cdots \to \infty$$

That is,

$$T_t^D\varphi_{m,\alpha}(x) = e^{-\lambda_{m,\alpha}t}\varphi_{m,\alpha}(x), \quad x \in D.$$

$$p_t^{D,\alpha}(x,y) = e^{-\lambda_{1,\alpha} t}\varphi_{1,\alpha}(x)\varphi_{1,\alpha}(y) + \sum_{m=2}^{\infty} e^{-\lambda_{m,\alpha} t}\varphi_{m,\alpha}(x)\varphi_{m,\alpha}(y)$$

Theorem (From "Intrinsic Ultracontractivity")

$$e^{-(\lambda_{2,\alpha}-\lambda_{1,\alpha})t} \le \sup_{x,y\in D}\left|\frac{e^{\lambda_{1,\alpha} t} p_t^{D,\alpha}(x,y)}{\varphi_{1,\alpha}(x)\varphi_{1,\alpha}(y)} - 1\right| \le C(D,\alpha)e^{-(\lambda_{2,\alpha}-\lambda_{1,\alpha})t}, \ t \ge 1.$$

For $\alpha = 2$ this is valid for "many" domains but not all. For $0 < \alpha < 2$, valid for any bounded domain.

Theorem (Implied by the Intrinsic Ultracontractivity result)

$$\lim_{t\to\infty} e^{t\lambda_{1,\alpha}} P_x\{\tau_D > t\} = \varphi_{1,\alpha}(x)\int_D \varphi_{1,\alpha}(y)dy, \tag{3}$$

$$\lim_{t\to\infty} \frac{1}{t}\log P_x\{\tau_D > t\} = -\lambda_{1,\alpha}, \tag{4}$$

uniformly for $x \in D$.

If I want to study the eigenfunction $\varphi_{1,\alpha}$ and $\lambda_{1,\alpha}$ and how these are affected by the geometry of the domain D, I should (better, must, ...) study the distribution of the exit time τ_D of the process. That is, study

$$P_x\{\tau_D > t\}$$

as a function of D, $x \in D$, $t > 0$.

But (modulo a technical point):

$$\begin{aligned} P_x\{\tau_D > t\} &= P_z\{X_s^\alpha \in D;\ \forall s, 0 < s \leq t\} \\ &= \lim_{m\to\infty} P_z\{X_{jt/m}^\alpha \in D, j = 1, 2, \ldots, m\} \\ &= \lim_{m\to\infty} \int_D \cdots \int_D p_{t/m}^\alpha(x - x_1) \cdots p_{t/m}^\alpha(x_m - x_{m-1}) dx_1 \ldots dx_m \end{aligned}$$

$$
\begin{aligned}
& P_x\{X^\alpha_{t_1} \in D, \ldots, X^\alpha_{t_m} \in D\} \\
&= \int_D \cdots \int_D \prod_{i=1}^m p^\alpha_{t_i - t_{i-1}}(x_i - x_{i-1})\, dx_1 \ldots dx_n \\
&= \int_0^\infty \ldots \int_0^\infty \left(\int_D \cdots \int_D \prod_{i=1}^m p^2_{s_i}(x_i - x_{i-1})\, dx_1 \ldots dx_n \right) \\
&\times \prod_{i=1}^n g_{\alpha/2}(t_i - t_{i-1}, s_i)\, ds_1 \ldots ds_m \\
&= \int_0^\infty \ldots \int_0^\infty P_x\{B_{2s_1} \in D, B_{2(s_1+s_2)} \in D, \ldots, B_{2(s_1+s_2+\cdots+s_n)} \in D\} \\
&\times \prod_{i=1}^m g_{\alpha/2}(t_i - t_{i-1}, s_i)\, ds_1 \ldots ds_m.
\end{aligned}
$$

Must study the function

$$
\Phi_m(x, D) = \int_D \cdots \int_D \prod_{i=1}^m p^2_{t_i}(x_i - x_{i-1})\, dx_1 \ldots dx_m, \qquad x_0 = x
$$

No order on t_i.

For $A \subset \mathbb{R}^d$, A^* =ball centered at the origin and same volume as A. $\chi_A^* = \chi_{A^*}$
$f : \mathbb{R}^d \to \mathbb{R}$,

$$f^*(x) = \int_0^\infty \chi^*_{\{|f|>t\}}(x)dt$$

(Compare this with)

$$|f(x)| = \int_0^\infty \chi_{\{|f|>t\}}(x)dt$$

Properties:

$$f^*(x) = f^*(y), \quad |x| = |y|, \quad f^*(x) \geq f^*(y), \quad |x| \leq |y|$$

$$\{x : f^*(x) > t\} = \{x : |f(x)| > t\}^* \text{ (same level sets)}$$

$$\Rightarrow m\{x : f^*(x) > \lambda\} = m\{x : |f(x)| > \lambda\}$$

Theorem (Luttinger 1973)

Let $f_1, \ldots, f_m$ be nonnegative functions in R^d and let $f_1^*, \ldots, f_m^*$ be their symmetric decreasing rearrangement. Then for any $x_0 \in D$ we have

$$\int_{D^m} \prod_{j=1}^{m} f_j(x_j - x_{j-1}) dx_1 \cdots dx_m \leq \int_{\{D^*\}^m} f_1^*(x_1) \prod_{j=2}^{m} f_j^*(x_j - x_{j-1}) dx_1 \cdots dx_m.$$

D^*=ball center at zero and and same volume as D

Theorem (R. B. Latala, Méndez, 2001 ($d = 2$), Méndez 2003, $d \geq 3$)

$D \subset \mathbb{R}^d$ convex of finite inradius r_D and S infinite strip of inradius r_D Let $f_1, \ldots, f_m$ be nonnegative radially symmetric decreasing on $\mathbb{R}^d$. For any $z_0 \in \mathbb{R}^d$,

$$\int_D \cdots \int_D \prod_{j=1}^{m} f_j(z_j - z_{j-1}) \, dz_1 \cdots dz_m \leq$$

$$\int_S \cdots \int_S f_1(z_1) \prod_{j=2}^{m} f_j(z_j - z_{j-1}) \, dz_1 \cdots dz_m.$$

Corollary (Isoperimetric property for stable and "all" symmetric Lévy Processes)

$$\Phi_m(x, D) \leq \Phi_m(0, D^*)$$

$$P_x\{\tau_D^\alpha > t\} \leq P_0\{\tau_{D^*}^\alpha > t\}$$

$$\sup_{x \in D} E_x(\tau_D^\alpha) \leq E_0(\tau_{D^*}^\alpha)$$

$$\lambda_{1,\alpha}(D^*) \leq \lambda_{1,\alpha}(D) \quad \textit{The Faber-Krahn Theorem}$$

$$Cap_\alpha(A) \geq Cap_\alpha(A^*),$$

(α-capacity version of a theorem of Polya–Szego. Proved by Watanabe 1984, conjectured by Mattila 1990, Proved by Betsakos 2003, P. Méndez 2006)

Corollary (Same for the Trace)

$$\begin{aligned} Z_t^\alpha(D) &= \sum_{m=1}^{\infty} e^{-t\lambda_{m,\alpha}(D)} = \int_D p_t^{\alpha,D}(x,x)dx \\ &\leq \int_{D^*} p_t^{\alpha,D^*}(x,x)dx \leq \sum_{m=1}^{\infty} e^{-t\lambda_{m,\alpha}(D^*)} = Z_t^\alpha(D^*) \end{aligned}$$

Amongst all regions of equal volume the ball minimizes surface area. It follows from "trace inequality" and

Theorem (M. Kac, "Can one hear the shape of a drum?")

With $\alpha = 2$, $|\partial D|$=*surface area of boundary of D,*

$$Z_t^2(D) \sim C_d t^{-d/2} vol(D) - C_d' t^{-(d-1)/2} |\partial D| + o(t^{-(d-1)/2}, \quad t \to 0$$

The first term is trivial from

$$\begin{aligned} P_t^{2,D}(x,y) &= \frac{1}{(4\pi t)^{d/2}} e^{\frac{-|x-y|^2}{4t}} P_x\{\tau_D > t | B_t = y\} \\ &= \text{free motion times Brownian bridge in D} \end{aligned}$$

$$\lim_{t\to 0} t^{\gamma}\int_0^{\infty} e^{-t\lambda}d\mu(\lambda) = A \Rightarrow \lim_{a\to\infty} a^{-\gamma}\mu[0,a) = \frac{A}{\Gamma(\gamma+1)}$$

Theorem (Weyl's Formula, $\alpha = 2$. $N_D(\lambda) = \#\{j \geq 1 : \lambda_j \leq \lambda\}$)

$$N_D(\lambda) \sim C_d vol(D)\lambda^{d/2}, \quad \lambda \to \infty$$

More difficult (and no probabilistic treatment exists):

$$N_D(\lambda) \sim C_d vol(D)\lambda^{d/2} - C_d'|\partial D|\lambda^{(d-1)/2} + o(\lambda^{(d-1)/2})$$

Theorem (R.B. and T. Kulczycki (2007). $0 < \alpha \leq 2$)

$$\int_D p_t^{\alpha,D}(x,x)dx \sim C_{d,\alpha}t^{-d/\alpha}vol(D) - C_d' t^{-(d-1)/\alpha}|\partial D| + o(t^{-(d-1)/\alpha}$$

as $t \to 0$. This gives Weyl's version for all $0 < \alpha \leq 2$.

The $$ Question: Is there an α–version of the more general Weyl?

Question

Amongst all convex domains $D \subset \mathbb{R}^d$ of inradius 1, which one has the largest exit time for Brownian motion? Also, lowest eigenvalue? **Answer:** *The infinite strip:*

$$S = \mathbb{R}^{d-1} \times (-1, 1)$$

Theorem (For D convex with inradius 1.)

$$\Phi_m(x, D) \leq \Phi_m(0, S), \quad x \in D$$

R.B. Méndez-Latala (2001), $d = 2$ and (2003), $d \geq 3$. (Convexity is essential here!)

Corollary (For D convex with inradius 1 and $0 < \alpha \leq 2$.)

$$P_x\{\tau_D > t\} \leq P_0\{\tau_S > t\} = P_0\{\tau_{(-1,1)} > t\} \quad (5)$$

$$\lambda_{1,\alpha}(-1, 1) \leq \lambda_{1,\alpha}(D) \quad (6)$$

Definition: $F : \mathbb{R}^d \to \mathbb{R}$ is said to be **log-concave** if

$$\log F(\beta x + (1-\beta)y) \geq \beta \log F(x) + (1-\beta) \log F(y), \quad x, y \in \mathbb{R}^d$$

or

$$F(\beta x + (1-\beta)y) \geq F(x)^{\beta} F(y)^{1-\beta}$$

Examples:

$$F(x) = \frac{1}{(4\pi)^{d/2}} e^{-|x|^2/4}$$

and

$$F(x) = \chi_D(x),$$

$D \subset \mathbb{R}^d$ is convex, are log–concave.

Theorem (Prékopa (1971))

Convolutions of log-concave functions are log-concave.

Corollary ($D \subset \mathbb{R}^d$ convex)

For Brownian motion, the function $\Phi_m(x, D)$ is log-concave.

Corollary (Brascamp-Lieb (1979))

For any bounded convex domain $D \subset \mathbb{R}^d$ and for Brownian motion, the function $P_x\{\tau_D > t\}$ is log-concave and therefore so is the "ground state" eigenfunction $\varphi_{1,2}(x)$. In fact, this holds for the "ground state" eigenfunction for the Scrödinger operator $-\Delta + V$ where $V : D \to [0, \infty)$ is convex.

Note: Unfortunately we cannot conclude the same for $0 < \alpha < 2$. Why?

Question ($D \subset \mathbb{R}^d$ convex, $0 < \alpha < 2$)

Are the functions $P_x\{\tau_D > t\}$ and/or $\varphi_{1,\alpha}(x)$ log-convex?

Known only for $\alpha = 1$, $D = (-1, 1)$. In fact, in this case the functions are concave, just like for $\alpha = 2$.

Several other partial results are known for special doubly symmetric domains in the plane.

Definition

$D \subset \mathbb{R}^d$ be a convex symmetric relative to each coordinate axes. J any line segment in D parallel to the x_1-axis intersecting ∂D only at the two points. $F : D \to \mathbb{R}$, is *mid–concave* on J if it is concave on mid half of J. F mid–concave along the x_1–axis if it is mid–concave on every such segment contained in D parallel to the x_1–axis. Same for mid–concavity along the x_2-axis, $\cdots$. F mid–concave on D if it is mid–concave along each coordinate axes.

Theorem (R.B.-Méndez-Kulczycki, 2006)

$Q \subset R^d$ a rectangle. $\Phi_m(x, Q) = P_x\{X^\alpha_{t_1} \in Q, \ldots, X^\alpha_{t_m} \in Q\}$ is mid–concave in Q for any $0 < \alpha \leq 2$. In addition, if $x = (x_1, \ldots, x_n) \in Q$, then

$$\frac{\partial}{\partial x_i} F(x) \geq 0, \text{ if } x_i < 0, \text{ and } \frac{\partial}{\partial x_i} F(x) \leq 0, \text{ if } x_i > 0.$$

But (recall, $x_0 = x$)

$$\Phi_m(x) = \int_{-1}^{1} \cdots \int_{-1}^{1} \prod_{i=1}^{m} p^{(2)}_{t_i}(x_{i-1} - x_i)\, dx_1 \ldots dx_m, \text{ not concave on } (-1, 1) \text{ for all } t_i.$$

The maximum (and the minimum) of the “first” non-constant Neumann eigenfunction for bounded convex domains are attained on the boundary and only on the boundary of the domain.

Many partial results: R.B.-K.Burdzy (1999), D.Jerison-N.Darishavilli (2000), M. Pascu (2001), R. Bass–K. Burdzy (2000), R.B.-M. Pang (2003), R.B. M.Pang-Pascu (2004), R.Atar K.Burdzy (2005)

Counterexample: K. Burdzy-W. Werner (2000), K. Burdzy (2005)

Believed to be true for any simply connected domain, conjectured to be true for any convex domain.

Unknown even for an arbitrary triangle in the plane!

Conjecture: The maximum and minimum for the first nonconstant eigenfunction for the semigroup of Brownian motion conditioned to remain forever in a convex domain are attained on the boundary and only on the boundary of the domain.

That is, the function φ_2/φ_1 attains its maximum and minimum on the boundary and only on the boundary of D.

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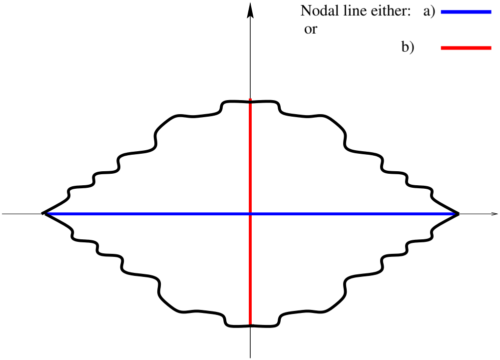

Theorem

Let D be a bounded domain in $\mathbb{R}^2$ which is symmetric and convex with respect to both axes.

(i) *If $z_1 = (x, y_1) \in D^+$, $z_2 = (x, y_2) \in D^+$ and $y_1 < y_2$, then*

$$\frac{P_{z_1}\{\tau_{D^+} > t\}}{P_{z_1}\{\tau_D > t\}} < \frac{P_{z_2}\{\tau_{D^+} > t\}}{P_{z_2}\{\tau_D > t\}},$$

for any $t > 0$. In particular, the function

$$\Psi(z,t) = \frac{P_z\{\tau_{D^+} > t\}}{P_z\{\tau_D > t\}},$$

for each $t > 0$ arbitrarily fixed, cannot have a maximum at an interior point of D^+.

(ii) *If $z_1 = (x_1, y) \in D^+$ and $z_2 = (x_2, y) \in D^+$ with $|x_2| \leq |x_1|$, then*

$$\frac{P_{z_1}\{\tau_{D^+} > t\}}{P_{z_1}\{\tau_D > t\}} \leq \frac{P_{z_2}\{\tau_{D^+} > t\}}{P_{z_2}\{\tau_D > t\}},$$

for any $t > 0$.

Corollary

$D \subset \mathbb{R}^2$ as in Theorem φ_2 be such that its nodal line is the intersection of the x-axis with the domain. Without LOG, $\varphi_2 > 0$ in D^+ and $\varphi_2 < 0$ in D^-. Set $\Psi = \varphi_2/\varphi_1$.

(i) *If $z_1 = (x, y_1) \in D^+$ and $z_2 = (x, y_2) \in D^+$ with $y_1 < y_2$, then*

$$\Psi(z_1) < \Psi(z_2).$$

(ii) *If $z_1 = (x, y_1) \in D^-$ and $z_2 = (x, y_2) \in D^-$ with $y_2 < y_1$, then*

$$\Psi(z_1) < \Psi(z_2).$$

In particular, Ψ cannot attain a maximum nor a minimum in the interior of D.

(iii) *If $z_1 = (x_1, y) \in D^+$ and $z_2 = (x_2, y) \in D^+$ with $|x_2| < |x_1|$, then*

$$\Psi(z_1) \leq \Psi(z_2). \tag{7}$$

Corollary (Exact analogue of D. Jerison and N. Nadirashvili (2000) for classical "hot–spots")

Suppose $D \subset \mathbb{R}^2$ is a bounded domain with piecewise smooth boundary which is symmetric and convex with respect to both coordinate axes and that φ_2 is as in Theorem 1.2. Then strict inequality holds in (7) unless D is a rectangle. The maximum and minimum of Ψ on $\overline{D}$ are achieved at the points where the y-axis meets ∂D and, except for the rectangle, at no other points.