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ON HOMOMORPHISMS OF MATRIX ALGEBRAS OF CONTINUOUS FUNCTIONS

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If X is a topological space we denote by $C(X) \otimes M_n$ the algebra of continuous functions from X to the algebra M_n of $n \times n$ complex matrices. A complete characterization of those topological spaces Y is given (in terms of vector bundles on Y) such that each unital algebra-homomorphism Φ: $C(X) \otimes M_n \to C(Y) \otimes M_{kn}$ is of the form $\alpha \circ (\Phi' \otimes \mathrm{id}_n)$ for some homomorphism Φ': $C(X) \to C(Y) \otimes M_k$ and some suitable inner (or $C(Y)$-linear) automorphism α of the algebra $C(Y) \otimes M_{kn}$. In particular this decomposition is assured provided that Y is a finite CW-complex of dimension $\leq 2k$ and $K^0(Y)$ does not have n-torsion.

Our interest in such homomorphisms arose in connection with a question of E. G. Effros [**1**] concerning the structure of inductive limits of C^*-algebras of the form $C(X) \otimes M_n$. In this context certain classes of homomorphisms related to a covering $X \to Y$ have been considered by C. Pasnicu [**5**]. When restricted to the case of automorphisms our results give nothing new (see [**4**], [**6**] and [**7**]).

1. Preliminaries. Let $\mathrm{GL}_n(C)$ be the general linear group (nonsingular $n \times n$ matrices over the complex field) and denote by 1_n its unit. Let $\mathrm{Vect}_m(Y)$ denote the set of isomorphism classes of complex vector bundles of rank m on the topological space Y. In $\mathrm{Vect}_m(Y)$ we have one naturally distinguished element—the class of the trivial bundle of rank m. Let $T_n \mathrm{Vect}_m(Y)$ be the subset of $\mathrm{Vect}_m(Y)$ given by all vector bundles E such that the direct sum $E \oplus E \oplus \cdots \oplus E$ (n-times) is isomorphic to the trivial bundle of rank nm.

If A, B are unital complex algebras we denote by $\mathrm{Hom}(A, B)$ the set of all unital algebra-homomorphisms from A to B. Two homomorphisms $\Phi_1, \Phi_2 \in \mathrm{Hom}(A, B)$ are said to be inner equivalent if $\Phi_2 = u\Phi_1 u^{-1}$ for some invertible element $u \in B$. Let $\mathrm{Hom}(A, B)/\sim$ be the set of classes of inner equivalent homomorphisms from A to B.

We need some elementary sheaf cohomology. Let G be a Lie group and let H be a closed subgroup of G. For each topological space Y the fibration $H \to G \to G/H$ induces the following exact sequence of pointed cohomology sets:

$$H^0(Y,H) \to H^0(Y,G) \to H^0(Y,G/H) \xrightarrow{\delta} H^1(Y,H) \to H^1(Y,G).$$

We have $H^0(Y,H) = C(Y,H)$ (continuous maps from Y to H) and $H^0(Y,G) = C(Y,G)$. These sets are pointed by the constant map $f = 1_G$ given by the unity of G. Similarly $H^0(Y,G/H) = C(Y,G/H)$ is pointed by the constant map $f = \{H\}$. The cohomology sets $H^1(Y,H)$ and $H^1(Y,G)$ are pointed by the trivial cocycles $(Y,1_H)$ and $(Y,1_G)$ respectively [2]. Given $f \in C(Y,G/H)$ the cocycle $\delta(f)$ represents the obstruction for lifting f to a function in $C(Y,G)$. By the exactness of the above sequence f has a continuous lifting if and only if $\delta(f) = (Y,1_H)$. The action of G on G/H induces an action of $C(Y,G)$ on $C(Y,G/H)$. If $f_1, f_2 \in C(Y,G/H)$ then $\delta(f_1) = \delta(f_2)$ if and only if $f_2 = gf_1$ for some $g \in C(Y,G)$.

2. Results.

PROPOSITION 1. *Let Y be a topological space. Then there is a bijection* $\mathrm{Hom}(M_n, C(Y) \otimes M_{kn})/ \sim \longrightarrow T_n \,\mathrm{Vect}_k(Y)$.

Proof. We describe the exact sequence induced by the following fibration:

$$\mathrm{GL}_k(C) \xrightarrow{\gamma} \mathrm{GL}_{kn}(C) \xrightarrow{j} \mathrm{GL}_{kn}(C)/\mathrm{GL}_k(C)$$

where the imbedding γ is given by

$$\gamma(u) = u \otimes 1_n, \quad M_{kn} = M_k \otimes M_n.$$

There is a commutative diagram of pointed sets:

$$\begin{array}{ccccccc}
C(Y,\mathrm{GL}_{kn}(C)) & \to & C(Y,\mathrm{GL}_{kn}(C)/\mathrm{GL}_k(C)) & \xrightarrow{\delta} & H^1(Y,\mathrm{GL}_k(C)) & \xrightarrow{\gamma_*} & H^1(Y,\mathrm{GL}_{kn}(C)) \\
\| & & \downarrow \alpha & & \downarrow \beta & & \downarrow \beta_1 \\
C(Y,\mathrm{GL}_{kn}(C)) & \to & \mathrm{Hom}(M_n, C(T) \otimes M_{kn}) & \xrightarrow{\delta'} & \mathrm{Vect}_k(Y) & \xrightarrow{\gamma'} & \mathrm{Vect}_{kn}(Y)
\end{array}$$

The vertical arrows are bijections. To describe α recall that

$$\mathrm{Hom}(M_n, M_{kn}) \simeq \mathrm{GL}_{kn}(C)/\mathrm{GL}_k(C)$$

as topological spaces, the homeomorphism being induced by the map $\eta\colon \mathrm{GL}_{kn}(C) \to \mathrm{Hom}(M_n, M_{kn})$ given by $\eta(v)(a) = v(1_k \otimes a)v^{-1}$, $a \in M_n$. Let η_1 be the map

$$C(Y,\mathrm{GL}_{kn}(C)/\mathrm{GL}_k(C)) \to C(Y,\mathrm{Hom}(M_n, M_{kn}))$$

induced by η. By definition we set $\alpha = \alpha_1\eta_1$ where

$$\alpha_1 : C(Y,\mathrm{Hom}(M_n, M_{kn})) \to \mathrm{Hom}(M_n, C(Y) \otimes M_{kn})$$

is given by $\alpha_1(\Psi)(a)(y) = \Psi(y)(a)$, $a \in M_n, y \in Y$. If in

$$\mathrm{Hom}(M_n, C(Y) \otimes M_{kn})$$

we distinguish the homomorphism $a \mapsto a \otimes 1_k$, α will be an isomorphism of pointed sets.The maps β and β_1 are the natural ones. Namely if (U_i, g_{ij}) is a GL_k-cocycle, then $\beta(U_i, g_{ij})$ is the isomorphism class of the vector bundle obtained by clutching the trivial bundles $U_i \times C^k$ with the transition functions (g_{ij}). The map β_1 is defined in a similar way. The other maps are defined to make the diagram commutative. If $v \in C(Y, \mathrm{GL}_{kn}(C))$ then $j'(v): M_n \to C(Y) \otimes M_{kn}$ is defined by

$$j'(v)(a)(y) = v(y)(1_k \otimes a)v(y)^{-1}, \qquad a \in M_n, \ y \in Y.$$

The map γ' takes the vector bundle E to the direct sum $E \oplus E \oplus \cdots \oplus E$ (n-times). After the above identifications, it follows that two homomorphisms $\Phi_1, \Phi_2 \in \mathrm{Hom}(M_n, C(Y) \otimes M_{kn})$ are inner equivalent if and only if $\delta'(\Phi_1) = \delta'(\Phi_2)$. The isomorphism class of the vector bundle $\delta'(\Phi_1)$ represents the obstruction for lifting Φ_1 to an invertible element in $C(Y) \otimes M_{kn}$. Also, by the exactness of the second row in the above diagram, the image of δ' is equal to $T_n \,\mathrm{Vect}_k(Y)$.

THEOREM 2. *Let X, Y be topological spaces. Then the following assertions are equivalent:*

(i) *The set $T_n \,\mathrm{Vect}_k(Y)$ reduces to the trivial bundle of rank k.*

(ii) *Each homomorphism $\Phi \in \mathrm{Hom}(C(X) \otimes M_n, C(Y) \otimes M_{kn})$ is inner equivalent to a homomorphism of the form $\Phi' \otimes \mathrm{id}_n$ for some $\Phi' \in \mathrm{Hom}(C(X), C(Y) \otimes M_k)$.*

Proof. The implication (i) $\Rightarrow$ (ii) follows easily from Proposition 1. Indeed, if we choose a point x in X and a homomorphism Φ_1 in $\mathrm{Hom}(M_n, C(Y) \otimes M_{kn})$ which is not inner equivalent to the homomorphism $a \mapsto 1_k \otimes a$ then the homomorphism $C(X) \otimes M_n \ni F \mapsto \Phi_1(F(x)) \in C(Y) \otimes M_{kn}$ failed to satisfy (ii).

To prove the other implication we assume, as a preliminary step, that Φ acts on matrices as an amplification:

$$\Phi(1_{C(X)} \otimes a) = 1_{C(Y)} \otimes 1_k \otimes a, \qquad a \in M_n.$$

Under this assumption we get

$$\begin{aligned}\Phi(f \otimes a) &= \Phi(f \otimes 1_n)\Phi(1 \otimes a) = \Phi(1 \otimes a)\Phi(f \otimes 1_n) \\ &= 1 \otimes 1_k \otimes a \cdot \Phi(f \otimes 1_n), \qquad a \in M_n, \ f \in C(X).\end{aligned}$$

The previous computation shows us that the algebra $\Phi(C(X) \otimes 1_n)$ lies in the relative commutant of $1_{C(Y)} \otimes 1_k \otimes M_n$ in $C(Y) \otimes M_k \otimes M_n$ which is equal to $C(Y) \otimes M_k \otimes 1_n$. It follows that there is a unique homomorphism $\Phi' \in \mathrm{Hom}(C(X), C(Y) \otimes M_k)$ such that $\Phi(f \otimes 1_n) = \Phi'(f) \otimes 1_n$. Using again our assumption on Φ we get $\Phi = \Phi' \otimes \mathrm{id}_n$.

Consider now an arbitrary homomorphism Φ and let Φ_1 denote its restriction to M_n. Using (i) it follows by Proposition 1 that there is some invertible element $u \in C(Y) \otimes M_{kn}$ such that

$$\Phi_1(a) = \Phi(1 \otimes a) = u(1 \otimes 1_k \otimes a)u^{-1}, \qquad a \in M_n.$$

Hence the homomorphism $u^{-1}\Phi u$ acts on matrices as an amplification.

REMARK 3. The assertion (i) in the above theorem holds provided that Y is homotopy equivalent to a finite CW-complex of dimension $\leq 2k$ and the K-theory group $K^0(Y)$ does not have n-torsion. This follows from the stability properties of vector bundles (see [**3**, Ch. 8, Th. 1.5]).

Note that $T_n \operatorname{Vect}_1(Y)$ is a subgroup of the group $(\operatorname{Vect}_1(Y), \otimes)$. We have a natural action of $T_n \operatorname{Vect}_1(Y)$ on $T_n \operatorname{Vect}_k(Y)$ given by $(L, E) \mapsto L \otimes E$. By similar methods one can prove the following

THEOREM 4. *Let X, Y be topological spaces. Then the following assertions are equivalent*:

(i) *$T_n \operatorname{Vect}_1(Y)$ acts transitively on $T_n \operatorname{Vect}_k(Y)$.*

(ii) *For any homomorphism $\Phi \in \mathrm{Hom}(C(X) \otimes M_n, C(Y) \otimes M_{kn})$ there is an automorphism α of $C(Y) \otimes M_{kn}$ which is $C(Y)$-linear such that $\alpha \circ \Phi = \Phi' \otimes \mathrm{id}_n$ for some homomorphism $\Phi' \in \mathrm{Hom}(C(X), C(Y) \otimes M_k)$.*

REFERENCES

[1] E. G. Effros, *On the structure of C^*-algebras: Some old and some new problems*, in *Operator Algebras and Applications*. Proc. Symp. Pure Math., Amer. Math. Soc., Providence, RI, 1982.

[2] F. Hirzebruch, *Topological Methods in Algebraic Geometry*, Springer-Verlag, New York, 1965.

[3] D. Husemoller, *Fibre Bundles*, McGraw-Hill Book Company, 1966.

[4] R. V. Kadison and J. R. Ringrose, *Derivations and automorphisms of operator algebras*, Commun. Math. Phys., **4** (1967), 32–63.

[5] C. Pasnicu, *On certain inductive limit C^*-algebras*, Indiana Univ. Math. J., **35** (1986), 269–288.

[6] J. Phillips and I. Raeburn, *Automorphisms of C^*-algebras and second Cech cohomology*, Indiana Univ. Math. J., **29** (1980), 799–822.

[7] K. Thomsen, *Automorphisms of homogeneous C^*-algebras*, Bull. Austral. Math. Soc., **33** (1986), 145–154.

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