Solutions to Mid1, MA266, Fall07

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Problem 1: Show that $y(t) = t^{-2}\ln(t)$ is a solution of $t^2y'' + 5ty' + 4y = 0$.

$$\begin{aligned}
y(t) &= t^{-2}\ln(t)\\
y'(t) &= -2t^{-3}\ln(t) + t^{-3}\\
y''(t) &= 6t^{-4}\ln(t) - 2t^{-4} - 3t^{-4}\\
t^2y'' &= 6t^{-2}\ln(t) - 5t^{-2}\\
5ty' &= -10t^{-2}\ln(t) + 5t^{-2}\\
4y &= 4t^{-2}\ln(t)
\end{aligned}$$

Adding up we get the result.

Problem 2: Solve the differential equation $ty' + 2y = \sin(t)/t$ with $y(\pi/2) = a$ for $t > 0$.

The equation is linear: $y' + \frac{2}{t}y = \sin(t)/t^2$

$$\begin{aligned}
\mu(t) &= \exp(\int \frac{2}{t}\mathrm{d}t) = \exp(2\ln(t)) = t^2\\
y(t) &= t^{-2}(\int t^2\sin(t)/t^2\mathrm{d}t)\\
&= t^{-2}(-\cos(t) + c)
\end{aligned}$$

Initial condition: $y(\pi/2) = (\frac{\pi}{2})^{-2}c = a$ so $c = a\pi^2/4$ and

$$\boxed{y(t) = t^{-2}(-\cos(t) + a\pi^2/4)}$$

Problem 3: Solve the differential equation $yy' + x^2 + x + 1 = 0$. The equation is separable:

$$\begin{aligned}
y\mathrm{d}y &= (-x^2 - x - 1)\mathrm{d}x\\
\int y\mathrm{d}y &= \int -x^2 - x - 1\,\mathrm{d}x\\
\frac{1}{2}y^2 &= -\frac{1}{3}x^3 - \frac{1}{2}x^2 - x + c
\end{aligned}$$

Hence

$$\boxed{y = \pm\sqrt{-\tfrac{2}{3}x^3 - x^2 - 2x + 2c}}$$

Problem 4: If $y(x)$ satisfies $y' = 3\sin(x)y + 5y$ and $y(0) = 2$ then $y(\pi) = ?$.

The equation is linear and separable, $x \geq 0$:

$$\begin{aligned}
\frac{\mathrm{d}y}{y} &= (3\sin(x) + 5)\mathrm{d}x\\
\int \frac{\mathrm{d}y}{y} &= \int (3\sin(x) + 5)\mathrm{d}x\\
\ln(y) &= -3\cos(x) + 5x + c\\
y &= C\exp(-3\cos(x) + 5x)
\end{aligned}$$

Initial condition $y(0) = Ce^{-3} = 2$ so $C = 2e^3$. Evaluate at π:

$$\boxed{y(\pi) = 2e^3e^{3+3\pi} = 2e^{6+5\pi}}$$

Problem 5: A tank of water that holds 100 gallons originally contains 80 gallons of water with 100 lbs of salt. Water containing 1 lbs/gal of salt is entering the tank at 2 gal/min. The mixture flows out of the tank at 1 gal/minute. Find the concentration of the mixture at the time the tank overflows.

Setting up the differential equation: The changing volume is $V(t) = 80 + t$. Time of overflow $V = 80 + t = 100, t = 20$.

$$\begin{aligned}
\mathrm{d}Q/\mathrm{d}t &= \text{solid rate in} - \text{solid rate out}\\
&= \text{liquid rate in} \times \text{Concentration in}\\
&\quad -\text{liquid rate out} \times \text{Concentration out}\\
&= 2 - \frac{1}{80+t}Q
\end{aligned}$$

This is linear:

$$\mu(t) = \exp(\int \frac{1}{80+t}\mathrm{d}t) = \exp(\ln(80+t))$$

$$\begin{aligned} &= 80+t \\ y(t) &= \frac{1}{80+t}\int 2(80+t)\mathrm{d}t = \frac{160t+t^2+c}{80+t} \end{aligned}$$

Initial condition. $Q(0) = c/80 = 100$ so $c = 8000$. $Q(20) = (3200+400+8000)/100 = 116$ and the concentration is

$$\boxed{\frac{116lbs}{100gal} = 1.16\frac{lbs}{gal}}$$

Problem 6: A population of mosquitos increases at a rate proportional to the current population and in absence of other factors, the population triples each month. Assume that there are initially 100 mosquitos and the rate at which they are killed is 25 per week. Solve the resulting differential equation. Discuss the behavior in the short term and the long term (use $e \approx 2.72$).

$\frac{\mathrm{d}P}{\mathrm{d}t} = rP - k$ has the solution

$$P(t) = (P_0 - k/r)e^{rt} + k/r$$

Assuming 4 weeks per months: $k = 100$. The population would triple if $k = 0$ so $r = \ln(3)$. Lastly, $P_0 = P(0) = 100$ which yields:

$$\begin{aligned} P(t) &= (100 - \frac{100}{\ln(3)})e^{\ln(3)t} + \frac{100}{\ln(3)} \\ &= (100 - \frac{100}{\ln(3)})3^t + \frac{100}{\ln(3)} \end{aligned}$$

Now $\ln(3) > \ln(e) = 1$ but close to it. Hence $\ln(3)$ slightly bigger than 1 and $100 - \frac{100}{\ln(3)}$ is slightly bigger than 0. So in the short term the solution is almost constant and in the long term it has exponential growth.

Problem 7: Determine the largest open interval on which the solution to the initial value problem $\sin(t)y' + \frac{y}{t^2-9} = \tan(t), y(-2) = 0$ is guaranteed by the Existence and Uniqueness Theorem to exist.

Transforming to a linear equation: $y' + \frac{y}{(\sin(t)(y^2-9)} = \frac{1}{\cos(t)}$. The coefficient functions $\frac{1}{\sin(t)(y^2-9)}$ and $\frac{1}{\cos(t)}$ are singular at $t = n\frac{\pi}{2}, n \in \mathbb{Z}$ and at ± 3. Hence the biggest interval of the real line minus these points containing -2 is

$$\boxed{-3 \le t \le -\frac{\pi}{2}}$$

Problem 8: Determine the critical (equilibrium) points of $y' = 12y^2(y^2-9)$, draw the phase line, and classify the critical points as stable, semi-stable or unstable.

The equilibrium solutions are $12y^2(y^2-9) = 0$ hence $y = 0, \pm 3$.

$$\text{The phase line is}\quad \begin{array}{ccl} \uparrow & & \\ \bullet & 3 & \text{unstable} \\ \downarrow & & \\ \bullet & 0 & \text{semi-stable} \\ \downarrow & & \\ \bullet & \text{-3} & \text{stable} \\ \uparrow & & \end{array}$$

Problem 9: Find the implicit solution of $\frac{x}{(x^2+y^2)^{3/2}}\mathrm{d}x + \frac{y}{(x^2+y^2)^{3/2}}\mathrm{d}y = 0$ with $y(3) = 4$.

Test for exact:

$$\begin{aligned} M_y &= \frac{\partial}{\partial y}\frac{x}{(x^2+y^2)^{3/2}} = \frac{-3xy}{(x^2+y^2)^{5/2}} \\ N_x &= \frac{\partial}{\partial x}\frac{y}{(x^2+y^2)^{3/2}} = \frac{-3xy}{(x^2+y^2)^{5/2}} \end{aligned}$$

so $M_y = N_x$ and the equation is exact.

Solve $\Psi_x(x,y) = M$: $\Psi = -\frac{1}{(x^2+y^2)^{1/2}} + h(y)$.
Solve $\Psi_y = N$: $\frac{y}{(x^2+y^2)^{3/2}} + h'(y) = \frac{y}{(x^2+y^2)^{3/2}}$ so $h'(y) = 0$ and $h(y) = c$

Implicit solution $\Psi(x,y) = -\frac{1}{(x^2+y^2)^{1/2}} = C$. Initial condition $\Psi(3,4) = -\frac{1}{(9+16)^{1/2}} = -\frac{1}{5} = C$.

Final answer:

$$\boxed{\frac{1}{(x^2+y^2)^{1/2}} = \frac{1}{5}}$$

Alternative method: multiply to get $x\mathrm{d}x + y\mathrm{d}y = 0$ hence $\frac{1}{2}(x^2+y^2) = c$ or $x^2+y^2 = C$ with initial condition $C = 25$. Answer: $x^2+y^2 = 25$.

Problem 10: Give the solution in implicit form of $(2y^2+2y^3)\mathrm{d}x + (2xy+3xy^2)\mathrm{d}y = 0$.

Test for exactness:

$$\begin{aligned} M_y &= \frac{\partial}{\partial y}(2y^2+2y^3) = 4y+6y^2 \\ N_x &= \frac{\partial}{\partial x}(2xy+3xy^2) = 2y+3y^2 \end{aligned}$$

So $M_y - N_x = 2y+3y^2 \neq 0$ but $\frac{M_y-N_x}{N} = \frac{1}{x}$ depends only on x. Thus have integrating factor $\mu(x)$, satisfying $\frac{\mathrm{d}\mu}{\mathrm{d}x} = \frac{1}{x}\mu$ or $\frac{\mathrm{d}\mu}{\mu} = \frac{\mathrm{d}x}{x}$. We solve $\ln(\mu) = \ln(x)$ and hence $\mu(x) = x$.

The new equation reads.

$(2xy^2+2xy^3)\mathrm{d}x + (2x^2y+3x^2y^2)\mathrm{d}y = 0$.

Integrating the first function with respect to x we obtain $\psi(x,y) = x^2y^2 + x^2y^3 + h(y)$ and since $\Psi_y = 2x^2y+3x^2y^2+h'(y)$ we have $h'(y) = 0, h(y) = c$ and the solution reads

$$\boxed{x^2y^2 + x^2y^3 = c}$$

Alternate method: Separate variables to get $\frac{2}{x}\mathrm{d}x = -\frac{2y+3y^2}{y^2+y^3}\mathrm{d}y$ and hence $2\ln(x) = -\ln(y^2+y^3) + c$ and $x^2 = \frac{1}{(y^2+y^3)}C$ or $x^2(y^2+y^3) = C$.