THE ℓ^p NORM OF THE RIESZ–TITCHMARSH TRANSFORM FOR EVEN INTEGER p

RODRIGO BAÑUELOS AND MATEUSZ KWAŚNICKI

ABSTRACT. The long-standing conjecture that for $p \in (1, \infty)$ the $\ell^p(\mathbb{Z})$ norm of the Riesz–Titchmarsh discrete Hilbert transform is the same as the $L^p(\mathbb{R})$ norm of the classical Hilbert transform, is verified when $p = 2n$ or $\frac{p}{p-1} = 2n$, for $n \in \mathbb{N}$. The proof, which is algebraic in nature, depends in a crucial way on the sharp estimate for the $\ell^p(\mathbb{Z})$ norm of a different variant of this operator for the full range of p. The latter result was recently proved by the authors in [2].

1. INTRODUCTION AND STATEMENT OF MAIN RESULT

There are several non-equivalent definitions of the *discrete Hilbert transform* on $\mathbb{Z}$. In this paper we focus on the one introduced by E.C. Titchmarsh in [18] and often called the Riesz–Titchmarsh operator:

$$\mathcal{R}a_n = \frac{1}{\pi} \sum_{m \in \mathbb{Z}} \frac{a_{n-m}}{m + \frac{1}{2}} \,. \tag{1.1}$$

A conjecture that has remained open for almost a century (see [19]) states that for $p \in (1, \infty)$ the norm of $\mathcal{R}$ on $\ell^p(\mathbb{Z})$ equals the norm of

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the (continuous) Hilbert transform

$$Hf(x) = \text{p.v.}\ \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f(x-y)}{y}\,dy \tag{1.2}$$

on $L^p(\mathbb{R})$. The latter was found by S. Pichorides in [15] to be $\cot\frac{\pi}{2p}$ when $p \geqslant 2$ and $\tan\frac{\pi}{2p}$ when $p \leqslant 2$. That is,

$$\|H\|_p = \cot\frac{\pi}{2p^*},$$

where $p^* = \max\{p, \frac{p}{p-1}\}$ and $\frac{p}{p-1}$ is the conjugate exponent of p. The main result of this paper resolves this conjecture when p is an even integer and, by duality, when p is the conjugate exponent of an even integer. More precisely, we prove the following result.

Theorem 1.1. *For $p = 2n$ or $p = \frac{2n}{2n-1}$, $n \in \mathbb{N}$, we have*

$$\|(\mathcal{R}a_n)\|_p \leqslant \cot\frac{\pi}{2p^*}\,\|(a_n)\|_p,$$

for every sequence (a_n) in ℓ^p. The constant is best possible. In particular, the operator norm of $\mathcal{R}$ on ℓ^p is equal to the operator norm of H on $L^p(\mathbb{R})$ for such p.

Although our proof is algebraic, it relies on the sharp ℓ^p bound for another natural candidate for the discrete Hilbert transform introduced by Hilbert in 1909, defined by

$$\mathcal{H}_0 a_n = \frac{1}{\pi}\sum_{m\in\mathbb{Z}\setminus\{0\}} \frac{a_{n-m}}{m}\,. \tag{1.3}$$

This bound was proved recently by the authors in [2] for every $p \in (1,\infty)$ using analytical and probabilistic techniques. For the reader's convenience, we state this result explicitly.

Theorem 1.2 ([2], Theorem 1.1)**.** *If $p \in (1,\infty)$, the ℓ^p norm of the operator $\mathcal{H}_0$ is given by*

$$\|\mathcal{H}_0\|_p = \cot\frac{\pi}{2p^*}. \tag{1.4}$$

In particular, the operator norm of $\mathcal{H}_0$ on ℓ^p is equal to the operator norm of H on $L^p(\mathbb{R})$.

We stress that in the proof of Theorem 1.1 we need the assertion of Theorem 1.2 for all $p \in (1,\infty)$ and not just for even integers p. Formula (1.4) when p is a power of 2 has been known for several year, see [14]. We refer to Remarks 4.1 and 4.2 below for further discussion.

We note, more generally, that one can consider the one-parameter family of operators defined by

$$T_t a_n = \begin{cases} \dfrac{\sin(\pi t)}{\pi} \displaystyle\sum_{m\in\mathbb{Z}} \dfrac{a_{n-m}}{m+t} & \text{if } t \in \mathbb{R}\setminus\mathbb{Z}, \\ (-1)^t a_{n+t} & \text{if } t\in\mathbb{Z}. \end{cases} \tag{1.5}$$

It is shown in Theorem 1.1 in [6] that for $t \geqslant 0$ these operators form a strongly continuous group of isometries on ℓ^2 with generator $\pi\mathcal{H}_0$. Note that when $t = \frac{1}{2}$, T_t corresponds to the Riesz–Titchmarsh operator. In [14], Theorem 5.6, the ℓ^p norm of these operators is studied for $t \in (0, 1)$ and $p \in (1,\infty)$ and it is proved that

$$\|T_t\|_p \geqslant \|\cos(t\pi)I + \sin(t\pi)H\|_p, \tag{1.6}$$

where I is the identity operator. In particular,

$$\|\mathcal{R}\|_p = \|T_{1/2}\|_p \geqslant \cot\frac{\pi}{2p^*}. \tag{1.7}$$

Conjecture 5.7 in [14] asserts that equality holds in (1.6). For all $t \in (0, 1)$, the right-hand side of (1.6) is evaluated explicitly (as a maximum of a certain function of one real variable) in Corollary 4.4 in [12].

For further history and references related to discrete Hilbert transforms and other classical discrete operators in harmonic analysis including Riesz transforms, we refer the reader to [1, 2, 3, 5, 7, 8, 9, 11, 16]. Some further historical remarks can be found in Section 3 below. For an early account of Riesz's proof on the boundedness of the Hilbert transform on $L^p(\mathbb{R})$ and of $\mathcal{H}_0$ on $\ell^p(\mathbb{Z})$ before publication in [17], we refer the reader to M.L. Cartwright's article [4] *Manuscripts of Hardy, Littlewood, Marcel Riesz and Titchmarsh*, Sections 6–8.

The remaining part of this article consists of six sections. First, in Section 2 we recall known properties of various variants of discrete Hilbert transforms. Section 3 briefly discusses other papers that involve related arguments. The idea of the proof of Theorem 1.1 is given in Section 4. The final three sections contain the actual proof: in Section 5 we introduce *skeletal decomposition* of integer powers of $\mathcal{R}a_n$, in Section 6 we use it to find an inequality involving the norms of $\mathcal{R}a_n$ on different ℓ^p spaces, and we complete the proof of the main result in Section 7.

2. Preliminaries

In this section we introduce basic notation and we gather some known results. For the convenience of the reader, we also provide short proofs where available.

We always assume that $p \in (1, \infty)$. By ℓ^p we denote the class of doubly infinite sequences $(a_n) = (a_n : n \in \mathbb{Z})$ such that

$$\|(a_n)\|_p^p = \sum_{n \in \mathbb{Z}} |a_n|^p$$

is finite. For a continuous linear operator $\mathcal{T}$ on ℓ^p we denote by $\|\mathcal{T}\|_p$ the operator norm of $\mathcal{T}$. We write a somewhat informal symbol $\mathcal{T} a_n$ for the n-th entry of the sequence obtained by applying $\mathcal{T}$ to the sequence (a_n), and $\|a_n\|_p$ for the norm of the sequence (a_n). We also commonly use the short-hand notation $\mathcal{T}[\varphi(n)]$ for the n-th entry of the application of the operator $\mathcal{T}$ to the sequence obtained by evaluating $\varphi(n)$ for $n \in \mathbb{Z}$. To improve readability, we often denote multiplication of numbers a and b by $a \cdot b$ rather than ab.

We work with a variant of the Riesz–Titchmarsh operator, studied, among others, by S. Kak [13], and thus sometimes called the *Kak–Hilbert transform*:

$$\mathcal{K} a_n = \frac{2}{\pi} \sum_{m \in 2\mathbb{Z}+1} \frac{a_{n-m}}{m} \,, \tag{2.1}$$

and a similarly modified operator $\mathcal{H}_0$, which we denote simply by $\mathcal{H}$:

$$\mathcal{H} a_n = \frac{2}{\pi} \sum_{m \in 2\mathbb{Z} \setminus \{0\}} \frac{a_{n-m}}{m} \,. \tag{2.2}$$

We also commonly use the operator

$$\mathcal{I} a_n = \frac{4}{\pi^2} \sum_{m \in 2\mathbb{Z}+1} \frac{a_{n-m}}{m^2} \,. \tag{2.3}$$

Note that all these operators are convolution operators with kernel in ℓ^p for every $p \in (1, \infty)$. In particular, by Hölder's inequality, $\mathcal{K} a_n$, $\mathcal{H} a_n$ and $\mathcal{I} a_n$ are well-defined when (a_n) is in ℓ^p for some $p \in (1, \infty)$.

The following (rather well-known) result allows us to write Theorem 1.1 in terms of $\mathcal{K}$ rather than $\mathcal{R}$, and Theorem 1.2 in terms of $\mathcal{H}$ rather than $\mathcal{H}_0$. The first equality, $\|\mathcal{K}\|_p = \|\mathcal{R}\|_p$, is proved in Theorem 3.1 in [6].

Lemma 2.1. *We have the following equalities of operator norms:*

$$\|\mathcal{K}\|_p = \|\mathcal{R}\|_p, \qquad \|\mathcal{H}\|_p = \|\mathcal{H}_0\|_p, \qquad \|\mathcal{I}\|_p = 1. \tag{2.4}$$

Proof. Let (a_n) be a sequence in ℓ^p, and write $b_n = a_{2n}$ and $c_n = a_{2n-1}$. Then

$$\mathcal{R}b_n = \frac{1}{\pi}\sum_{m\in\mathbb{Z}} \frac{b_{n-m}}{m+\frac{1}{2}} = \frac{2}{\pi}\sum_{m\in\mathbb{Z}} \frac{a_{2n-2m}}{2m+1} = \frac{2}{\pi}\sum_{k\in 2\mathbb{Z}+1} \frac{a_{2n+1-k}}{k} = \mathcal{K}a_{2n+1},$$

$$\mathcal{R}c_n = \frac{1}{\pi}\sum_{m\in\mathbb{Z}} \frac{c_{n-m}}{m+\frac{1}{2}} = \frac{2}{\pi}\sum_{m\in\mathbb{Z}} \frac{a_{2n-2m-1}}{2m+1} = \frac{2}{\pi}\sum_{k\in 2\mathbb{Z}+1} \frac{a_{2n-k}}{k} = \mathcal{K}a_{2n}.$$

It follows that

$$\begin{aligned}\|\mathcal{K}a_n\|_p^p &= \|\mathcal{K}a_{2n+1}\|_p^p + \|\mathcal{K}a_{2n}\|_p^p \\ &= \|\mathcal{R}b_n\|_p^p + \|\mathcal{R}c_n\|_p^p \\ &\leqslant \|\mathcal{R}\|_p^p(\|b_n\|_p^p + \|c_n\|_p^p) \\ &= \|\mathcal{R}\|_p^p(\|a_{2n}\|_p^p + \|a_{2n-1}\|_p^p) = \|\mathcal{R}\|_p^p\|a_n\|_p^p,\end{aligned}$$

and hence $\|\mathcal{K}\|_p \leqslant \|\mathcal{R}\|_p$. On the other hand, if $a_{2n-1} = c_n = 0$ for every $n\in\mathbb{Z}$, then $\mathcal{K}a_{2n} = 0$ for every $n \in \mathbb{Z}$, and we find that

$$\|\mathcal{R}b_n\|_p = \|\mathcal{K}a_{2n+1}\|_p = \|\mathcal{K}a_n\|_p \leqslant \|\mathcal{K}\|_p\|a_n\|_p,$$

so that $\|\mathcal{R}\|_p \leqslant \|\mathcal{K}\|_p$. We have thus proved that $\|\mathcal{R}\|_p = \|\mathcal{K}\|_p$. A very similar argument shows that $\|\mathcal{H}_0\|_p = \|\mathcal{H}\|_p$.

The operator $\mathcal{J}$ is the convolution with a probability kernel:

$$\sum_{m\in 2\mathbb{Z}+1} \frac{1}{m^2} = 2\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \frac{2\pi^2}{8} = \frac{\pi^2}{4},$$

and so $\|\mathcal{J}\|_p \leqslant 1$ follows from Jensen's inequality. Finally, if $a_n = 1$ when $|n| \leqslant N$ and $a_n = 0$ otherwise, then a simple estimate shows that for every positive integer k we have

$$\begin{aligned}\limsup_{N\to\infty} \frac{\|\mathcal{J}a_n\|_p^p}{\|a_n\|_p^p} &\geqslant \limsup_{N\to\infty} \frac{1}{2N+1}\sum_{n=-N+k}^{N-k}(\mathcal{J}a_n)^p \\ &\geqslant \limsup_{N\to\infty} \frac{2N+1-2k}{2N+1} \times \frac{4}{\pi^2}\sum_{m=-k}^{k}\frac{1}{m^2} = \frac{4}{\pi^2}\sum_{m=-k}^{k}\frac{1}{m^2},\end{aligned}$$

and the right-hand side can be arbitrarily close to 1. Thus, $\|\mathcal{J}\|_p \geqslant 1$, and the proof is complete. □

Lemma 2.1, Theorems 1.1 and 1.2, and formula (1.7), give the following result.

Corollary 2.2. *(i) For $p \in (1,\infty)$, we have*

$$\cot\frac{\pi}{2p^*} = \|\mathcal{H}_0\|_p = \|\mathcal{H}\|_p \leqslant \|\mathcal{R}\|_p = \|\mathcal{K}\|_p.$$

(ii) For $p = 2n$ or $p = \frac{2n}{2n-1}$, $n \in \mathbb{N}$, we have

$$\cot \frac{\pi}{2p^*} = \|\mathcal{H}_0\|_p = \|\mathcal{H}\|_p = \|\mathcal{R}\|_p = \|\mathcal{K}\|_p.$$

For a sequence (a_n) in ℓ^2, we denote by $\mathcal{F}a(t)$ the Fourier series with coefficients a_n:

$$\mathcal{F}a(t) = \sum_{n \in \mathbb{Z}} a_n e^{-int}$$

for $t \in \mathbb{R}$. As this is a 2π-periodic function, we only consider $t \in (-\pi, \pi)$.

The following three results are also rather well-known.

Lemma 2.3. *For every sequence (a_n) in ℓ^2 we have:*

$$\mathcal{F}[\mathcal{K}a](t) = -i \operatorname{sign} t \, \mathcal{F}a(t), \tag{2.5}$$

$$\mathcal{F}[\mathcal{H}a](t) = (1 - \tfrac{2}{\pi}|t|)(-i \operatorname{sign} t)\mathcal{F}a(t), \tag{2.6}$$

$$\mathcal{F}[\mathcal{I}a](t) = (1 - \tfrac{2}{\pi}|t|)\mathcal{F}a(t). \tag{2.7}$$

Proof. Observe that $\mathcal{K}$, $\mathcal{H}$ and $\mathcal{I}$ are convolution operators, with kernels in ℓ^2 (and even in ℓ^1 for $\mathcal{I}$). Thus, they are Fourier multipliers, and the corresponding symbols are simply Fourier series with coefficients given by convolution kernels of $\mathcal{K}$, $\mathcal{H}$ and $\mathcal{I}$. Evaluation of these kernels reduces to well-known formulas.

For $\mathcal{K}$, we have

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} (-i \operatorname{sign} t) e^{imt} dt = \begin{cases} \frac{2}{\pi} \frac{1}{m} & \text{if } m \text{ is odd,} \\ 0 & \text{if } m \text{ is even.} \end{cases}$$

For $\mathcal{H}$,

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} (1 - \tfrac{2}{\pi}|t|)(-i \operatorname{sign} t) e^{imt} dt = \begin{cases} 0 & \text{if } m \text{ is odd or } m = 0, \\ \frac{2}{\pi} \frac{1}{m} & \text{if } m \text{ is even and } m \neq 0. \end{cases}$$

Finally, for $\mathcal{I}$,

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} (1 - \tfrac{2}{\pi}|t|) e^{imt} dt = \begin{cases} \frac{4}{\pi^2} \frac{1}{m^2} & \text{if } m \text{ is odd,} \\ 0 & \text{if } m \text{ is even.} \end{cases}$$ □

Lemma 2.4. *We have*

$$\|\mathcal{K}\|_2 = \|\mathcal{H}\|_2 = 1. \tag{2.8}$$

Proof. This is an immediate consequence of Lemma 2.3 and Parseval's identity. □

Lemma 2.5. *For every sequence (a_n) in ℓ^p we have*

$$\mathcal{H}a_n = \mathcal{I}\mathcal{K}a_n = \mathcal{K}\mathcal{I}a_n. \tag{2.9}$$

We offer two independent proofs of this simple result.

Proof #1: We have

$$\begin{aligned}\mathcal{I}\mathcal{K}a_n = \mathcal{K}\mathcal{I}a_n &= \frac{8}{\pi^3}\sum_{m\in 2\mathbb{Z}+1}\sum_{k\in 2\mathbb{Z}+1}\frac{a_{n-m-k}}{m^2 k} \\ &= \frac{8}{\pi^3}\sum_{m\in 2\mathbb{Z}+1}\sum_{j\in 2\mathbb{Z}}\frac{a_{n-j}}{m^2(j-m)} \\ &= \frac{8}{\pi^3}\sum_{j\in 2\mathbb{Z}}\left(\sum_{m\in 2\mathbb{Z}+1}\frac{1}{m^2(j-m)}\right)a_{n-j}.\end{aligned}$$

Note that all sums are absolutely convergent. The inner sum can be evaluated explicitly: we have

$$\begin{aligned}\sum_{m\in 2\mathbb{Z}+1}\frac{1}{m^2(j-m)} &= \frac{1}{j}\sum_{m\in 2\mathbb{Z}+1}\frac{1}{m^2} + \frac{1}{j^2}\sum_{m\in 2\mathbb{Z}+1}\left(\frac{1}{m}+\frac{1}{j-m}\right) \\ &= \frac{1}{j}\frac{\pi^2}{4} + \frac{1}{j^2}\sum_{k=0}^{\infty}\left(\frac{1}{j+2k+1}+\frac{1}{j-2k-1}\right) \\ &= \frac{\pi^2}{4}\frac{1}{j} + \frac{1}{j^2}\frac{\pi}{2}\tan\frac{j\pi}{2}.\end{aligned}$$

Since for $j \in 2\mathbb{Z}$ we have $\tan\frac{j\pi}{2} = 0$, it follows that

$$\mathcal{I}\mathcal{K}a_n = \mathcal{K}\mathcal{I}a_n = \frac{8}{\pi^3}\sum_{j\in 2\mathbb{Z}}\frac{\pi^2}{4}\frac{1}{j}\,a_{n-j} = \mathcal{H}a_n,$$

as desired. □

Proof #2: By Lemma 2.3, if (a_n) is in ℓ^2, then

$$\mathcal{F}[\mathcal{I}\mathcal{K}a](t) = (1-\tfrac{2}{\pi}|t|)\mathcal{F}[\mathcal{K}a](t) = -i(1-\tfrac{2}{\pi}|t|)\operatorname{sign} t\,\mathcal{F}a(t) = \mathcal{F}[\mathcal{H}a](t)$$

and, similarly,

$$\mathcal{F}[\mathcal{K}\mathcal{I}a](t) = -i\operatorname{sign} t\,\mathcal{F}[\mathcal{I}a](t) = -i(1-\tfrac{2}{\pi}|t|)\operatorname{sign} t\,\mathcal{F}a(t) = \mathcal{F}[\mathcal{H}a](t),$$

as desired. Extension to ℓ^p follows by continuity. □

The next lemma provides a key *product rule*. It is similar to an analogous result for $\mathcal{H}a_n \cdot \mathcal{H}b_n$, which was used to estimate the operator norm of $\mathcal{H}$ on ℓ^p when p is a power of 2 in [14]. The corresponding result for the continuous Hilbert transform is the well-known formula

$$Hf \cdot Hg = H(f\cdot Hg) + H(Hf\cdot g) + f\cdot g; \tag{2.10}$$

see pp. 253–254 in [10].

Lemma 2.6. *For every sequences* (a_n) *and* (b_n) *in* ℓ^p *we have*

$$\mathcal{K}a_n \cdot \mathcal{K}b_n = \mathcal{K}[a_n \cdot \mathcal{H}b_n] + \mathcal{K}[\mathcal{H}a_n \cdot b_n] + \mathcal{I}[a_n b_n]. \tag{2.11}$$

Again, we offer two proofs of this result.

Proof #1: We have

$$\begin{aligned}\mathcal{K}[a_n \cdot \mathcal{H}b_n] &= \frac{4}{\pi^2} \sum_{m\in 2\mathbb{Z}+1} \sum_{k\in 2\mathbb{Z}\setminus\{0\}} \frac{a_{n-m}b_{n-m-k}}{mk} \\ &= \frac{4}{\pi^2} \sum_{m\in 2\mathbb{Z}+1} \sum_{j\in(2\mathbb{Z}+1)\setminus\{m\}} \frac{a_{n-m}b_{n-j}}{m(j-m)}.\end{aligned}$$

Similarly,

$$\mathcal{K}[\mathcal{H}a_n \cdot b_n] = \frac{4}{\pi^2} \sum_{m\in 2\mathbb{Z}+1} \sum_{j\in(2\mathbb{Z}+1)\setminus\{m\}} \frac{a_{n-j}b_{n-m}}{m(j-m)}.$$

Changing the roles of m and j in the latter equality and combining both identities, we find that

$$\begin{aligned}&\mathcal{K}[a_n \cdot \mathcal{H}b_n] + \mathcal{K}[\mathcal{H}a_n \cdot b_n] \\ &\quad= \frac{4}{\pi^2} \sum_{m\in 2\mathbb{Z}+1} \sum_{j\in(2\mathbb{Z}+1)\setminus\{m\}} \left(\frac{1}{m(j-m)} + \frac{1}{j(m-j)}\right) a_{n-m}b_{n-j} \\ &\quad= \frac{4}{\pi^2} \sum_{m\in 2\mathbb{Z}+1} \sum_{j\in(2\mathbb{Z}+1)\setminus\{m\}} \frac{a_{n-m}b_{n-j}}{mj}.\end{aligned}$$

For $j = m$ (which is excluded from the double sum), the expression under the sum is equal to $a_{n-m}b_{n-m}/m^2$, appearing in the expression for $\mathcal{I}[a_n \cdot b_n]$. It follows that

$$\begin{aligned}\mathcal{K}[a_n \cdot \mathcal{H}b_n] + \mathcal{K}[\mathcal{H}a_n \cdot b_n] + \mathcal{I}[a_n \cdot b_n] &= \frac{4}{\pi^2} \sum_{m\in 2\mathbb{Z}+1} \sum_{j\in 2\mathbb{Z}+1} \frac{a_{n-m}b_{n-j}}{mj} \\ &= \left(\frac{2}{\pi} \sum_{m\in 2\mathbb{Z}+1} \frac{a_{n-m}}{m}\right)\left(\frac{2}{\pi} \sum_{j\in 2\mathbb{Z}+1} \frac{b_{n-j}}{j}\right) = \mathcal{K}a_n \cdot \mathcal{K}b_n,\end{aligned}$$

as desired. □

Proof #2: Suppose that (a_n) and (b_n) are in ℓ^2. Recall that

$$a_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathcal{F}a(t)e^{int}dt, \qquad b_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathcal{F}b(s)e^{ins}ds.$$

Therefore,

$$\begin{aligned} a_n \cdot b_n &= \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \mathcal{F}a(t)\mathcal{F}b(s) e^{int+ins} dt ds \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \mathcal{F}a(t)\mathcal{F}b(r-t) dt \right) e^{inr} dr, \end{aligned}$$

and thus

$$\mathcal{F}[a \cdot b](r) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathcal{F}a(t)\mathcal{F}b(r-t) dt.$$

Let $S(t)$ and $I(t)$ denote 2π-periodic functions such that $S(t) = \operatorname{sign} t$ and $I(t) = 1 - \frac{2}{\pi}|t|$ for $t \in (-\pi, \pi)$. Applying the above formula to $(\mathcal{K}a_n)$ and $(\mathcal{K}b_n)$ rather than to (a_n) and (b_n), we find that

$$\begin{aligned} \mathcal{F}[\mathcal{K}a \cdot \mathcal{K}b](r) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} (-iS(t))(-iS(r-t)) \mathcal{F}a(t)\mathcal{F}b(r-t) dt \\ &= -\frac{1}{2\pi} \int_{-\pi}^{\pi} S(t)S(r-t) \cdot \mathcal{F}a(t)\mathcal{F}b(r-t) dt. \end{aligned}$$

Similarly,

$$\begin{aligned} \mathcal{F}[\mathcal{K}[a \cdot \mathcal{H}b]](r) &= -\frac{1}{2\pi} \int_{-\pi}^{\pi} S(r)I(r-t)S(r-t) \cdot \mathcal{F}a(t)\mathcal{F}b(r-t) dt, \\ \mathcal{F}[\mathcal{K}[\mathcal{H}a \cdot b]](r) &= -\frac{1}{2\pi} \int_{-\pi}^{\pi} S(r)I(t)S(t) \cdot \mathcal{F}a(t)\mathcal{F}b(r-t) dt, \\ \mathcal{F}[\mathcal{I}[a \cdot b]](r) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} I(r) \cdot \mathcal{F}a(t)\mathcal{F}b(r-t) dt. \end{aligned}$$

The desired result follows now from the elementary identity

$$S(t)S(s) = S(t+s)I(s)S(s) + S(t+s)I(t)S(t) - I(s+t) \tag{2.12}$$

after substituting $s = r - t$. The proof of (2.12) is straightforward, but tedious. Both sides are 2π-periodic with respect to both t and s, so we can restrict our attention to $t, s \in (-\pi, \pi)$. Additionally, both changing (t, s) to $(-t, -s)$ or to (s, t) does not change neither side of (2.12), so it is sufficient to consider the case when $t < s$ and $t + s > 0$. If $0 < t < s < \pi$, then

$$\begin{aligned} &S(t+s)I(s)S(s) + S(t+s)I(t)S(t) - I(s+t) \\ &\quad = 1 \cdot (1 - \tfrac{2}{\pi}s) \cdot 1 + 1 \cdot (1 - \tfrac{2}{\pi}t) \cdot 1 - (1 - \tfrac{2}{\pi}(t+s)) \\ &\qquad = 1 - \tfrac{2}{\pi}s + 1 - \tfrac{2}{\pi}t - 1 + \tfrac{2}{\pi}(t+s) = 1 = S(t)S(s). \end{aligned}$$

Similarly, if $-\pi < t < 0 < s < \pi$ and $t + s > 0$, then

$$\begin{aligned} &S(t+s)I(s)S(s) + S(t+s)I(t)S(t) - I(s+t) \\ &\qquad = 1 \cdot (1 - \tfrac{2}{\pi}s) \cdot 1 + 1 \cdot (1 + \tfrac{2}{\pi}t) \cdot (-1) - (1 - \tfrac{2}{\pi}(t+s)) \\ &\qquad\qquad = 1 - \tfrac{2}{\pi}s - 1 - \tfrac{2}{\pi}t - 1 + \tfrac{2}{\pi}(t+s) = -1 = S(t)S(s). \end{aligned}$$

This exhausts all possibilities, and hence the proof is complete. Extension to general (a_n) and (b_n) in ℓ^p follows by continuity. □

3. Historical remarks

The key idea of our proof goes back to the original work of Titchmarsh [18]. Formula (2.32) in [18] asserts that if $b_n = \mathcal{R}a_{-n}$, then

$$\mathcal{R}[b_{-n}^2] = \mathcal{H}_0[a_n^2] + 2a_n \cdot \mathcal{H}_0 a_n.$$

This identity is proved by a direct calculation, and it is essentially equivalent to our Lemma 2.6. It is used to prove that if $\mathcal{R}$ is bounded on ℓ^p, then it is bounded on ℓ^{2p}, and in fact

$$\|\mathcal{R}\|_{2p} \leqslant \|\mathcal{R}\|_p + \sqrt{\tfrac{2}{\pi}(5p+3)\|\mathcal{R}\|_p + 2\|\mathcal{R}\|_p^2}\,; \tag{3.1}$$

see formula (2.37) in [18]. Since $\mathcal{R}$ is a unitary operator on ℓ^2, by induction it follows that $\mathcal{R}$ is bounded on ℓ^p when p is a power of 2.

The estimate (3.1) is not sharp, and thus it led to suboptimal bounds on the norm of $\mathcal{R}$. A very similar argument, but with sharp bounds, was applied in [9] for the continuous Hilbert transform. In this work formula (2.10) was used to prove the optimal bound $\|H\|_p \leqslant \cot \frac{\pi}{2p^*}$ when p or the conjugate exponent of p is a power of 2. Let us briefly discuss the argument used in [9].

By (2.10), we have

$$|Hf|^2 = |f|^2 + 2H[f \cdot Hf].$$

Combining this with Hölder's inequality, we find that

$$\begin{aligned} \|Hf\|_{2p}^2 = \|(Hf)^2\|_p &\leqslant \|f^2\|_p + 2\|H[f \cdot Hf]\|_p \\ &\leqslant \|f\|_{2p}^2 + 2\|H\|_p \|f \cdot Hf\|_p \\ &\leqslant \|f\|_{2p}^2 + 2\|H\|_p \|f\|_{2p} \|Hf\|_{2p} \\ &\leqslant \|f\|_{2p}^2 + 2\|H\|_p \|H\|_{2p} \|f\|_{2p}^2, \end{aligned}$$

or, equivalently,

$$\|H\|_{2p} \leqslant \|H\|_p + \sqrt{1 + \|H\|_p^2}.$$

The claimed bound $\|H\|_p \leqslant \cot \frac{\pi}{2p^*}$ for $p = 2^n$ follows from $\|H\|_2 = 1$ by induction on n and the trigonometric identity

$$\cot \tfrac{\alpha}{2} = \cot \alpha + \sqrt{1 + \cot^2 \alpha}.$$

As already mentioned, essentially the same argument was used in [14] for the sharp upper bound of the operator norm of $\mathcal{H}$ on ℓ^p when p is a power of 2.

In our case, in order to estimate the operator norm of $\mathcal{K}$ on ℓ^p, we develop the expression $(\mathcal{K}a_n)^k$ repeatedly using Lemma 2.6. This idea can be traced back to the original Riesz's work [17], where the development of $(f + iHf)^k$ was used to find the suboptimal bound

$$\|H\|_{2k} \leqslant \frac{2k}{\log 2}$$

for the continuous Hilbert transform for even integers $p = 2k$, see Sections 3 and 7 in [17]. As a further historical note we mention that in response to a letter from Riesz dated December 23, 1923, describing his proof, Hardy writes back on January 5, 1924 and says: "Most elegant & beautiful, it is amazing that none of us would have seen it before (even for $p = 4!$)," see pp. 489 & 502 in Cartwright's article [4].

4. Idea of the proof

As mentioned above, the key idea of our work is to develop the expression $(\mathcal{K}a_n)^k$ repeatedly using the product rule given in Lemma 2.6. The difficult part is to do it in a right way, and know when to stop. Here is a brief description of our method.

In the first step, we simply take two factors $\mathcal{K}a_n$ from $(\mathcal{K}a_n)^k$, we apply the product rule (2.11) to it, and we expand the resulting expression. In each of the next steps, our expression is a finite sum of finite products, and we apply repeatedly the following procedure to each of the increasing number of summands.

- Whenever there is no factor of the form $\mathcal{K}[\ldots]$ other than $\mathcal{K}a_n$, or the entire summand is of the form $\mathcal{K}[\ldots]$, we stop.
- Otherwise there is exactly one factor of the form $\mathcal{K}[\ldots]$ other than $\mathcal{K}a_n$. We choose this factor and one of the remaining factors $\mathcal{K}a_n$, we apply the product rule (2.11), and expand the result.

The above procedure leads to a finite sum of terms which are either of the form $\mathcal{K}[\ldots]$, or of the form $\mathcal{I}[\ldots](\mathcal{K}a_n)^j$ with no other appearance of $\mathcal{K}$.

What remains to be done is to use Hölder's inequality and known expressions for the operator norms of $\mathcal{H}$ and $\mathcal{I}$ to bound the $\ell^{p/k}$ norm of $(\mathcal{K}a_n)^k$, and then simplify the result. We illustrate the above approach by explicit calculations for $k = 2, 3, 4$.

For $k = 2$ we simply have

$$(\mathcal{K}a_n)^2 = \mathcal{K}a_n \cdot \mathcal{K}a_n = 2\mathcal{K}[a_n \cdot \mathcal{H}a_n] + \mathcal{I}[a_n^2].$$

Therefore, whenever $\|a_n\|_p \leqslant 1$, we have

$$\|\mathcal{K}a_n\|_p^2 = \|(\mathcal{K}a_n)^2\|_{p/2} \leqslant 2\|\mathcal{K}\|_{p/2}\|\mathcal{H}\|_p + \|\mathcal{I}\|_{p/2}.$$

We already know that $\|\mathcal{I}\|_p = 1$ and $\|\mathcal{H}\|_p = \cot \frac{\pi}{2p}$ for every $p \geqslant 2$. Thus, if $\|\mathcal{K}\|_{p/2} = \cot \frac{\pi}{p}$ for some $p \geqslant 4$, then

$$\|\mathcal{K}\|_p^2 \leqslant 2\cot \tfrac{\pi}{p} \cdot \cot \tfrac{\pi}{2p} + 1 = (\cot \tfrac{\pi}{2p})^2. \tag{4.1}$$

In fact, the above argument easily implies that the operator norm of $\mathcal{K}$ on ℓ^p is $\cot \frac{\pi}{2p}$ when p is a power of 2. This is essentially the same argument as the one used for the continuous Hilbert transform in [9] and described in Section 3, as well as the one applied in [14] for the discrete Hilbert transform $\mathcal{H}_0$.

Remark 4.1. The above argument shows that a sharp estimate on $\ell^{p/2}$ implies sharp estimate on ℓ^p, and it does not really require the result of [2]. Indeed: even without that result we know that $\|\mathcal{H}\|_p \leqslant \|\mathcal{I}\|_p\|\mathcal{K}\|_p = \|\mathcal{K}\|_p$, and so instead of (4.1) we obtain that

$$\|\mathcal{K}\|_p^2 \leqslant 2\cot \tfrac{\pi}{p} \cdot \|\mathcal{K}\|_p + 1.$$

This is sufficient to show that if $\|\mathcal{K}\|_{p/2} \leqslant \cot \frac{\pi}{p}$, then $\|\mathcal{K}\|_p \leqslant \cot \frac{\pi}{2p}$.

The calculations get more involved for $k = 3$. In this case we have

$$\begin{aligned}
(\mathcal{K}a_n)^3 &= \mathcal{K}a_n \cdot \big(\mathcal{K}a_n \cdot \mathcal{K}a_n\big) \\
&\overset{r}{=} \mathcal{K}a_n \cdot \big(2\mathcal{K}[a_n \cdot \mathcal{H}a_n] + \mathcal{I}[a_n^2]\big) \\
&\overset{e}{=} 2\mathcal{K}a_n \cdot \mathcal{K}[a_n \cdot \mathcal{H}a_n] + \mathcal{K}a_n \cdot \mathcal{I}[a_n^2] \\
&\overset{r}{=} 2\mathcal{K}[a_n \cdot \mathcal{H}[a_n \cdot \mathcal{H}a_n]] + 2\mathcal{K}[\mathcal{H}a_n \cdot a_n \cdot \mathcal{H}a_n] + 2\mathcal{I}[a_n \cdot a_n \cdot \mathcal{H}a_n] \\
&\qquad + \mathcal{K}a_n \cdot \mathcal{I}[a_n^2],
\end{aligned}$$

where $\overset{r}{=}$ marks an application of the product rule (2.11), and $\overset{e}{=}$ denotes simple expansion. Therefore, whenever $\|a_n\|_p \leqslant 1$, we have

$$\begin{aligned}
\|\mathcal{K}a_n\|_p^3 = \|(\mathcal{K}a_n)^3\|_{p/3} &\leqslant 2\|\mathcal{K}\|_{p/3}\|\mathcal{H}\|_{p/2}\|\mathcal{H}\|_p + 2\|\mathcal{K}\|_{p/3}\|\mathcal{H}\|_p\|\mathcal{H}\|_p \\
&\qquad + 2\|\mathcal{I}\|_{p/3}\|\mathcal{H}\|_p + \|\mathcal{K}\|_p\|\mathcal{I}\|_{p/2}.
\end{aligned}$$

If $\|\mathcal{K}\|_{p/3} = \cot \frac{3\pi}{2p}$, then it follows that

$$\|\mathcal{K}\|_p^3 \leqslant 2\cot \tfrac{3\pi}{2p} \cdot \cot \tfrac{\pi}{p} \cdot \cot \tfrac{\pi}{2p} + 2\cot \tfrac{3\pi}{2p} \cdot \cot \tfrac{\pi}{2p} \cdot \cot \tfrac{\pi}{2p} + 2\cot \tfrac{\pi}{2p} + \|\mathcal{K}\|_p.$$

It is an elementary, but non-obvious fact that this inequality becomes an equality if $\|\mathcal{K}\|_p = \cot \frac{\pi}{2p}$, and it is relatively easy to see that the inequality in fact implies that $\|\mathcal{K}\|_p \leqslant \cot \frac{\pi}{2p}$.

Remark 4.2. We stress that in the above proof that $\|\mathcal{K}\|_{p/3} \leqslant \cot \frac{3\pi}{2p}$ implies $\|\mathcal{K}\|_p \leqslant \cot \frac{\pi}{2p}$, we use the result of [2] in its full strength: we need an estimate for the operator norm of $\mathcal{H}$ on $\ell^{p/2}$, and the argument would not work without this ingredient. In other words, the approach used for $k = 2$ in Remark 4.1 no longer applies for $k = 3$.

The case $k = 4$ is, of course, is not interesting, as it is sufficient to apply the result for $k = 2$ twice. Nevertheless, it is instructive to study $k = 4$ only to better understand the method. The initial steps are the same as for $k = 3$, and we re-use the above calculation:

$$\begin{aligned}
(\mathcal{K}a_n)^4 &= \mathcal{K}a_n \cdot \bigl(2\mathcal{K}[a_n \cdot \mathcal{H}[a_n \cdot \mathcal{H}a_n]] + 2\mathcal{K}[\mathcal{H}a_n \cdot a_n \cdot \mathcal{H}a_n] \\
&\qquad\qquad + \mathcal{I}[a_n^2 \cdot \mathcal{H}a_n] + \mathcal{K}a_n \cdot \mathcal{I}[a_n^2]\bigr) \\
&\overset{e}{=} 2\mathcal{K}a_n \cdot \mathcal{K}[a_n \cdot \mathcal{H}[a_n \cdot \mathcal{H}a_n]] \\
&\qquad + 2\mathcal{K}a_n \cdot \mathcal{K}[\mathcal{H}a_n \cdot a_n \cdot \mathcal{H}a_n] \\
&\qquad\qquad + \mathcal{K}a_n \cdot \mathcal{I}[a_n^2 \cdot \mathcal{H}a_n] + (\mathcal{K}a_n)^2 \cdot \mathcal{I}[a_n^2] \\
&\overset{r}{=} 2\mathcal{K}[a_n \cdot \mathcal{H}[a_n \cdot \mathcal{H}[a_n \cdot \mathcal{H}a_n]]] + 2\mathcal{K}[\mathcal{H}a_n \cdot a_n \cdot \mathcal{H}[a_n \cdot \mathcal{H}a_n]] \\
&\qquad\qquad + 2\mathcal{I}[a_n \cdot a_n \cdot \mathcal{H}[a_n \cdot \mathcal{H}a_n]] \\
&\qquad + 2\mathcal{K}[a_n \cdot \mathcal{H}[\mathcal{H}a_n \cdot a_n \cdot \mathcal{H}a_n]] + 2\mathcal{K}[\mathcal{H}[a_n] \cdot \mathcal{H}a_n \cdot a_n \cdot \mathcal{H}a_n] \\
&\qquad\qquad + 2\mathcal{I}[a_n \cdot \mathcal{H}a_n \cdot a_n \cdot \mathcal{H}a_n] \\
&\qquad\qquad + \mathcal{K}a_n \cdot \mathcal{I}[a_n^2 \cdot \mathcal{H}a_n] + (\mathcal{K}a_n)^2 \cdot \mathcal{I}[a_n^2].
\end{aligned}$$

Again this leads to an inequality that involves only $\|\mathcal{K}\|_p$ and $\|\mathcal{K}\|_{p/4}$ and known constants, and solving this inequality shows that if $\|\mathcal{K}\|_{p/4} \leqslant \cot \frac{2\pi}{p}$, then $\|\mathcal{K}\|_p \leqslant \cot \frac{\pi}{2p}$. However, the calculations are even more involved than in the case $k = 3$, and so we stop here.

Remark 4.3. It is important not to develop further the expressions such as $(\mathcal{K}a_n)^{k-2} \cdot \mathcal{I}[a_n^2]$ using the product rule (2.11). For $k = 4$ that would lead to expressions involving the unknown operator norms of $\mathcal{K}$ on $\ell^{p/2}$. This could be circumvented by re-using appropriately the result for $k = 2$. However, already when $k = 5$ further development of $(\mathcal{K}a_n)^3 \cdot \mathcal{I}[a_n^2]$ would lead to expressions involving the unknown operator norms of $\mathcal{K}$ on $\ell^{p/3}$ or $\ell^{p/2}$, and the method would break.

Clearly, the expressions become rapidly more complicated as k grows, and we need a systematic way to handle them. This is done using *skeletons* introduced in the next section.

5. Skeletons, frames and buildings

The following auxiliary definition allows us to conveniently enumerate the terms in the development of $(\mathcal{K}a_n)^k$.

Definition 5.1. For a positive integer k, we define the set $\mathbb{S}_k$ of *skeletons* of size k inductively:

$$\begin{aligned} \mathbb{S}_1 &= \big\{\{1\}\big\}, \\ \mathbb{S}_{k+1} &= \big\{\{S, k+1\} : S \in \mathbb{S}_k\big\} \cup \big\{S \cup \{\{k+1\}\} : S \in \mathbb{S}_k\big\}. \end{aligned} \tag{5.1}$$

We define the set $\mathbb{F}$ of *frames* to be the minimal collection of numbers and sets with the following properties:

- every integer is a frame;
- every finite set of frames is a frame.

More precisely, we let $\mathbb{F}_0$ be the set of integers, that is, *frames of depth* 0, and for every positive integer k we define the set $\mathbb{F}_k$ of *frames of depth at most k* to be the collection of all finite sets of frames of depth less than k. Finally, we define the *size* $|F|$ of a frame F inductively by

$$|F| = \begin{cases} 1 & \text{if } F \text{ is an integer,} \\ \displaystyle\sum_{f \in F} |f| & \text{if } F \text{ is a set.} \end{cases} \tag{5.2}$$

Clearly, there are 2^k skeletons of size k, and every skeleton is a frame. The size of a frame F is just the number of integers ('bones') that appear in a textual (roster notation) or graphical (Venn diagram) representation of F, while the depth of F is the maximal number of nested brackets in the textual representation of F. It is straightforward to see that every skeleton of size k is a frame of size k (with variable depth). By inspection, we find that the initial sets of skeletons are:

$$\begin{aligned} \mathbb{S}_1 &= \big\{\{1\}\big\} \\ \mathbb{S}_2 &= \big\{\{\{1\}, 2\}, \{1, \{2\}\}\big\} \\ \mathbb{S}_3 &= \big\{\{\{\{1\}, 2\}, 3\}, \{\{1, \{2\}\}, 3\}, \{\{1\}, 2, \{3\}\}, \{1, \{2\}, \{3\}\}\big\} \\ \mathbb{S}_4 &= \big\{\{\{\{\{1\}, 2\}, 3\}, 4\}, \{\{\{1, \{2\}\}, 3\}, 4\}, \{\{\{1\}, 2, \{3\}\}, 4\}, \\ &\qquad \{\{1, \{2\}, \{3\}\}, 4\}, \{\{\{1\}, 2\}, 3, \{4\}\}, \{\{1, \{2\}\}, 3, \{4\}\}, \\ &\qquad\qquad \{\{1\}, 2, \{3\}, \{4\}\}, \{1, \{2\}, \{3\}, \{4\}\}\big\}. \end{aligned}$$

Definition 5.2. For a sequence (a_n) in ℓ^p and a frame F, we define the *building* $(\mathcal{H}^{\{F\}}a_n)$ with frame $\{F\}$ inductively by

$$\mathcal{H}^{\{F\}}a_n = \begin{cases} a_n & \text{if } F \text{ is an integer,} \\ \mathcal{H}\Big[\prod_{f\in F}\mathcal{H}^{\{f\}}a_n\Big] & \text{if } F \text{ is a set.} \end{cases} \tag{5.3}$$

If the frame F is a set containing more than one element, we extend the above definition, so that the building $(\mathcal{H}^F a_n)$ with frame F is given by

$$\mathcal{H}^F a_n = \prod_{f\in F}\mathcal{H}^{\{f\}}a_n. \tag{5.4}$$

In other words, construction of the building $(\mathcal{H}^F a_n)$ with frame F corresponds to replacing every integer in the textual representation of the frame F by (a_n), every pair of corresponding curly brackets $\{\ldots\}$ — except the outermost ones — by $\mathcal{H}[\ldots]$, and every comma by multiplication. For example,

$$\begin{aligned}
\mathcal{H}^{\{1\}}a_n &= a_n, \\
\mathcal{H}^{\{\{1\},2\}}a_n &= \mathcal{H}a_n\cdot a_n, \\
\mathcal{H}^{\{\{\{1\},2\},3\}}a_n &= \mathcal{H}[\mathcal{H}a_n\cdot a_n]\cdot a_n, \\
\mathcal{H}^{\{\{1\},2,\{3\}\}}a_n &= \mathcal{H}a_n\cdot a_n\cdot\mathcal{H}a_n = a_n\cdot(\mathcal{H}a_n)^2, \\
\mathcal{H}^{\{\{1,\{2\}\},3,\{4\}\}}a_n &= \mathcal{H}[a_n\cdot\mathcal{H}a_n]\cdot a_n\cdot\mathcal{H}a_n.
\end{aligned}$$

Our next results provides a skeletal decomposition of $(\mathcal{K}a_n)^k$.

Proposition 5.3. *For every sequence (a_n) in ℓ^p and every positive integer k we have*

$$(\mathcal{K}a_n)^k = \sum_{S\in\mathbb{S}_k}\mathcal{K}[\mathcal{H}^S a_n] + \sum_{j=1}^{k-1}\sum_{S\in\mathbb{S}_j}\mathcal{I}[a_n\cdot\mathcal{H}^S a_n]\cdot(\mathcal{K}a_n)^{k-j-1}. \tag{5.5}$$

Proof. We proceed by induction with respect to k. For $k=1$ formula (5.5) takes form

$$\mathcal{K}a_n = \sum_{S\in\mathbb{S}_1}\mathcal{K}[\mathcal{H}^S a_n],$$

which is obviously true, because $\mathbb{S}_1$ has only one skeleton $S=\{1\}$ and $\mathcal{K}[\mathcal{H}^{\{1\}}a_n] = \mathcal{K}a_n$. Suppose that (5.5) holds for some positive integer

k. Then

$$(5.6)\quad \begin{aligned}(\mathcal{K}a_n)^{k+1} &= (\mathcal{K}a_n)^k \cdot \mathcal{K}a_n \\ &= \sum_{S\in\mathbb{S}_k} \mathcal{K}[\mathcal{H}^S a_n]\cdot\mathcal{K}a_n + \sum_{j=1}^{k-1}\sum_{S\in\mathbb{S}_j} \mathcal{I}[a_n\cdot\mathcal{H}^S a_n]\cdot(\mathcal{K}a_n)^{k-j}.\end{aligned}$$

By Lemma 2.6 (the product rule), for every $S\in\mathbb{S}_k$ we have

$$\begin{aligned}\mathcal{K}[\mathcal{H}^S a_n]\cdot\mathcal{K}a_n &= \mathcal{K}[\mathcal{H}^S a_n\cdot\mathcal{H}a_n] + \mathcal{K}[\mathcal{H}[\mathcal{H}^S a_n]\cdot a_n] + \mathcal{I}[\mathcal{H}^S a_n\cdot a_n] \\ &= \mathcal{K}[\mathcal{H}^{S\cup\{\{k+1\}\}} a_n] + \mathcal{K}[\mathcal{H}^{\{S,k+1\}} a_n] + \mathcal{I}[a_n\cdot\mathcal{H}^S a_n].\end{aligned}$$

The term $\mathcal{I}[a_n\cdot\mathcal{H}^S a_n]$, summed over all $S\in\mathbb{S}_k$, corresponds to the term $j=k$ in the latter sum in (5.6). On the other hand, by Definition 5.1, the sum of the remaining terms $\mathcal{K}[\mathcal{H}^{S\cup\{\{k+1\}\}} a_n]$ and $\mathcal{K}[\mathcal{H}^{\{S,k+1\}} a_n]$ over $S\in\mathbb{S}_k$ is equal to the sum of $\mathcal{K}[\mathcal{H}^S a_n]$ over $S\in\mathbb{S}_{k+1}$. Therefore,

$$(\mathcal{K}a_n)^{k+1} = \sum_{S\in\mathbb{S}_{k+1}} \mathcal{K}[\mathcal{H}^S a_n] + \sum_{j=1}^{k}\sum_{S\in\mathbb{S}_j} \mathcal{I}[a_n\cdot\mathcal{H}^S a_n]\cdot(\mathcal{K}a_n)^{k-j},$$

as desired. This completes the proof by induction. □

6. Norms and constants

By Lemma 2.1, the operator norm of $\mathcal{I}$ on ℓ^p is equal to 1. By the same lemma and the result of [2] (given in Theorem 1.2), the operator norm of $\mathcal{H}$ on ℓ^p is equal to

$$(6.1)\qquad C_p = \cot\tfrac{\pi}{2p^*} = \begin{cases}\tan\tfrac{\pi}{2p} & \text{when } 1<p\leqslant 2,\\ \cot\tfrac{\pi}{2p} & \text{when } 2\leqslant p<\infty.\end{cases}$$

The following definition will allow us to bound the $\ell^{p/|F|}$ norm of the building $(\mathcal{H}^F a_n)$ with frame F.

Definition 6.1. For a frame F, we define the corresponding *building norm* $C_p^{\{F\}}$ inductively by

$$(6.2)\qquad C_p^{\{F\}} = \begin{cases}1 & \text{if } F \text{ is an integer},\\ C_{p/|F|}\prod_{f\in F} C_p^{\{f\}} & \text{if } F \text{ is a set}.\end{cases}$$

If the frame F is a set containing more than one element, we extend the above definition, so that

$$(6.3)\qquad C_p^F a_n = \prod_{f\in F} C_p^{\{f\}}.$$

For example, we have

$$\begin{aligned} C_p^{\{1\}} &= 1, \\ C_p^{\{\{1\},2\}} &= C_p, \\ C_p^{\{\{\{1\},2\},3\}} &= C_{p/2} \cdot C_p, \\ C_p^{\{\{1\},2,\{3\}\}} &= C_p \cdot C_p, \\ C_p^{\{\{\{\{1\},2\},3\},4\}} &= C_{p/3} \cdot C_{p/2} \cdot C_p, \\ C_p^{\{\{1,\{2\}\},3,\{4\}\}} &= C_{p/2} \cdot C_p \cdot C_p. \end{aligned}$$

The relation between the norm of the building $\mathcal{H}^F a_n$ and the corresponding building norm is provided by the following result.

Lemma 6.2. *For every sequence (a_n) in ℓ^p and every frame F, we have*

$$\|\mathcal{H}^{\{F\}} a_n\|_{p/|F|} \leqslant C_p^{\{F\}} \|a_n\|_p^{|F|}.$$

If the frame F is a set, then, more generally,

$$\|\mathcal{H}^F a_n\|_{p/|F|} \leqslant C_p^F \|a_n\|_p^{|F|}.$$

Proof. The proof of the first part of the lemma proceeds by induction with respect to the depth of F. If F is an integer (a frame of depth 0), then $\mathcal{H}^{\{F\}} a_n = a_n$ has norm equal to $\|a_n\|_p = C_p^{\{F\}} \|a_n\|_p$. Suppose now that

$$\|\mathcal{H}^{\{F\}} a_n\|_{p/|F|} \leqslant C_p^{\{F\}} \|a_n\|_p^{|F|}$$

for every frame F of depth less than k for some positive integer k. Let F be a frame of depth k. Then

$$\|\mathcal{H}^{\{F\}} a_n\|_{p/|F|} = \left\| \mathcal{H}\left[\prod_{f \in F} \mathcal{H}^{\{f\}} a_n\right] \right\|_{p/|F|} \leqslant \|\mathcal{H}\|_{p/|F|} \left\| \prod_{f \in F} \mathcal{H}^{\{f\}} a_n \right\|_{p/|F|}.$$

By definition, the size $|F|$ of frame F is the sum of sizes $|f|$ of all frames $f \in F$. Thus, Hölder's inequality implies that

$$\|\mathcal{H}^{\{F\}} a_n\|_{p/|F|} \leqslant \|\mathcal{H}\|_{p/|F|} \prod_{f \in F} \|\mathcal{H}^{\{f\}} a_n\|_{p/|f|}.$$

Using the equality $\|\mathcal{H}\|_{p/|F|} = C_{p/|F|}$ and the induction hypothesis, we find that

$$\|\mathcal{H}^{\{F\}} a_n\|_{p/|F|} \leqslant C_{p/|F|} \prod_{f \in F} C_p^{\{f\}} \|a_n\|_p^{|f|} = C_p^{\{F\}} \|a_n\|_p^{|F|},$$

as desired. This completes the proof by induction.

The other part of the lemma follows now by the same argument as above, but without the final application of $\mathcal{H}$:

$$\begin{aligned}\|\mathcal{H}^F a_n\|_{p/|F|} &= \Big\|\prod_{f\in F}\mathcal{H}^{\{f\}}a_n\Big\|_{p/|F|}\\ &\leqslant \prod_{f\in F}\|\mathcal{H}^{\{f\}}a_n\|_{p/|f|} = \prod_{f\in F} C_p^{\{f\}}\|a_n\|_p^{|f|} = C_p^F\|a_n\|_p^{|F|},\end{aligned}$$

and the proof is complete. □

The skeletal decomposition allows us to find an inequality that links the operator norms of $\mathcal{K}$ on ℓ^p and $\ell^{p/k}$.

Lemma 6.3. *Let k be a positive integer and $p \geqslant k$. The operator norms of $\mathcal{K}$ on ℓ^p and $\ell^{p/k}$ satisfy the inequality*

$$\|\mathcal{K}\|_p^k \leqslant \Big(\sum_{S\in\mathbb{S}_k} C_p^S\Big)\|\mathcal{K}\|_{p/k} + \sum_{j=1}^{k-1}\Big(\sum_{S\in\mathbb{S}_j} C_p^S\Big)\|\mathcal{K}\|_p^{k-j-1}. \tag{6.4}$$

Proof. Suppose that (a_n) is a sequence in ℓ^p, k is a positive integer and $p \geqslant k$. By Proposition 5.3 and triangle inequality, we have

$$\|(\mathcal{K}a_n)^k\|_{p/k} \leqslant \sum_{S\in\mathbb{S}_k}\|\mathcal{K}[\mathcal{H}^S a_n]\|_{p/k} + \sum_{j=1}^{k-1}\sum_{S\in\mathbb{S}_j}\|\mathcal{I}[a_n\cdot\mathcal{H}^S a_n]\cdot(\mathcal{K}a_n)^{k-j-1}\|_{p/k}.$$

The left-hand side is equal to $\|\mathcal{K}a_n\|_p^k$. Applying Hölder's inequality to the expressions in the right-hand side, we find that

$$\begin{aligned}\|\mathcal{K}a_n\|_p^k &\leqslant \sum_{S\in\mathbb{S}_k}\|\mathcal{K}[\mathcal{H}^S a_n]\|_{p/k}\\ &\qquad + \sum_{j=1}^{k-1}\sum_{S\in\mathbb{S}_j}\|\mathcal{I}[a_n\cdot\mathcal{H}^S a_n]\|_{p/(j+1)}\|(\mathcal{K}a_n)^{k-j-1}\|_{p/(k-j-1)}\\ &= \sum_{S\in\mathbb{S}_k}\|\mathcal{K}[\mathcal{H}^S a_n]\|_{p/k} + \sum_{j=1}^{k-1}\sum_{S\in\mathbb{S}_j}\|\mathcal{I}[a_n\cdot\mathcal{H}^S a_n]\|_{p/(j+1)}\|\mathcal{K}a_n\|_p^{k-j-1}.\end{aligned}$$

Since the operator norm of $\mathcal{I}$ is equal to 1, we obtain

$$\|\mathcal{K}a_n\|_p^k \leqslant \sum_{S\in\mathbb{S}_k}\|\mathcal{K}\|_{p/k}\|\mathcal{H}^S a_n\|_{p/k} + \sum_{j=1}^{k-1}\sum_{S\in\mathbb{S}_j}\|a_n\cdot\mathcal{H}^S a_n\|_{p/(j+1)}\|\mathcal{K}a_n\|_p^{k-j-1}.$$

Another application of Hölder's inequality leads to

$$\|\mathcal{K}a_n\|_p^k \leqslant \sum_{S\in\mathbb{S}_k}\|\mathcal{K}\|_{p/k}\|\mathcal{H}^S a_n\|_{p/k} + \sum_{j=1}^{k-1}\sum_{S\in\mathbb{S}_j}\|a_n\|_p\|\mathcal{H}^S a_n\|_{p/j}\|\mathcal{K}a_n\|_p^{k-j-1}.$$

By Lemma 6.2, we have

$$\|\mathcal{K}a_n\|_p^k \leqslant \sum_{S\in\mathbb{S}_k} \|\mathcal{K}\|_{p/k}\, C_p^S \|a_n\|_p^k + \sum_{j=1}^{k-1} \sum_{S\in\mathbb{S}_j} \|a_n\|_p\, C_p^S \|a_n\|_p^j \|\mathcal{K}a_n\|_p^{k-j-1}.$$

It remains to apply the supremum over all sequences (a_n) in ℓ^p with norm less than or equal to 1 to both sides of the above inequality. $\square$

7. Cotangents

In order to prove Theorem 1.1, we need one more technical ingredient. We remark that throughout this section the constant C_p is used only with $p \geqslant 2$, so that $C_p = \cot \frac{\pi}{2p}$.

Lemma 7.1. *If k is a positive integer, $p \geqslant 2k$ and x is a real number such that*

$$x^k \leqslant \biggl(\sum_{S\in\mathbb{S}_k} C_p^S\biggr) C_{p/k} + \sum_{j=1}^{k-1} \biggl(\sum_{S\in\mathbb{S}_j} C_p^S\biggr) x^{k-j-1}, \tag{7.1}$$

then $x \leqslant C_p$.

Proof. Let $f_k(x)$ denote the right-hand side of (7.1) (for a fixed $p \geqslant 2k$). Thus, we need to prove that if $x^k \leqslant f_k(x)$, then $x \leqslant C_p$; or, equivalently: if $x > C_p$, then $f_k(x)/x^k < 1$.

The function $f_k(x)$ is a non-zero polynomial of x of degree less than k with non-negative coefficients, and hence $f_k(x)/x^k$ is a decreasing function of $x > 0$. It follows that

$$\frac{f_k(x)}{x^k} < \frac{f_k(C_p)}{C_p^k}$$

when $x > C_p$. Thus, it is sufficient to prove that $f_k(C_p) = C_p^k$, that is, for $x = C_p$ we have equality in (7.1).

The proof is based on the formula for the cotangent of sum, which we re-write in a form that resembles the product rule (2.11):

$$\cot\alpha \cdot \cot\beta = \cot(\alpha+\beta)\cot\beta + \cot(\alpha+\beta)\cot\alpha + 1$$

whenever $\alpha, \beta > 0$ and $\alpha + \beta < \pi$. Setting $\alpha = \frac{i\pi}{2p}$ and $\beta = \frac{j\pi}{2p}$ with positive integers i, j such that $p \geqslant 2(i+j)$, we find that

$$C_{p/i}C_{p/j} = C_{p/(i+j)}C_{p/j} + C_{p/(i+j)}C_{p/i} + 1, \tag{7.2}$$

which is exactly what is needed to show that $f_k(C_p) = C_p^k$. We proceed by induction with respect to k.

For $k=1$ and $p \geqslant 2$, we have

$$f_1(x) = \Bigg(\sum_{S \in \mathbb{S}_1} C_p^S\Bigg) C_p = C_p^{\{1\}} C_p = C_p,$$

and hence indeed $f_1(C_p) = C_p$. Suppose now that for some positive integer k we have $p \geqslant 2k+2$ and $f_k(C_p) = C_p^k$. Observe that

$$C_p^{k+1} = C_p f_k(C_p) = \Bigg(\sum_{S \in \mathbb{S}_k} C_p^S\Bigg) C_{p/k} C_p + \sum_{j=1}^{k-1} \Bigg(\sum_{S \in \mathbb{S}_j} C_p^S\Bigg) C_p^{k-j}.$$

Recall that every skeleton of size $k+1$ is equal to either $\{S, k+1\}$ or $S \cup \{\{k+1\}\}$ for some skeleton S of size k. This property and the definition of the constant C_p^S imply that

$$\begin{aligned}
f_{k+1}(x) &= \Bigg(\sum_{S \in \mathbb{S}_{k+1}} C_p^S\Bigg) C_{p/(k+1)} + \sum_{j=1}^{k} \Bigg(\sum_{S \in \mathbb{S}_j} C_p^S\Bigg) x^{k-j} \\
&= \Bigg(\sum_{S \in \mathbb{S}_k} (C_{p/k} C_p^S + C_p^S C_p)\Bigg) C_{p/(k+1)} + \sum_{j=1}^{k} \Bigg(\sum_{S \in \mathbb{S}_j} C_p^S\Bigg) x^{k-j} \\
&= \Bigg(\sum_{S \in \mathbb{S}_k} C_p^S\Bigg) (C_{p/(k+1)} C_{p/k} + C_{p/(k+1)} C_p + 1) + \sum_{j=1}^{k-1} \Bigg(\sum_{S \in \mathbb{S}_j} C_p^S\Bigg) x^{k-j}.
\end{aligned}$$

Applying the cotangent of sum identity (7.2) with $i=1$ and $j=k$, we obtain

$$f_{k+1}(x) = \Bigg(\sum_{S \in \mathbb{S}_k} C_p^S\Bigg) C_p C_{p/k} + \sum_{j=1}^{k-1} \Bigg(\sum_{S \in \mathbb{S}_j} C_p^S\Bigg) x^{k-j}.$$

We conclude that

$$f_{k+1}(C_p) = \Bigg(\sum_{S \in \mathbb{S}_k} C_p^S\Bigg) C_p C_{p/k} + \sum_{j=1}^{k-1} \Bigg(\sum_{S \in \mathbb{S}_j} C_p^S\Bigg) C_p^{k-j} = C_p f_k(C_p) = C_p^{k+1},$$

as desired. This completes the proof by induction. $\square$

Corollary 7.2. *If k is a positive integer, $p \geqslant 2k$ and $\|\mathcal{K}\|_{p/k} \leqslant C_{p/k}$, then $\|\mathcal{K}\|_p \leqslant C_p$.*

Proof. By Lemma 6.3, the number $x = \|\mathcal{K}\|_p$ satisfies (7.1), and therefore, by Lemma 7.1, $x \leqslant C_p$. $\square$

Corollary 7.3. *If p is a positive even integer, then $\|\mathcal{K}\|_p = C_p$.*

Proof. By Lemma 2.4, we have $\|\mathcal{K}\|_2 = 1 = C_2$. We apply Corollary 7.2 with $k = \frac{p}{2}$ to find that $\|\mathcal{K}\|_p \leqslant C_p$. Furthermore, $\|\mathcal{K}\|_p = \|\mathcal{R}\|_p \geqslant C_p$ by (1.7) and Lemma 2.1, and so equality follows. $\square$

Proof of Theorem 1.1. Since $\|\mathcal{R}\|_p = \|\mathcal{K}\|_p$ by Lemma 2.1, the desired result when p is an even integer is an immediate consequence of Corollary 7.3. Extension to the case when the conjugate exponent of p is an even integer follows by duality. $\square$

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