MULTI-DEGREE BOUNDS ON THE BETTI NUMBERS OF REAL VARIETIES AND SEMI-ALGEBRAIC SETS AND APPLICATIONS

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Abstract. We prove new bounds on the Betti numbers of real varieties and semi-algebraic sets that have a more refined dependence on the degrees of the polynomials defining them than results known before. Our method also unifies several different types of results under a single framework, such as bounds depending on the total degrees, on multi-degrees, as well as in the case of quadratic and partially quadratic polynomials. The bounds we present in the case of partially quadratic polynomials offer a significant improvement over what was previously known.

We give several applications of our results, including a generalization of the polynomial partitioning theorem due to Guth and Katz, which has become a very important tool in discrete geometry in the multi-degree setting, and give an application of this result proving a theorem that interpolates between two different kinds of partitions of the plane.

Finally, we extend a result of Basu and Barone on bounding the number of connected components of real varieties defined by two polynomials of differing degrees to the sum of all Betti numbers, thus making progress on an open problem posed in their paper – on extending their bounds on the number of connected components to higher Betti numbers as well.

Contents

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1. Introduction

Throughout this paper R will denote a fixed real closed field and C the algebraic closure of R. For any semi-algebraic subset $S \subset \mathrm{R}^k$ we denote by $b_i(S, \mathbb{Z}_2)$ the dimension of the i-th homology group, $\mathrm{H}_i(S, \mathbb{Z}_2)$, and by $b(S, \mathbb{F}) = \sum_{i \geq 0} b_i(S, \mathbb{F})$ (we refer the reader to [17, Chapter 6] for definition of homology groups of semi-algebraic sets defined over arbitrary real closed fields).

Remark 1. Notice that by the universal coefficient formula, $b_i(S, \mathbb{Z}_2) \geq b_i(S)$ (where $b_i(S)$ denotes the rank of the i-th homology group of S with integer coefficients), and thus an upper bound on $b_i(S, \mathbb{Z}_2)$ is automatically an upper bound on $b_i(S)$.

1.1. **Background.** The problem of bounding the Betti numbers of real algebraic varieties as well as semi-algebraic subsets of R^k, in terms of the format of their defining formulas, has been an active topic of investigation for a long time starting from the first results bounding the Betti numbers of real varieties proved by Oleĭnik and Petrovskiĭ [47], Thom [53] and Milnor [44]. Later these results were extended to more general semi-algebraic sets [12, 27, 29]. These results were based on Morse-theoretic arguments involving bounding the number of critical points of a Morse function on a bounded, non-singular real algebraic hypersurface, using Bezout's theorem, arguments involving infinitesimal perturbations, and use of inequalities coming from the Mayer-Vietoris exact sequence. The bounds were singly exponential in the dimension of the ambient space and polynomial in the number of polynomials used in the definition of the given semi-algebraic set, and also in the *maximum of the total degrees of these polynomials* (see Theorems 1, 2, 3, below for precise statements).

In another direction, bounds which are *polynomial* in the dimension were proved for a restricted class of semi-algebraic sets. Barvinok [10] proved a polynomial bound on the Betti numbers of semi-algebraic sets (see Theorem 4 for a precise statement), which were sharpened in [15, 40], and also extended to a more general setting in [19] (see Theorem 8 for a precise statement). These results were proved using different techniques than a simple counting of critical points. A spectral sequence argument first proposed by Agrachev [6, 5, 4] plays an important role in some of the latter results.

Much more recently, because of certain new techniques developed in incidence geometry, more refined bounds than those mentioned above were needed. In particular, it was not enough to prove bounds which depended on the maximum of the degrees of the polynomials, and it was necessary to prove bounds with a more refined dependence on the sequence of degrees. Nearly optimal bounds on the zero-th Betti numbers (i.e. the number of connected components) of semi-algebraic sets was proved later in [9, 8] which has proved useful in applications. However, the techniques used to prove the results in [9, 8] are not sufficient for bounding the higher Betti numbers. Extending the bounds proved in [9, 8] to sum of all the Betti numbers (i.e. not just the zero-th Betti number) remains a challenging open problem in real algebraic geometry.

The first contribution of the current paper is to develop a single framework which allows one to prove the bounds on general semi-algebraic sets, as well as those defined by quadratic or even partially quadratic polynomials (Theorems 11 16, 17, 18, 19). Moreover, we improve the known bounds in all of these cases. In the process, we also answer an open question of Lerario [40] on the asymptotic behavior of the Betti numbers of generic complete intersections of projective quadrics over C (Remark 23).

Additionally, the framework allows us to prove bounds in terms of the *multi-degrees* of the polynomials instead of the total degrees (Theorems 12, 13, 14, 15, 20, 21). We give several applications in which this new flexibility proves to be important (Theorems 26, 27, 28, 29). Note that there have been some other applications of multi-degree bounds in special cases (see for instance [32] for a recent algorithmic application).

As mentioned above, extending the bounds proved in [9, 8] to sum of all the Betti numbers remains an open problem. The second contribution of the current paper is extending the result [9] to the sum of all the Betti numbers to the case of degree sequence of length bounded by 2 (Theorems 31 and 32).

Finally, we give another application in which "multi-degree" bounds play an important role. We prove a generalization of the polynomial partitioning theorem of Guth and (N.H.) Katz [33] in the plane. This theorem and its various generalizations has proved to be very useful in proving tight bounds on several quantitative problems on bounding the number of incidences between finite sets of points and varieties in R^k (generalizing the classic result of Szemerédi and Trotter [52] on point-line incidences in R^2). The polynomial partitioning theorem is proved using another classical partitioning result due to Stone and Tukey [51], often referred to as the "polynomial ham-sandwich" theorem, and produces for every finite subset $S \subset \mathrm{R}^k$ and a parameter $r > 0$, a "partitioning-polynomial" of *total* degree bounded by $r^{1/k}$ (see Definition 5 and Theorem 36 for the exact statement). We prove multi-degree analogs of both the polynomial ham-sandwich theorem and Theorem 36 (see Theorems 39 and 40). The resulting flexibility allows us to prove (in the case $k = 2$), the existence of a parametrized family of partitions with two competing measures of quality (see Definition 4 and Theorem 38). While this result does not immediately produce an improvement in any incidence problem, the new flexibility in producing partitions could potentially be useful in such problems (especially if it is possible to extend this theorem to higher dimensions).

1.2. **Prior Results.** In this section we state more precisely the prior results mention in the previous section. We first fix some notation that we will use for the rest of the paper.

1.2.1. *Basic notation and definition.*

Notation 1. For $P \in \mathrm{R}[X_1, \ldots, X_k]$ (resp. $P \in \mathrm{C}[X_1, \ldots, X_k]$) we denote by $\mathrm{Zer}(P, \mathrm{R}^k)$ (resp. $\mathrm{Zer}(P, \mathrm{C}^k)$) the set of zeros of P in R^k(resp. C^k). More generally, for any finite set $\mathcal{P} \subset \mathrm{R}[X_1, \ldots, X_k]$ (resp. $\mathcal{P} \subset \mathrm{C}[X_1, \ldots, X_k]$), we denote by $\mathrm{Zer}(\mathcal{P}, \mathrm{R}^k)$ (resp. $\mathrm{Zer}(\mathcal{P}, \mathrm{C}^k)$) the set of common zeros of $\mathcal{P}$ in R^k (resp. C^k).

For a homogeneous polynomial $P \in \mathrm{R}[X_0, \ldots, X_k]$, (resp. $P \in \mathrm{C}[X_0, \ldots, X_k]$) we denote by $\mathrm{Zer}(P, \mathbb{P}^k_{\mathrm{R}})$ (resp. $\mathrm{Zer}(P, \mathbb{P}^k_{\mathrm{C}})$) the set of zeros of P in $\mathbb{P}^k_{\mathrm{R}}$. (resp. $\mathbb{P}^k_{\mathrm{C}}$). And, more generally, for any finite set of homogeneous polynomials $\mathcal{P} \subset \mathrm{R}[X_0, \ldots, X_k]$, (resp. $\mathcal{P} \subset \mathrm{C}[X_0, \ldots, X_k]$), we denote by $\mathrm{Zer}(\mathcal{P}, \mathbb{P}^k_{\mathrm{R}})$ (resp. $\mathrm{Zer}(\mathcal{P}, \mathbb{P}^k_{\mathrm{C}})$) the set of common zeros of $\mathcal{P}$ in $\mathbb{P}^k_{\mathrm{R}}$. (resp. $\mathbb{P}^k_{\mathrm{C}}$).

Notation 2. For any finite family of polynomials $\mathcal{P} \subset \mathrm{R}[X_1, \ldots, X_k]$, we call an element $\sigma \in \{0, 1, -1\}^{\mathcal{P}}$, a *sign condition* on $\mathcal{P}$. For any semi-algebraic set $Z \subset \mathrm{R}^k$, and a sign condition $\sigma \in \{0, 1, -1\}^{\mathcal{P}}$, we denote by $\mathrm{Reali}(\sigma, Z)$ the semi-algebraic set defined by

$$\{\mathbf{x} \in Z \mid \mathbf{sign}(P(\mathbf{x})) = \sigma(P), P \in \mathcal{P}\},$$

and call it the *realization* of σ on Z. More generally, we call any Boolean formula Φ with atoms, $P\{=, >, <\}0, P \in \mathcal{P}$, to be a $\mathcal{P}$*-formula*. We call the realization of Φ, namely the semi-algebraic set

$$\mathrm{Reali}\left(\Phi, \mathrm{R}^k\right) \quad = \quad \left\{\mathbf{x} \in \mathrm{R}^k \mid \Phi(\mathbf{x})\right\}$$

a $\mathcal{P}$*-semi-algebraic set*. Finally, we call a Boolean formula without negations, and with atoms $P\{\geq, \leq\}0$, $P \in \mathcal{P}$, to be a $\mathcal{P}$*-closed formula*, and we call the realization, $\mathrm{Reali}\left(\Phi, \mathrm{R}^k\right)$, a $\mathcal{P}$*-closed semi-algebraic set*.

1.2.2. *General Bounds.* The first results on bounding the Betti numbers of real varieties were proved by Oleĭnik and Petrovskiĭ [47], Thom [53] and Milnor [44]. Using a Morse-theoretic argument and Bezout's theorem they proved:

Theorem 1. [47, 53, 44] *Let* $\mathcal{Q} \subset \mathrm{R}[X_1, \ldots, X_k]$ *with* $\deg(Q) \leq d, Q \in \mathcal{Q}$*. Then,*

$$b(\mathrm{Zer}(\mathcal{Q}, \mathrm{R}^k), \mathbb{Z}_2) \leq d(2d-1)^{k-1}. \tag{1.1}$$

Remark 2. Theorem 1 is proved (see for example proof of Theorem 11.5.3 in [21] for an exposition) by first replacing the given variety by a bounded basic, closed semi-algebraic set having the same homotopy type as $\mathrm{Zer}(\mathcal{Q}, \mathrm{R}^k)$ defined by a single polynomial inequality of total degree at most twice the maximum of the degrees of the polynomials in $\mathcal{Q}$. Moreover, the critical points of the coordinate function X_1 are shown to be non-degenerate and their number can be bounded using Bezout's theorem. If one wants to obtain a bound in terms of the *multi-degrees* (i.e. the tuple of degrees in each of the variables) of the polynomials in $\mathcal{Q}$, it is possible to mimic the same proof as above, and finally use the *multi-homogeneous Bezout's theorem* in order to bound the number of critical points. The method used in this paper (see proof of Theorem 11) is different, and gives slightly better bounds (see Remark 18).

Remark 3. Also, note that the bound in Theorem 1 holds for dimensions of the homology groups with coefficients in any field and was proved in that generality. The same is true for some of the other results surveyed below. However, the new results in this paper give bounds only for Betti numbers over the field $\mathbb{Z}_2$ (because our technique for proving them involves using Smith inequalities cf. Theorem 22), and are thus correspondingly weaker. On the other hand they do imply via the universal coefficients theorem (see Remark 1) the same bounds on the ranks of the homology groups with integer coefficients. Moreover, $\mathbb{Z}_2$-homology is very natural in the context of real algebraic geometry. We will state all bounds for the $\mathbb{Z}_2$-Betti numbers from now on without comment (except in Section 6 below).

Theorem 1 was later generalized to arbitrary semi-algebraic sets defined by quantifier-free formulas in two steps. In the first step, Theorem 1 was extended to a particular – namely $\mathcal{P}$-closed semi-algebraic sets, where $\mathcal{P} \subset \mathrm{R}[X_1, \ldots, X_k]$ is a finite family of polynomials. The following theorem (which makes more precise an earlier result appearing in [12]) appears in [16].

Theorem 2. [16] *If $S \subset \mathrm{R}^k$ is a $\mathcal{P}$-closed semi-algebraic set, then*

$$b(S, \mathbb{Z}_2) \leq \sum_{i=0}^{k} \sum_{j=0}^{k-i} \binom{s+1}{j} 6^j d(2d-1)^{k-1}, \tag{1.2}$$

where $s = \mathrm{card}(\mathcal{P}) > 0$ and $d = \max_{P \in \mathcal{P}} \deg(P)$.

Using an additional ingredient (namely, a technique to replace an arbitrary semi-algebraic set by a locally closed one with a very controlled increase in the number of polynomials used to describe the given set), Gabrielov and Vorobjov [27] extended Theorem 2 to arbitrary $\mathcal{P}$-semi-algebraic sets with only a small increase in the bound. Their result in conjunction with Theorem 2 gives the following theorem.

Theorem 3. [29] *If $S \subset \mathrm{R}^k$ is a $\mathcal{P}$-semi-algebraic set, then*

$$b(S, \mathbb{Z}_2) \leq \sum_{i=0}^{k} \sum_{j=0}^{k-i} \binom{2ks+1}{j} 6^j d(2d-1)^{k-1}, \tag{1.3}$$

where $s = \mathrm{card}(\mathcal{P})$ and $d = \max_{P \in \mathcal{P}} \deg(P)$.

1.2.3. *Quadratic and partially quadratic case.* Semi-algebraic sets defined by few quadratic inequalities are topologically simpler. This was first noticed by Agrachev [6, 5, 4] who proved a bound which is polynomial in the number of variables and exponential in the number of inequalities for *generic* quadratic inequalities. The technique introduced by Agrachev was very important in later developments as well. Independently, using a different technique (closer to the spirit of Morse theoretic arguments) Barvinok [10] proved the following theorem (no genericity assumption is required).

Theorem 4. [10] *Let $S \subset \mathrm{R}^k$ be defined by $P_1 \geq 0, \ldots, P_s \geq 0,\ \deg(P_i) \leq 2,\ 1 \leq i \leq s$. Then,*

$$b(S, \mathbb{Z}_2) \leq k^{O(s)}.$$

This bound was later sharpened in [15] and further sharpened in the case of algebraic sets by Lerario in [40, Theorem 15], where the following nearly optimal result was proved.

Theorem 5. [40] *Let $\mathcal{Q} \subset \mathrm{R}[X_0, \ldots, X_k]$ be a set of ℓ quadratic forms, and $V = \mathrm{Zer}(\mathcal{Q}, \mathbb{P}^k_{\mathrm{R}})$ be the* projective *variety defined by $\mathcal{Q}$. Then,*

$$b(V, \mathbb{Z}_2) \leq (O(k))^{\ell-1}.$$

Theorem 4 was later extended in [14] where the following theorem was proved. Notice that this bound is polynomial even in the number of inequalities (for fixed ℓ).

Theorem 6. [14] *Let ℓ be any fixed number and let* R *be a real closed field. Let $S \subset \mathrm{R}^k$ be defined by $P_1 \geq 0, \ldots, P_s \geq 0,\ \deg(P_i) \leq 2$. Then,*

$$b_{k-\ell}(S, \mathbb{Z}_2) \leq \binom{s}{\ell} k^{O(\ell)}.$$

Theorem 6 was further improved using bounds on the Betti numbers of non-singular complete intersections and the Smith inequality (Theorem 22) in [15] where the following theorem is proved.

Theorem 7. [15] *Let $\mathcal{P} = \{P_1, \ldots, P_s\} \subset \mathrm{R}[X_1, \ldots, X_k],\ s \leq k$. Let $S \subset \mathrm{R}^k$ be defined by*

$$P_1 \geq 0, \ldots, P_s \geq 0$$

with $\deg(P_i) \leq 2$. Then, for $0 \leq i \leq k-1$,

$$b_i(S, \mathbb{Z}_2) \leq \frac{1}{2}\left(\sum_{j=0}^{min\{s,k-i\}} \binom{s}{j}\binom{k+1}{j} 2^j\right).$$

In particular, for $2 \leq s \leq k/2$, we have

$$b_i(S, \mathbb{Z}_2) \leq \frac{1}{2} 3^s \binom{k+1}{s} \leq \frac{1}{2}\left(\frac{3e(k+1)}{s}\right)^2.$$

Finally, in [19] the authors also prove a result that generalizes the bounds on Betti numbers of general semi-algebraic sets (defined by s polynomials having degrees bounded by d, cf. Theorem 2), as well as the bounds in the quadratic case (cf. Theorems 4, 6 and 7). More precisely they prove:

Theorem 8. [19] *Let $\mathcal{P}_1 \subset \mathrm{R}[X_1,\ldots,X_{k_1}]$, a finite set of polynomials with*

$$\deg_X(P) \leq d, P \in \mathcal{P}_1, \mathrm{card}(\mathcal{P}_1) = s,$$

and let $\mathcal{P}_2 \subset \mathrm{R}[X_1,\ldots,X_{k_1},Y_1,\ldots,Y_{k_2}]$, a finite set of polynomials with

$$\deg_X(P) \leq d, \deg_Y(P) \leq 2, P \in \mathcal{P}_2, \mathrm{card}(\mathcal{P}_2) = m,$$

Let $S \subset \mathrm{R}^{k_1+k_2}$ be a $(\mathcal{P}_1 \cup \mathcal{P}_2)$-closed semi-algebraic set. Then

$$b(S,\mathbb{Z}_2) \leq k_2^2(O(k_2+s+m)k_2d)^{k_1+2m}. \tag{1.4}$$

In particular, for $m \leq k_2$, $b(S,\mathbb{Z}_2) \leq k_2^2(O(s+k_2)k_2d)^{k_1+2m}$.

Remark 4. In particular, if in Theorem 8, $\mathcal{P}_1 = \emptyset$ (and hence $s = 0$), and $m, k_1 < k_2$, we get

$$b(S,\mathbb{Z}_2) \leq k_2^2(O(m+k_2)k_2d)^{k_1+2m}. \tag{1.5}$$

Remark 5. The main tool used in the proof of Theorem 8 was a technique introduced by Agrachev in [4, 5, 6] and later exploited by several authors [19, 3, 41, 40] for bounding the Betti numbers of semi-algebraic sets defined by quadratic polynomials.

The techniques used in the proof of the theorems corresponding to Theorem 8 in the current paper (namely, Theorems 16, 18 and 19) are quite different – involving the method of infinitesimal perturbations, Mayer-Vietoris inequalities as explained in [17, Chapter 7], and bounds on the Betti numbers of real affine varieties defined by partially quadratic polynomials proved in Proposition 12 below.

1.2.4. *Generic vs special.* In many quantitative results in algebraic geometry, one assumes that the given system of polynomials is generic. Bounding the topological complexity of varieties defined by generic systems of polynomials (over R as well as C) is often easy. However, such a result does not imply a bound in the non-generic situation. The following example which appears in [26] is very well-known and shows that even for the zero-th Betti number, the "generic" bound might not hold for all special systems.

Example 1. [26] Let $k = 3$ and let

$$\begin{aligned} Q_1 &= X_3, \\ Q_2 &= X_3, \\ Q_3 &= \sum_{i=1}^{2}\left(\prod_{j=1}^{d}(X_i - j)^2\right). \end{aligned}$$

The real variety defined by $\mathcal{Q} = \{Q_1, Q_2, Q_3\}$ is 0-dimensional, and has d^2 isolated (in R^3) points. However, a "generic" system of three polynomials in $\mathrm{R}[X_1, X_2, X_3]$ having degrees $1, 1, 2d$ will have by Bezout's theorem at most $2d$ isolated points as its real zeros. Observe, that even though the real variety $\mathrm{Zer}(\mathcal{Q}, \mathrm{R}^3)$ is zero-dimensional, the complex variety $\mathrm{Zer}(\mathcal{Q}, \mathrm{C}^3)$ is not, which accounts for this discrepancy. We refer the reader to [8] for a Bezout-type inequality that works over R as well.

There has been some work on bounding the number of connected components of real algebraic varieties defined by systems of polynomials satisfying certain genericity conditions. For example, the following theorem appears in [25].

Theorem 9. [25] *Let* $(P_1,\ldots,P_s) \subset \mathbb{Q}[X_1,\ldots,X_k]$ *(with* $s \leq k-1$*) generate a radical ideal and define a smooth algebraic variety* $V \subset \mathrm{C}^k$ *of dimension* k'*. Denote by* $d_1,\ldots,d_s$ *the respective degrees of* $P_1,\ldots,P_s$ *and by* d *the maximum of* $d_1,\ldots,d_s$*. The number of connected components of* $V \cap \mathrm{R}^k$ *is bounded by*

$$d_1 \cdots d_s \sum_{i=0}^{k'} (d-1)^{k-s-i} \binom{k-i}{k-i-s}.$$

Moreover, if $(P_1,\ldots,P_s)$ *is a regular sequence, the number of connected components of* $V \cap \mathrm{R}^k$ *is bounded by*

$$d_1 \cdots d_s \sum_{i=0}^{k'} (d-1)^{k-s-i} \binom{k-i-1}{k-i-s}.$$

Even though, bounds on generic systems do not immediately produce a bound on the Betti numbers of general semi-algebraic sets, with extra effort such bounds can be used to prove (possibly worse) bounds for general semi-algebraic sets. This is in fact the approach taken in this paper, but the approach already appears in the paper by Benedetti, Loeser and Risler [20], which is the starting point of the results presented in the current paper. Using a clever reduction from the general case to the generic case they prove the following theorem.

Theorem 10. [20, Proposition 2.6] *Let* $\mathcal{P} = \{P_1,\ldots,P_\ell\}$ *with* $\deg(P_i) \leq d,\ 1 \leq i \leq \ell,\ P_i \in \mathrm{R}[X_1,\ldots,X_k]$*. Then,*

$$b_0(\mathrm{Zer}(\mathcal{P},\mathrm{R}^k),\mathbb{Z}_2) \leq \lambda(d,k,\ell),$$

where

$$\lambda(d,k,\ell) = Q_1(d,k) + 2Q_2(d,k) + \cdots + 2^{k-\ell-1}Q_{k-\ell}(d,k) + 2^{k-\ell}\mu_d(\ell),$$

and each Q_i *is a polynomial in* d *of degree* $k-i+1$*, the leading coefficient of* Q_i *is a polynomial in* k *of degree* $\ell-1$ *with leading coefficient* $(\ell+1)/2$*, and the other terms polynomials in* k *of degree* $\max\{\ell-1,1\}$*, and* $\mu_d(\ell) = d(2d-1)^{\ell-1}$*.*

Remark 6. Of special interest here is that for every fixed ℓ, and k large enough (depending on ℓ), and for d tending to infinity, $\lambda(d,k,\ell)$ is asymptotically equal to $\left(\frac{1}{2}(\ell+1)k^{\ell-1} + O_\ell(k^{\ell-2})\right)d^k + O_{k,\ell}(d^{k-1})$ [20, Corollary 2.7] where the implied constant in the notation O_ℓ (resp. $O_{k,\ell}$) depends only on ℓ (resp. k,ℓ) (compare with Theorem 11 and Remark 7 below).

In this paper, we consider the problem of bounding the sum of all the Betti numbers of real varieties and semi-algebraic sets with a more refined dependence on the degrees of the polynomials. These refinements are of two kinds. First, we allow different blocks of variables to have different degrees (see Theorems 12 and 13). Second, we allow different polynomials to have different degrees (see Theorems 14 and 15) We also improve using our techniques existing bounds on the sum of the Betti numbers of real varieties and semi-algebraic sets in terms of the number and total degrees of polynomials defining them (Theorem 11) as well as in the partially quadratic case (Theorems 16, 18 19, 20, and 21). We apply the results mentioned above to prove refined bounds on the Betti numbers of pull-backs and direct images under polynomial maps (Theorems 26, 27, 28), and as an application of the last result (i.e. Theorem 28) give a better bound (than possible below) on the Betti numbers of the space of affine subspaces of a fixed dimension that meet a given semi-algebraic subset of R^k (related to an important problem studied in discrete geometry). As another application of our results we give a multi-degree generalization of the polynomial partitioning theorem of Guth and Katz (Theorem 37) and use it to prove a theorem that interpolates between different polynomial partitions of the plane (Theorem 38). Some of the definitions regarding the measure of "quality" of partitions introduced here might be of independent interest. Finally, we make some progress on extending the theorem on refined bounds on the number of connected components of real varieties defined by polynomials having different degrees, to a bound on the sum of the Betti numbers using results proved in this paper and some other ingredients (namely stratified Morse theory and Lefschetz duality from topology of manifolds). This is reported in Theorem 32.

The rest of the paper is organized as follows. In Section 2 we state the main results proved in this paper. In Section 3, we state some preliminary results that are needed in the proofs of the man theorems. In Section 4, we prove the main theorems of the paper. In Section 5 , we prove bounds on the Betti numbers of pull-backs, direct images, and the space of transversals of semi-algebraic sets. In Section 6, we prove a refined bound on the Betti numbers of varieties defined by polynomials having two different degree bounds. Finally, in Section 7 we discuss an application of the results of our paper to the polynomial partitioning problem and state and prove a generalization of a well-known theorem of Guth and Katz in the plane.

2. Main Results

2.1. **Betti numbers of sets defined by polynomials of bounded total degree.** We begin with the classical case of bounding the sum of the Betti numbers of varieties and semi-algebraic sets in terms of the total degrees of the polynomials defining them. This is the classical situation already considered by many authors and already surveyed in Section 1.2, but our methods produce slight improvements which we record here. We prove the following theorems.

Theorem 11. *Let $\mathcal{Q} = \{Q_1, \ldots, Q_\ell\} \subset \mathrm{R}[X_1, \ldots, X_k]$ be a finite set of polynomials whose (total) degrees are bounded by d with $\ell > 0$. Let V denote $\mathrm{Zer}(\mathcal{Q}, \mathrm{R}^k)$. Then, $b(V, \mathbb{Z}_2)$ is bounded by*

(2.1)
$$\min\left(\sum_{j=1}^{k-1}\binom{\ell}{j}2^j(F_1(d',k,j)+F_2(d',k,j))+\binom{\ell}{k}2^k d'^k+3, \frac{1}{2}(1+(2d-1)^k)\right),$$

where

$$\begin{aligned} F_1(d',k,j) &= 1+(-1)^{k-j+1}+ \\ & \quad 2d'^{j-1}\cdot\left(\sum_{h=0}^{k-j}\sum_{i=0}^{h}(-1)^{k-j+h}\binom{k}{h+j}\binom{j+i-2}{i}2^{h-i}d'^{i}\right) \\ &\leq 2\binom{k-2}{j-2}d^{k-1}+(O(d))^{k-2}, \\ F_2(d,k,j) &= 1+(-1)^{k-j+1}+\binom{k-1}{j-1}(d'^{k}+k-1) \\ &\leq \binom{k-1}{j-1}d^{k}+O(1)^{k}, \end{aligned}$$

and d' is the least even integer $\geq d$.

In particular, if k is fixed and $\ell \leq k$, then for large d we have

$$b(V,\mathbb{Z}_2) \leq \sum_{j=1}^{\ell} 2^j\binom{\ell}{j}\binom{k-1}{j-1}d^k+(O(d))^{k-1}. \tag{2.2}$$

Remark 7. Writing the bound in Theorem 11 as a polynomial in d, the leading coefficient is

$$\begin{aligned} \sum_{j=1}^{\ell} 2^j\binom{\ell}{j}\binom{k-1}{j-1} &\leq (3^\ell-1)\left(\sum_{j=1}^{\ell}\binom{k-1}{j-1}\right) \\ &\leq \ell(3^\ell-1)\binom{k-1}{\ell-1} \text{ for } \ell<k/2. \end{aligned}$$

Thus, for every fixed ℓ and every k sufficiently large (depending on ℓ), and as d tends to infinity, the bound in (2.2) is asymptotically equal to

$$\left(\frac{\ell(3^\ell-1)}{(\ell-1)!}k^{\ell-1}+O_\ell(k^{\ell-2})\right)d^k+O_{k,\ell}(d^{k-1}),$$

where the implied constant in the notation O_ℓ (resp. $O_{k,\ell}$) depends only on ℓ (resp. k,ℓ). Notice that for $\ell > 8$,

$$\frac{\ell(3^\ell-1)}{(\ell-1)!} < \frac{1}{2}(\ell+1)$$

(cf. Remark 6 following Theorem 10). Thus, for fixed ℓ and k (sufficiently large) the bound in Theorem 11 is asymptotically better (as d tends to infinity) than the bound in Theorem 10.

Remark 8. Notice that the bound in Theorem 11 is strictly better than the Oleĭnik-Petrovskiĭ-Thom-Milnor bound (Theorem 1) for all values of ℓ, d and k, with $d,k > 1$, with equality in the case $d=1$ or $k=1$. Assuming that $d,k>1$, we have that

$$\frac{1}{2}(1+(2d-1)^k) < d(2d-1)^{k-1}$$

since

$$1+(2d-1)^k < (2d-1)^{k-1}+(2d-1)^k = 2d(2d-1)^{k-1}.$$

Remark 9. Even though the results are incomparable, it is still interesting to note that using earlier results of Adolphson and Sperber [1] (who used methods involving bounding exponential sums), (N. M.) Katz [38] proved a bound of

$$6 \cdot 2^r \cdot (2 + (1 + rd))^{k+1} \tag{2.3}$$

on $\sum_{i \geq 0} \dim_{\mathbb{Q}_\ell} \mathrm{H}^i_c(V, \mathbb{Q}_\ell)$ (here $\mathrm{H}^*_c(V, \mathbb{Q}_\ell)$ denotes the ℓ-adic cohomology groups with compact support), where $V \subset \mathrm{C}^k$ is an affine variety defined by r polynomials in $\mathrm{C}[X_1, \ldots, X_k]$ of total degrees bounded by d. While this result is incomparable with the results proved in this paper, and cannot be derived using our methods, notice that the bound in (2.3) has an exponent of $k+1$ which is worse than the bound in Theorem 11 (in the case ℓ, k are fixed and d is large).

2.2. **Betti numbers of sets defined by polynomials of bounded multi-degrees.** We now consider the multi-degree case.

Notation 3. Given, $\mathbf{k} = (k_1, \ldots, k_p), \mathbf{d} = (d_1, \ldots, d_p) \in \mathbb{N}^p$, and $j > 0$, we denote by $k = \sum_{i=1}^p k_i$ and

$$G_{\mathrm{gen}}(\mathbf{d}, \mathbf{k}, j) = 1 + (-1)^{k-j+1} + (k-j+2)^2 \binom{k}{j-1} \binom{k}{\mathbf{k}}^{-1} \frac{(1+p)^{3k-j+1}}{p(p+2)} d_1^{k_1} \cdots d_p^{k_p}.$$

Theorem 12. *Let $\mathcal{Q} = \{Q_1, \ldots, Q_\ell\} \subset \mathrm{R}[\mathbf{X}^{(1)}, \ldots, \mathbf{X}^{(p)}]$ be a finite set of polynomials with $\ell > 0$, where for $1 \leq i \leq p$, $\mathbf{X}^{(i)} = (X_1^{(i)}, \ldots, X_{k_i}^{(i)})$, and $\deg_{\mathbf{X}^{(i)}}(Q) \leq d_i$, $d_i \geq 2$, for all $Q \in \mathcal{Q}$. Let also $V = \mathrm{Zer}(\mathcal{Q}, \mathrm{R}^k)$, where $k = \sum_{i=1}^p k_i$. Denote by $\mathbf{d} = (d_1, \ldots, d_p)$ and $\mathbf{k} = (k_1, \ldots, k_p)$. Then,*

$$\begin{aligned} b(V, \mathbb{Z}_2) &\leq G_{\min}(\mathbf{d}, \mathbf{k}, \ell) \\ &\leq O(1)^k p^{3k} d_1^{k_1} \cdots d_p^{k_p}, \end{aligned}$$

where $G_{\min}(\mathbf{d}, \mathbf{k}, \ell)$ equals

$$\min\left(3 + \sum_{j=1}^{k} \binom{\ell}{j} 2^j (G_{\mathrm{gen}}(\mathbf{d}', \mathbf{k}, j) + G_{\mathrm{gen}}(\mathbf{d}', \mathbf{k}, j+1)),\ \frac{1}{2} G_{\mathrm{gen}}(2\mathbf{d}, \mathbf{k}, 1)\right),$$

$\mathbf{d}' = (d'_1, \ldots, d'_p)$, and for $1 \leq i \leq p$, d'_i is the least even integer $\geq d_i$.

Theorem 13. *Let $\mathcal{P} = \{P_1, \ldots, P_s\} \subset \mathrm{R}[\mathbf{X}^{(1)}, \ldots, \mathbf{X}^{(p)}]$ be a finite set of polynomials with $s > 0$, where for $1 \leq i \leq p$, $\mathbf{X}^{(i)} = (X_1^{(i)}, \ldots, X_{k_i}^{(i)})$, and $\deg_{\mathbf{X}^{(i)}}(P) \leq d_i$, $d_i \geq 2$, for all $P \in \mathcal{P}$. Denote by $\mathbf{d} = (d_1, \ldots, d_p)$ and $\mathbf{k} = (k_1, \ldots, k_p)$. Then, for each $i, 0 \leq i \leq k-1$,*

$$\begin{aligned} \sum_{\sigma \in \{0,1,-1\}^{\mathcal{P}}} b_i(\mathrm{Reali}(\sigma, \mathrm{R}^k), \mathbb{Z}_2) &\leq \sum_{j=1}^{k-i} \binom{s}{j} 4^j G_{\min}(\mathbf{d}, \mathbf{k}, j) \\ &\leq O(1)^k s^{k-i} p^{3k} d_1^{k_1} \cdots d_p^{k_p}. \end{aligned}$$

Furthermore, if S is any $\mathcal{P}$-closed semi-algebraic set, then

$$\begin{aligned} b(S, \mathbb{Z}_2) &\leq \sum_{i=0}^{k} \sum_{j=1}^{k-i} \binom{s+1}{j} 6^j G_{\min}(\mathbf{d}, \mathbf{k}, j) \\ &\leq O(1)^k s^k p^{3k} d_1^{k_1} \cdots d_p^{k_p}. \end{aligned}$$

2.3. **Betti numbers of semi-algebraic sets defined by polynomials with different multi-degrees.** We now consider the case when different polynomials are allowed to have different multi-degrees.

Notation 4. For a matrix $\mathbf{d} \in \mathbb{Z}^{\ell \times k}$ and $I \subset [1,\dots \ell], J \subset [1,k]$, denote by $\mathbf{d}_{I,J}$ the sub-matrix extracted from $\mathbf{d}$ by taking the rows indexed by I and columns indexed by J. We denote

$$(2.4) \quad K_{\mathrm{gen}}(\mathbf{d}) = \left(\sum_{j=\ell}^{k} \sum_{J \in \binom{[1,k]}{j}} (-1)^{k-j} \sum_{\substack{\boldsymbol{\alpha}=(\alpha_1,\dots,\alpha_\ell)\in\mathbb{Z}^\ell_{>0} \\ \alpha_1+\cdots+\alpha_\ell=j}} N(\mathbf{d}_{[1,\ell],J}, \boldsymbol{\alpha}) \right),$$

where the function N is defined in Eqn. (3.5).

Theorem 14. *Let* $\mathbf{d} = \mathbb{Z}^{\ell\times k}_{\geq 2}$. *Let for* $1 \leq i \leq \ell$, $B_i = [0, d_{i,1}] \times \cdots \times [0, d_{i,k}] \subset \mathbb{Z}^k$. *Let* $\mathcal{Q} = \{Q_1,\dots,Q_\ell\} \subset \mathrm{R}[X_1,\dots,X_k]$, *with* $\mathrm{supp}(Q_i) \subset B_i, 1 \leq i \leq \ell$, *and let* $V = \mathrm{Zer}(\mathcal{Q}, \mathrm{R}^k)$.

Then,

$$b(V, \mathbb{Z}_2) \leq K(\mathbf{d}),$$

where

$$K(\mathbf{d}) = 3 + \sum_{i=1}^{k} \sum_{I \subset [1,\ell], \mathrm{card}(I)=i} 2^{i+1} K_{\mathrm{gen}}(\mathbf{d}''_{I,[1,k]}),$$

$$\mathbf{d}'' = \begin{bmatrix} \mathbf{d}' \\ \mathbf{d}' \end{bmatrix},$$

and $\mathbf{d}' = [d'_{i,j}]_{\substack{1\leq i\leq \ell \\ 1 \leq j \leq k}}$, *with* $d'_{i,j}$ *the least even number* $\geq d_{i,j}$ *for* $1 \leq i \leq \ell, 1 \leq j \leq k$.

Theorem 15. *Let* $\mathbf{d} = \mathbb{Z}^{\ell\times k}_{\geq 2}$. *Let for* $1 \leq i \leq \ell$, $B_i = [0, d_{i,1}] \times \cdots \times [0, d_{i,k}] \subset \mathbb{Z}^k$. *Let* $\mathcal{P} = \{P_1,\dots,P_s\} \subset \mathrm{R}[X_1,\dots,X_k]$, *with* $\mathrm{supp}(P_i) \subset B_i, 1 \leq i \leq s$, $s > 0$.

Then, for each $i, 0 \leq i \leq k-1$,

$$\sum_{\sigma \in \{0,1,-1\}^{\mathcal{P}}} b_i(\mathrm{Reali}(\sigma, \mathrm{R}^k), \mathbb{Z}_2) \leq \sum_{j=1}^{k-i} \binom{s}{j} 4^j K(\mathbf{d}).$$

Furthermore, if S *is any* $\mathcal{P}$*-closed semi-algebraic set, then*

$$b(S, \mathbb{Z}_2) \leq \sum_{i=0}^{k} \sum_{j=1}^{k-i} \binom{s+1}{j} 6^j K(\mathbf{d}).$$

Example 2. We give here an example in which Theorem 14 can be applied. Let $\mathcal{Q} = \{Q_1,\dots,Q_\ell\} \subset \mathrm{R}[X_1,\dots,X_k]$ with $\ell \leq k$. Suppose that for each $i, 1 \leq i \leq \ell$, $\deg_{X_i}(Q_i) \leq d_i, \deg_{X_j}(Q_i) = O(1), j \neq i$. Moreover, assume that $d_1 \leq d_2 \leq \cdots \leq d_\ell$. (Such systems are not as unnatural as it might seem at first glance. In fact, polynomials having a similar degree structure, i.e. with supports which are contained in parallelepipeds that are long in one direction and short in the others, play an important role in the proof of Theorem 38 in Section 7 below.)

Then using Theorem 14, one obtains immediately that

$$b(\mathrm{Zer}(\mathcal{Q},\mathrm{R}^k),\mathbb{Z}_2) \leq O(1)^k \left(\sum_{\substack{\boldsymbol{\alpha}\in\mathbb{Z}^\ell_{>0} \\ \alpha_1+\cdots+\alpha_\ell=k}} \mathrm{Cont}(\ell,k,\boldsymbol{\alpha},\mathbf{1}) \right) d_1\cdots d_\ell^{k-\ell+1}, \tag{2.5}$$

where for any $m,n>0$, and $\mathbf{r}=(r_1,\ldots,r_m)\in\mathbb{Z}^m_{>0}$, $\mathbf{c}=(c_1,\ldots,c_n)\in\mathbb{Z}^n_{>0}$, $\sum_i r_i = \sum_j c_j$, $\mathrm{Cont}(m,n,\mathbf{r},\mathbf{c})$ denotes the number of matrices in $\mathbb{Z}^{m\times n}_{\geq 0}$ with the vector of row-sums equal to $\mathbf{r}$, and the vector of column-sums equal to $\mathbf{c}$ (such matrices are often referred to as *contingency tables*). Note that the quantity

$$\left(\sum_{\substack{\boldsymbol{\alpha}\in\mathbb{Z}^\ell_{>0} \\ \alpha_1+\cdots+\alpha_\ell=k}} \mathrm{Cont}(\ell,k,\boldsymbol{\alpha},\mathbf{1}) \right)$$

appearing in (2.5) depends only on k and ℓ, is independent of the d_i's, and is bounded by $2^{O(k^2)}$ using results in [11] on the asymptotic number of contingency tables.

Notice that the dependence on the various degrees d_i in the bound above is similar to the bound proved in [8] on the number of semi-algebraically connected components of a real variety defined by polynomials of increasing *total* degrees, with some added assumptions on the dimensions of the intermediate varieties defined by some of the subsets of the polynomials. There are no dimensional restrictions for the bound in (2.5) to hold, and moreover the inequality in (2.5) gives a bound on the sum of all Betti numbers not just on the zero-th one. However, the degree restrictions in the assumption for (2.5) is much stronger than just requiring that for each i, the total degree of the polynomial Q_i is bounded by $d_i + O(1)$ which would suffice for the result in [8] to hold.

Finally, note that using Alexander duality, the bound in (2.5) is also an upper bound on $b(\mathrm{R}^k \setminus \mathrm{Zer}(\mathcal{Q},\mathrm{R}^k),\mathbb{Z}_2)$.

2.4. **Betti numbers of sets defined by quadratic and partially quadratic polynomials.** In the following theorems we improve the result in Theorem 8. In Theorems 16 and 18, we assume that the set $\mathcal{P}_1$ is empty, and we are able to provide more precise bounds in this situation. In Theorem 19 the hypothesis is the same as in Theorem 8, and we improve the bound in Theorem 8 a significant way – namely the dependence of the bound on m.

We first introduce the following notation.

Notation 5. In the following theorems, we will denote by $k = k_1 + k_2$ and

$$H_{\mathrm{gen}}(d,k_1,k_2,j) = 2 + (-1)^{k-j+1} + j2^j(k_1+k_2)^{j-1}\left(2d(k_1+k_2)+1\right)^{k_1}.$$

Remark 10. Notice that for $j, k_1 < k_2$,

$$H_{\mathrm{gen}}(d,k_1,k_2,j) \leq (O(k_2))^{j-1}(O(dk_2))^{k_1}.$$

Theorem 16. *Let* $\mathcal{Q} = \{Q_1,\ldots,Q_\ell\} \subset \mathrm{R}[X_1,\ldots,X_{k_1},Y_1,\ldots,Y_{k_2}]$ *be a finite set of polynomials with* $\ell>0$, $\deg_{\mathbf{X}}(Q)\leq d$, $d\geq 2$, *and* $\deg_{\mathbf{Y}}(Q)\leq 2$ *for all* $Q\in\mathcal{Q}$. *Let* V *denote* $\mathrm{Zer}(\mathcal{Q},\mathrm{R}^k)$. *Then,*

$$b(V,\mathbb{Z}_2) \quad \leq \quad H(d,k_1,k_2,\ell),$$

where

$$H(d,k_1,k_2,\ell) = 3 + \sum_{j=1}^{k} \binom{\ell}{j} 2^j (H_{\mathrm{gen}}(d',k_1,k_2,j) + H_{\mathrm{gen}}(d',k_1,k_2,j+1)), \tag{2.6}$$

where d' is the least even integer $\geq d$. In particular, for $\ell, k_1 \leq k_2$,

$$b(V,\mathbb{Z}_2) \quad \leq \quad (O(k_2))^{\ell+k_1} d^{k_1}. \tag{2.7}$$

Remark 11. Notice that in the case $k_1 = 0$ (thus the polynomials in $\mathcal{Q}$ are *fully* quadratic), the bound in inequality (2.7) reduces to $(O(k_2))^{\ell}$ almost recovering (i.e. up to a factor k) the bound in Theorem 5 (albeit for affine varieties).

Remark 12. It might also be possible with more work (using the same ideas as in the proof of Theorem 25 taking into account signs) to remove a factor of k_2 from the bound in Theorem 16 (cf. Remark 24), and we leave this as an open question.

For projective varieties in $\mathbb{P}^k_{\mathrm{R}}$ defined by a fixed number of homogeneous quadratic polynomials we have the following bound that is asymptotically a slight improvement over the tightest bound known previously [40, Theorem 15] (namely, the bound $(O(k))^{\ell-1}$).

Theorem 17. *For each fixed $\ell > 0$, and for each set $\mathcal{P} \subset \mathrm{R}[X_0,\ldots,X_k]$ of homogeneous polynomials of degree 2 of $\mathrm{card}(\mathcal{P}) \leq \ell$,*

$$\begin{aligned} b(\mathrm{Zer}(\mathcal{P},\mathbb{P}^k_{\mathrm{R}}) &\leq 3 + \sum_{i=1}^{k} \binom{\ell}{i} 2^i H'_{\mathrm{gen}}(k,i) \\ &\leq \left(O\left(\frac{k}{\ell}\right)\right)^{\ell-1}, \end{aligned} \tag{2.8}$$

where

$$H'_{\mathrm{gen}}(k,i) = (1+(-1)^{k-i+1})(k-i+1) + \\ (-1)^{k-i}\left(\sum_{h=0}^{i-1}(-2)^h \left(\sum_{j=i}^{k}(-1)^{j+1}\binom{j}{h}\right) + (k-i+1)\right).$$

Theorem 18. *Let $\mathcal{P} = \{P_1,\ldots,P_s\} \subset \mathrm{R}[X_1,\ldots,X_{k_1},Y_1,\ldots,Y_{k_2}]$ be a finite set of polynomials with $s > 0$, $\deg_{\mathbf{X}}(P) \leq d$, $d \geq 2$, and $\deg_{\mathbf{Y}}(P) \leq 2$ for all $P \in \mathcal{P}$.*

Then, for each i, $0 \leq i \leq k-1$,

$$\begin{aligned} \sum_{\sigma \in \{0,1,-1\}^{\mathcal{P}}} b_i(\mathrm{Reali}(\sigma,\mathrm{R}^k),\mathbb{Z}_2) &\leq \sum_{j=1}^{k-i} \binom{s}{j} 4^j H(d',k_1,k_2,j) \\ &\leq (O(k_2))^{s+k_1} d^{k_1} \text{ for } s,k_1 < k_2. \end{aligned} \tag{2.9}$$

where d' is the least even integer $\geq d$. Furthermore, if S is any $\mathcal{P}$-closed semi-algebraic set, then

$$\begin{aligned} b(S,\mathbb{Z}_2) &\leq \sum_{i=0}^{k}\sum_{j=1}^{k-i} \binom{s+1}{j} 6^j H(d',k_1,k_2,j) \\ &\leq (k_1+k_2+1)(O(k_2))^{s+k_1+1} d^{k_1}, \text{ for } k_1 < k_2, \\ &= (O(k_2))^{s+k_1+2} d^{k_1}. \end{aligned} \tag{2.10}$$

where d' is the least even integer $\geq d$.

Remark 13. Notice that the bound in inequality (2.10) in Theorem 18 is significantly better than the previous best bound known on this quantity (namely, inequality (1.5) in Remark 4).

Theorem 19. *With the same notation as in Theorem 8, for each i, $0 \leq i \leq k-1$ and assuming $m \leq k_2$, $\sum_{\sigma \in \{0,1,-1\}^{\mathcal{P}_1 \cup \mathcal{P}_2}} b_i(\mathrm{Reali}(\sigma, \mathrm{R}^k), \mathbb{Z}_2)$ is bounded by*

$$\sum_{j=1}^{k-i} \sum_{\substack{0 \leq j_1 \leq \min(s,k_1) \\ 0 \leq j_2 \leq \min(m+1, k_1+k_2-j_1) \\ j_1+j_2=j}} \binom{s}{j_1}\binom{m+1}{j_2} 5^j H(2d, k_1, k_2, j_2+1)$$

$$\leq \quad (O(k_2))^{k_1+m+2}(O(sd))^{k_1}, \tag{2.11}$$

for $m, k_1 < k_2$.

Furthermore, if S is any $(\mathcal{P}_1 \cup \mathcal{P}_2)$-closed semi-algebraic set, then

$$b(S, \mathbb{Z}_2) \leq \sum_{i=0}^{k} \sum_{\substack{0 \leq j_1 \leq \min(s,k_1) \\ 0 \leq j_2 \leq \min(m+1, k_1+k_2-j_1) \\ j_1+j_2=j \leq k-i}} \binom{s}{j_1}\binom{m+1}{j_2} 7^j H(2d, k_1, k_2, j_2+1)$$

$$\leq \quad (O(k_2))^{k_1+m+3}(O(sd))^{k_1}, \text{ for } m, k_1 < k_2. \tag{2.12}$$

Remark 14. Notice that the bound in inequalities (2.11) and (2.12) in Theorem 19 is significantly better than the corresponding bounds in Theorem 8 (namely, in the dependence on m and the exponent of k_2).

2.5. **Betti numbers of semi-algebraic sets defined by partially quadratic polynomials with several blocks of variables.** Lastly, we consider the case of partially quadratic polynomials, with the non-quadratically bounded variables allowed to have different degrees.

Notation 6. In the following theorems, we will denote by $k = k_1 + k_2$, $\mathbf{d} = (d_1, \ldots, d_{k_1}) \in \mathbb{N}^{k_1}$ and

$$M_{\mathrm{gen}}(\mathbf{d}, k_1, k_2, j) = 2 + (-1)^{k-j+1} + j2^j k_1!(k_1+k_2)^{j-1}\left(2(k_1+k_2)+1\right)^{k_1} d_1 \cdots d_{k_1}.$$

Theorem 20. *Let $\mathcal{Q} = \{Q_1, \ldots, Q_\ell\} \subset \mathrm{R}[X_1, \ldots, X_{k_1}, Y_1, \ldots, Y_{k_2}]$ be a finite set of polynomials with $\ell > 0$, $\deg_{X_i}(Q) \leq d_i$, $d_i \geq 2$, and $\deg_{\mathbf{Y}}(Q) \leq 2$ for all $Q \in \mathcal{Q}$. Let V denote $\mathrm{Zer}(\mathcal{Q}, \mathrm{R}^k)$. Then,*

$$b(V, \mathbb{Z}_2) \leq M(\mathbf{d}, k_1, k_2, \ell)$$

where

$$M(\mathbf{d}, k_1, k_2, \ell) = 3 + \sum_{j=1}^{k} \binom{\ell}{j} 2^j (M_{\mathrm{gen}}(\mathbf{d}', k_1, k_2, j) + M_{\mathrm{gen}}(\mathbf{d}', k_1, k_2, j+1)),$$

and where $\mathbf{d}' = (d'_1, \ldots, d'_{k_1})$ and for $1 \leq i \leq k_1$, d'_i is the least even integer $\geq d_i$.

In particular, for $\ell, k_1 \leq k_2$,

$$b(V, \mathbb{Z}_2) \leq (O(k_2))^{\ell+k_1} d_1 \cdots d_{k_1}.$$

Theorem 21. *Let $\mathcal{P} = \{P_1, \ldots, P_s\} \subset \mathrm{R}[X_1, \ldots, X_{k_1}, Y_1, \ldots, Y_{k_2}]$ be a finite set of polynomials with $s > 0$, $\deg_{X_i}(P) \leq d_i$, $d_i \geq 2$, and $\deg_{\mathbf{Y}}(P) \leq 2$ for all $P \in \mathcal{P}$. Then, for each i, $0 \leq i \leq k-1$,*

$$\begin{aligned}\sum_{\sigma \in \{0,1,-1\}^{\mathcal{P}}} b_i(\mathrm{Reali}(\sigma, \mathrm{R}^k), \mathbb{Z}_2) &\leq \sum_{j=1}^{k-i} \binom{s}{j} 4^j M(\mathbf{d}', k_1, k_2, j) \\ &\leq (O(k_2))^{s+k_1} d_1 \cdots d_{k_1} \text{ for } s, k_1 < k_2,\end{aligned}$$

where $\mathbf{d}' = (d'_1, \ldots, d'_{k_1})$ and for $1 \leq i \leq k_1$, d'_i is the least even integer $\geq d_i$. Furthermore, if S is any $\mathcal{P}$-closed semi-algebraic set, then

$$\begin{aligned}b(S, \mathbb{Z}_2) &\leq \sum_{i=0}^{k} \sum_{j=1}^{k-i} \binom{s+1}{j} 6^j M(\mathbf{d}', k_1, k_2, j) \\ &\leq (O(k_2))^{s+k_1+2} d_1 \cdots d_{k_1}, \text{ for } s, k_1 < k_2,\end{aligned}$$

where $\mathbf{d}' = (d'_1, \ldots, d'_{k_1})$ and for $1 \leq i \leq k_1$, d'_i is the least even integer $\geq d_i$.

3. Preliminaries

We first recall some preliminary results that we will need in the paper.

3.1. Real algebraic preliminaries.

Notation 7. For R a real closed field we denote by $\mathrm{R}\langle\varepsilon\rangle$ the real closed field of algebraic Puiseux series in ε with coefficients in R. We use the notation $\mathrm{R}\langle\varepsilon_1, \ldots, \varepsilon_m\rangle$ to denote the real closed field $\mathrm{R}\langle\varepsilon_1\rangle\langle\varepsilon_2\rangle \cdots \langle\varepsilon_m\rangle$. Note that in the unique ordering of the field $\mathrm{R}\langle\varepsilon_1, \ldots, \varepsilon_m\rangle$, $0 < \varepsilon_m \ll \varepsilon_{m-1} \ll \cdots \ll \varepsilon_1 \ll 1$.

Notation 8. For elements $x \in \mathrm{R}\langle\varepsilon\rangle$ which are bounded over R we denote by $\lim_\varepsilon x$ to be the image in R under the usual map that sets ε to 0 in the Puiseux series x.

Notation 9. If R' is a real closed extension of a real closed field R, and $S \subset \mathrm{R}^k$ is a semi-algebraic set defined by a first-order formula with coefficients in R, then we will denote by $\mathrm{Ext}(S, \mathrm{R}') \subset \mathrm{R}'^k$ the semi-algebraic subset of R'^k defined by the same formula. It is well-known that $\mathrm{Ext}(S, \mathrm{R}')$ does not depend on the choice of the formula defining S [17].

Notation 10. For $x \in \mathrm{R}^k$ and $r \in \mathrm{R}$, $r > 0$, we will denote by $B_k(x, r)$ the open Euclidean ball centered at x of radius r. If R' is a real closed extension of the real closed field R and when the context is clear, we will continue to denote by $B_k(x, r)$ the extension $\mathrm{Ext}(B_k(x, r), \mathrm{R}')$. This should not cause any confusion. We also denote by $\mathbf{S}^{k-1}(x, r)$ the $(k-1)$-dimensional sphere in R^k, centered at x and of radius r.

3.2. Topological preliminaries.

3.2.1. *Mayer-Vietoris inequalities.* Let S_1, S_2 be two closed semi-algebraic sets, and $\mathbb{F}$ any field of coefficients. We will use heavily the following inequalities which are consequences of the exactness of the Mayer-Vietoris sequence.

$$b_i(S_1 \cup S_2, \mathbb{F}) \leq b_i(S_1, \mathbb{F}) + b_i(S_2, \mathbb{F}) + b_{i-1}(S_1 \cap S_2, \mathbb{F}), \tag{3.1}$$

$$b_i(S_1 \cap S_2, \mathbb{F}) \leq b_i(S_1, \mathbb{F}) + b_i(S_2, \mathbb{F}) + b_{i+1}(S_1 \cup S_2, \mathbb{F}), \tag{3.2}$$

$$b_i(S_1, \mathbb{F}) + b_i(S_2, \mathbb{F}) \leq b_i(S_1 \cup S_2, \mathbb{F}) + b_i(S_1 \cap S_2, \mathbb{F}). \tag{3.3}$$

The following generalization in the case of more than two sets will also be useful for us (see for example [17, Proposition 7.33]).

Proposition 1. *Let $S_1,\ldots,S_s \subset \mathrm{R}^k$, $s \geq 1$, be closed semi-algebraic sets contained in a closed semi-algebraic set T of dimension k'. For $S_{\leq t} = \bigcap_{1\leq j\leq t} S_j$, and $S^{\leq t} = \bigcup_{1\leq j\leq t} S_j$. Also, for $J \subset \{1,\ldots,s\}$, $J \neq \emptyset$, let $S_J = \bigcap_{j\in J} S_j$, and $S^J = \bigcup_{j\in J} S_j$. Finally, let $S^{\emptyset} = T$. Then*

(A) For $0 \leq i \leq k'$,

$$b_i(S^{\leq s},\mathbb{F}) \leq \sum_{j=1}^{i+1} \sum_{\substack{J\subset\{1,\ldots,s\}\\ \mathrm{card}(J)=j}} b_{i-j+1}(S_J,\mathbb{F}).$$

(B) For $0 \leq i \leq k'$,

$$b_i(S_{\leq s},\mathbb{F}) \leq \sum_{j=1}^{k'-i} \sum_{\substack{J\subset\{1,\ldots,s\}\\ \mathrm{card}(J)=j}} b_{i+j-1}(S^J,\mathbb{F}) + \binom{s}{k'-i} b_{k'}(S^{\emptyset},\mathbb{F}).$$

3.2.2. *Smith inequality.* Let X be a compact space (or a regular complex) equipped with an involution map $c : X \to S$. Let $\mathrm{Fix}(c) \subset X$ denote the subspace of fixed points of X. The Smith exact sequence (see for example [22, page 126]) then implies that

$$b(\mathrm{Fix}(c),\mathbb{Z}_2) \quad\leq\quad b(X,\mathbb{Z}_2). \tag{3.4}$$

Taking the involution c to be the complex conjugation we obtain:

Theorem 22 (Smith inequality for affine sub-varieties of C^k defined over R)**.** *Let $\mathcal{Q} \subset \mathbb{R}[X_1,\ldots,X_k]$ be a finite set of polynomials. Then,*

$$b(\mathrm{Zer}(\mathcal{Q},\mathrm{R}^k),\mathbb{Z}_2) \quad\leq\quad b(\mathrm{Zer}(\mathcal{Q},\mathrm{C}^k),\mathbb{Z}_2).$$

Remark 15. Even though the Smith exact sequence from which the Smith inequality is derived is usually stated (see (see for example [22, page 126]) for regular complexes (and thus compact spaces) X, it has been extended to non-compact but *finitistic* (see [22, page 133] for definition) spaces using Čech cohomology. We avoid this complication by giving a direct reduction to the closed and bounded case for affine sub-varieties of C^k defined over R, the only case that is of interest to us in this paper.

Proof of Theorem 22. See Appendix. □

3.2.3. *Descent spectral sequence.* The following theorem proved in [28] allows one to bound the Betti numbers of the image of a closed and bounded semi-algebraic set S under a polynomial map $\mathbf{F}$ in terms of the Betti numbers of the iterated fibered product of S over $\mathbf{F}$. More precisely:

Theorem 23. [28] *Let $S \subset \mathrm{R}^k$ be a closed and bounded semi-algebraic set, and $\mathbf{F} = (F_1,\ldots,F_m) : \mathrm{R}^k \to \mathrm{R}^m$ be a polynomial map. For for all $p, 0 \leq p \leq m$,*

$$b_p(\mathbf{F}(S),\mathbb{Z}_2) \leq \sum_{\substack{i,j\geq 0\\ i+j=p}} b_i(\underbrace{S\times_{\mathbf{F}}\cdots\times_{\mathbf{F}} S}_{(j+1)},\mathbb{Z}_2).$$

3.3. **Mixed volume.** Mixed volumes of (Newton) polytopes of polynomials play a very important role in the role of toric varieties, and will play an important role in this paper (see Theorem 24 below). We recall here the definition and certain elementary property of mixed volumes that we will need later in the paper referring the reader to [46, Section A.4] for any missing detail.

3.3.1. *Definition of mixed volume.*

Definition 1 (Mixed volume)**.** Given compact, convex sets $K_1, \ldots, K_m \subset \mathbb{R}^m$, and $\lambda_1, \ldots, \lambda_m \geq 0$, $\lambda_1 K_1 + \cdots + \lambda_m K_m$ is also a compact, convex subset of $\mathbb{R}^m$, and $\frac{1}{m!}\mathrm{vol}_m(\lambda_1 K_1 + \cdots + \lambda_m K_m)$ is given by a polynomial in $\lambda_1, \ldots, \lambda_m$. The coefficient of $\lambda_1 \cdots \lambda_m$ in the polynomial $\frac{1}{m!}\mathrm{vol}_m(\lambda_1 K_1 + \cdots + \lambda_m K_m)$ is called the *mixed volume* of $K_1, \ldots, K_m$ and denoted by $\mathrm{MV}(K_1, \ldots, K_m)$.

We will use a few basic properties of mixed volume that we list below (see [46, Section A.4] for an exposition).

(A) (Linearity)

$$\mathrm{MV}(K_1, \ldots, K_{i-1}, \lambda' K_i' + \lambda'' K_i'', K_{i+1}, \ldots, K_m) = \\ \lambda' \mathrm{MV}(K_1, \ldots, K_i', \ldots, K_m) + \lambda'' \mathrm{MV}(K_1, \ldots, K_i'', \ldots, K_m).$$

(B) (Monotonicity) $K_i' \subset K_i''$ implies that

$$\mathrm{MV}(K_1, \ldots, K_{i-1}, K_i', K_{i+1}, \ldots, K_m) \leq \mathrm{MV}(K_1, \ldots, K_{i-1}, K_i'', K_{i+1}, \ldots, K_m).$$

(C) If $K_1 = \cdots = K_m = K$, then

$$\mathrm{MV}(K_1, \ldots, K_m) = \mathrm{vol}_m(K).$$

Since in many of our applications we will be interesting in obtaining upper bounds on mixed-volumes of certain special polytopes – namely products of simplices or boxes, the following simple consequences of Properties (A), (B) and (C) will be useful.

Lemma 1. *For $a_i \geq 0, 1 \leq i \leq m$, let*

$$K_i = \underbrace{\{0\} \times \{0\}}_{i-1} \times [0, a_i] \times \underbrace{\{0\} \times \{0\}}_{m-i-1}.$$

Then,

$$\mathrm{MV}(K_1, \ldots, K_m) = \frac{a_1 \cdots a_m}{m!}.$$

Proof. First observe that

$$\begin{aligned} \frac{1}{m!}\mathrm{vol}_m(\lambda_1 K_1 + \cdots + \lambda_m K_m) &= \frac{1}{m!}\mathrm{vol}_m([0, \lambda_1 a_1] \times \cdots \times [0, \lambda_m a_m]) \\ &= \left(\frac{a_1 \cdots a_m}{m!}\right) \lambda_1 \cdots \lambda_m. \end{aligned}$$

It then follows from Definition 1 that

$$\mathrm{MV}(K_1, \ldots, K_m) = \frac{a_1 \cdots a_m}{m!}.$$

□

Proposition 2. *Let* $B_1, \ldots, B_\ell \subset \mathbb{R}^k, 1 \leq \ell \leq k$, *and for* $1 \leq i \leq \ell$, $B_i = [0, d_{i,1}] \times \cdots \times [0, d_{i,k}]$. *We denote by* $\mathbf{d} = (d_{i,j})_{\substack{1 \leq i \leq \ell \\ 1 \leq j \leq k}}$. *Let* $\alpha_1, \ldots, \alpha_\ell \in \mathbb{Z}_{>0}$, *with* $\sum_{i=1}^{\ell} \alpha_i = k$, *and denote* $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_\ell)$. *Let*

$$N(\mathbf{d}, \boldsymbol{\alpha}) = \mathrm{MV}(\underbrace{B_1, \ldots, B_1}_{\alpha_1}, \ldots, \underbrace{B_\ell, \ldots, B_\ell}_{\alpha_\ell}).$$

Then,

$$N(\mathbf{d}, \boldsymbol{\alpha}) = \sum_{\substack{\mathbf{A} = (a_{ij}) \in \{0,1\}^{\ell \times k} \\ \sum_{1 \leq j \leq k} a_{ij} = \alpha_i, 1 \leq i \leq \ell \\ \sum_{1 \leq i \leq \ell} a_{ij} = 1, 1 \leq j \leq k}} \mathbf{d}^{\mathbf{A}}. \tag{3.5}$$

Denoting by $\binom{[1,k]}{\boldsymbol{\alpha}}$ *the set of all partitions of* $[1, k]$ *into disjoint subsets* $J_1, \ldots, J_\ell$ *with* $\mathrm{card}(J_i) = \alpha_i, 1 \leq i \leq \ell$,

$$N(\mathbf{d}, \boldsymbol{\alpha}) \leq \max_{(J_1, \ldots, J_\ell) \in \binom{[1,k]}{\boldsymbol{\alpha}}} \left(\prod_{\substack{1 \leq i \leq \ell \\ j \in J_i}} d_{i,j} \right). \tag{3.6}$$

In the special case, when for $1 \leq i \leq \ell, 1 \leq j \leq k$, $d_{i,j} = d_i$,

$$N(\mathbf{d}, \boldsymbol{\alpha}) = d_1^{\alpha_1} \cdots d_\ell^{\alpha_\ell}. \tag{3.7}$$

Proof. Eqns. (3.5) and (3.6) follow from Lemma 1 and properties (A), (B), and (C) of mixed volumes stated previously. Eqn. (3.7) is a simple consequence of properties (A), and (C) of mixed volumes stated earlier. □

Corollary 1. *With the same notation as in Proposition 2, denoting for* $1 \leq i \leq \ell$,

$$d_i = \max_{J \subset [1,k], \mathrm{card}(J) = \alpha_i} \prod_{j \in J} d_{i,j},$$

$$N(\mathbf{d}, \boldsymbol{\alpha}) \leq d_1^{\alpha_1} \cdots d_\ell^{\alpha_\ell}. \tag{3.8}$$

Proof. Immediate from Eqn. (3.5) in Proposition 2. □

3.4. **Topology of complex varieties.**

3.4.1. *Euler-Poincaré characteristics of generic intersections in* C^n. In this section we recall a fundamental result due to Khovanskiĭ [35] that we will exploit heavily later in the paper. This result in conjunction with Theorem 22 (Smith inequality) allows us to bound the Betti numbers of generic algebraic varieties in R^k in terms of the Newton polytopes of the defining polynomials (under a weak hypothesis on the Newton polytopes stated in Property 1 below).

Before recalling Khovanskiĭ's result we first introduce some more notation.

Notation 11. Let $\mathbf{k}$ be any field. For $P = \sum_{\boldsymbol{\alpha} \in \mathbb{N}^k} c_{\boldsymbol{\alpha}} \mathbf{X}^{\boldsymbol{\alpha}} \in \mathbf{k}[X_1, \ldots, X_k]$, we denote by $\mathrm{supp}(P) \subset \mathbb{Q}^k$ the convex hull of the set $\{\boldsymbol{\alpha} \in \mathbb{N}^k \mid c_{\boldsymbol{\alpha}} \neq 0\}$.

Property 1. Given a tuple $\boldsymbol{\Delta} = (\Delta_1, \ldots, \Delta_\ell)$, where for $i = 1, \ldots, \ell$, $\Delta_i \subset \mathbb{Q}^k$ is a convex polytope, we say that $\boldsymbol{\Delta}$ satisfies Property 1 if for each non-empty subset $L \subset [1, \ell]$, $\dim(\sum_{i \in L} \Delta_i)$ is at least $k - \ell + \mathrm{card}(L)$. We say that a tuple of polynomials $\mathcal{P} = (P_1, \ldots, P_\ell), P_i \in \mathbf{k}[X_1, \ldots, X_k]$ satisfies the same property if the tuple $\mathrm{supp}(\mathcal{P}) = (\mathrm{supp}(P_1), \ldots, \mathrm{supp}(P_\ell))$ satisfies the above property.

The following two special cases where Property 1 holds will be important later and we record this fact here.

Remark 16. Notice that if each Δ_i is a standard k-dimensional simplex in $\mathbb{R}^k$ of side length d_i (i.e. the convex hull of $\mathbf{0}, (d_i, 0 \ldots, 0), \ldots, (0, \ldots, 0, d_i)$, with $d_i > 0$), then the tuple $(\Delta_1, \ldots, \Delta_\ell)$ satisfies Property 1. The same is true if each $\Delta_i = [0, d_{i,1}] \times \cdots \times [0, d_{i,k}]$, where $\mathbf{d} = (d_{i,j})_{\substack{1 \leq i \leq \ell \\ 1 \leq j \leq k}} \in \mathbb{Z}_{>0}^{\ell \times k}$.

We first need a notation.

Notation 12. For $\boldsymbol{\alpha} \in \mathbb{Z}_{\geq 0}^p$, and polytopes $\Delta_1, \ldots, \Delta_p \subset \mathbb{R}^k$, and any monomial $M(X_1, \ldots, X_p) = \mathbf{X}^{\boldsymbol{\alpha}}$ of degree k,

$$M(\Delta_1, \ldots, \Delta_p) = k! \operatorname{MV}(\underbrace{\Delta_1, \ldots, \Delta_1}_{\alpha_1}, \ldots, \underbrace{\Delta_p, \ldots, \Delta_p}_{\alpha_p}),$$

and the definition is extended to any form $H \in \mathbb{Q}[X_1, \ldots, X_p]$ of degree k, by linearity. Finally, for any rational function $F(X_1, \ldots, X_p)$, we define $F(\Delta_1, \ldots, \Delta_p) = F_k(\Delta_1, \ldots, \Delta_p)$ where F_k is the degree k homogeneous component of the Taylor expansion of F at $\mathbf{0}$.

Theorem 24. [35] *Let $\mathcal{P} = (P_1, \ldots, P_\ell)$, where each $P_j \in \mathrm{C}[X_1, \ldots, X_k]$, and such that $\mathcal{P}$ satisfies Property 1, and the coefficients of the polynomials P_j are generic. Let $V = \operatorname{Zer}(\mathcal{P}, \mathrm{C}^k)$. Then,*

$$\chi(V) = \sum_{I \subset [1,k]} \prod_{j=1}^{\ell} \frac{\Delta_j^I}{1 + \Delta_j^I},$$

where Δ_j^I is the face of Δ_j obtained by setting $X_i = 0$ for all $i \in I$ (cf. Notation 12).

3.4.2. *Betti numbers of smooth complete intersections in* C^k. The following proposition is well known but we include a proof in the Appendix for the sake of completeness, and also because one finds usually the corresponding statement for non-singular projective varieties only in standard textbooks of algebraic geometry (see for example [55]).

Proposition 3. *Let $V \subset \mathrm{C}^k$ be either a zero-dimensional variety or a connected non-singular affine variety of dimension $k - \ell \geq 0$. Then,*

$$b(V, \mathbb{Z}_2) = 1 + (-1)^{k-\ell+1} + (-1)^{k-\ell} \chi(V, \mathbb{Z}_2). \tag{3.9}$$

Proof. See Appendix. □

A similar result also holds for non-singular projective complete intersection varieties as well, which we state without proof since this fact is very well known.

Proposition 4. *Let $V \subset \mathbb{P}_{\mathrm{C}}^k$ be a connected non-singular projective variety of dimension $k - \ell$. Then, for $k \geq 1$,*

$$b(V, \mathbb{Z}_2) \quad = \quad (1 + (-1)^{k-\ell+1})(k - \ell + 1) + (-1)^{k-\ell} \chi(V, \mathbb{Z}_2).$$

Remark 17. Note that there are well-known formulas (see for example [34, 20]) for the Betti numbers of *projective* varieties which are non-singular complete intersections, in terms of the degree sequence defining them. The formulas for the Betti numbers for the affine parts of these varieties are then easily deducible (by subtracting the part at infinity using Lefschetz duality). However, this approach is not applicable in many of the situations considered in the current paper since the generic affine intersections that we consider might necessarily be singular at infinity. For example, let $k = 3$, and $\Delta = [0,1] \times [0,1] \times [0,1]$. Then, a generic polynomial $P \in \mathrm{C}[X_1, X_2, X_3]$ with $\mathrm{supp}(P) \subset \Delta$, defines a non-singular hypersurface in C^3, but defines a singular curve in the projective plane at infinity defined by the (homogeneous) equation $X_1 X_2 X_3 = 0$ (with three singular points – namely, $(1:0:0), (0:1:0), (0:0:1)$). One could take more complicated compactifications of affine space – for example, multi-projective spaces – which would solve this problem, at the cost of increasing the complexity of the process of subtracting the added part, a process which moreover would be different in each of the cases that we consider. Because of these reasons it is convenient for us to have directly an expression for the Betti numbers of generic complex *affine* intersections – which is afforded by Theorem 24 in conjunction with Proposition 3.

We stated previously that one of our main tools that we are going to exploit heavily is Theorem 24, and this what we proceed to do in this section. We use Theorem 24 in conjunction with Proposition 3 and Theorem 22 (Smith inequalities) to obtain bounds on the sum of the ($\mathbb{Z}_2$) Betti numbers of certain generic affine complete intersection sub-varieties of C^k and R^k that are of interest to us. These include varieties defined by generic polynomials having prescribed total degrees, or multi-degrees, or with a fixed block of variables appearing at most quadratically with the remaining having prescribed degrees etc. These results are stated separately since the precise calculations and the bounds obtained in each case is different (though the main idea used to obtain these bounds is the same).

3.5. **Some applications of Khovanskiĭ's theorem.** In this section we use Theorem 24 to obtain bounds on the Betti numbers of generic affine intersections in several cases of interest to us. Since some of the calculations are long and technical, for the sake of readability, we defer the proofs of some of the propositions to the Appendix.

We begin by observing that as a special case, when $\ell = k$, Theorem 24 gives us a theorem of Bernstein and Kouchnirenko, namely:

Proposition 5. [39] *Let $\mathcal{P} = \{P_1, \ldots, P_k\} \subset \mathrm{C}[X_1, \ldots, X_k]$ be a finite set such that $\mathcal{P}$ satisfies Property 1, and the coefficients of the polynomials P_i are sufficiently generic. Then, $\mathrm{Zer}(\mathcal{P}, \mathrm{C}^k)$ is a finite set, and*

$$\mathrm{card}(\mathrm{Zer}(\mathcal{P}, \mathrm{C}^k)) = k!\, \mathrm{MV}(\Delta_1, \ldots, \Delta_k),$$

where $\Delta_i = \mathrm{supp}(P_i), 1 \leq i \leq k$. Moreover, if additionally $\mathcal{P} \subset \mathrm{R}[X_1, \ldots, X_k]$, then

$$\mathrm{card}(\mathrm{Zer}(\mathcal{P}, \mathrm{R}^k)) \leq k!\, \mathrm{MV}(\Delta_1, \ldots, \Delta_k).$$

Proof. Immediate from Theorem 24. □

Proposition 5 deals with the generic zero-dimensional case. Another result that follows immediately from Theorem 24 is the following well known expression giving the sum of the Betti numbers of a generic affine hypersurface in C^k defined by one polynomial of degree d. Note that this proposition could be also be deduced using an argument involving counting multiplicities of Milnor fibers (see for example, [24, page 152]).

Proposition 6. *Let $P \in \mathrm{C}[X_1,\ldots,X_k], k > 0$ be a generic polynomial of total degree d. Then,*

$$b(\mathrm{Zer}(P,\mathrm{C}^k),\mathbb{Z}_2) \quad = \quad 1+(d-1)^k. \tag{3.10}$$

If $P \in \mathrm{R}[X_1,\ldots,X_k]$,

$$b(\mathrm{Zer}(P,\mathrm{R}^k),\mathbb{Z}_2) \quad \leq \quad 1+(d-1)^k. \tag{3.11}$$

Proof. See Appendix. □

Remark 18. Notice that if $P \in \mathrm{R}[X_1,\ldots,X_k]$ and defines a bounded, non-singular hypersurface in R^k, then Proposition 6 gives a better bound than just counting critical points of a linear functional on $\mathrm{Zer}(P,\mathrm{R}^k)$. The latter gives a bound of $d(d-1)^{k-1} > 1+(d-1)^k$ for all $d,k>1$, since

$$\begin{aligned} d(d-1)^{k-1} - (1+(d-1)^k) &= (d-1)^{k-1}-1 \\ &> 0, \text{ for all } d,k>1. \end{aligned}$$

We now consider the case of generic affine intersections of hypersufaces in C^k defined by polynomials of possibly different degrees. The same result as in Proposition 7 could *in principle* be deduced from Hirzebruch's formula for the Euler-Poincaré characteristics of generic complex intersections in $\mathbb{P}^k_{\mathrm{C}}$, but the calculation would be much more complicated (see for example [20, 40]). We did not find the explicit expression (as opposed to being given implicitly via recurrence relations or generating functions) given in (3.12) below in the existing literature.

Proposition 7. *Let $\mathcal{P} = \{P_1,\ldots,P_\ell\} \subset \mathrm{C}[X_1,\ldots,X_k], k \geq \ell > 0$ be a set of generic polynomials with $\deg(P_i) = d_i$.*

Then,

(3.12)

$$b(\mathrm{Zer}(\mathcal{P},\mathrm{C}^k),\mathbb{Z}_2) = 1+(-1)^{k-\ell+1}+d_1\cdots d_\ell\cdot\left(\sum_{j=0}^{k-\ell}(-1)^{k+j+1}\binom{k}{j+\ell}h_j(d_1,\ldots,d_\ell)\right),$$

if $\ell < k$, and

$$b(\mathrm{Zer}(\mathcal{P},\mathrm{C}^k),\mathbb{Z}_2) = d_1\cdots d_k \tag{3.13}$$

if $\ell = k$, where we denote by

$$h_d(X_1,\ldots,X_n) = \sum_{\substack{i_1,\ldots,i_n\geq 0\\ i_1+\cdots+i_n=d}} X_1^{i_1}\cdots X_n^{i_n},$$

the complete homogeneous symmetric function of degree d.

Moreover, if $d_1 = \cdots = d_\ell = d$ then

$$b(\mathrm{Zer}(\mathcal{P},\mathrm{C}^k),\mathbb{Z}_2) \leq 1+(-1)^{k-\ell+1}+\binom{k-1}{\ell-1}(d^k+k-1) \text{ if } \ell<k, \tag{3.14}$$

$$b(\mathrm{Zer}(\mathcal{P},\mathrm{C}^k),\mathbb{Z}_2) \leq d^k \text{ if } \ell = k. \tag{3.15}$$

Additionally, if $\mathcal{P} \subset \mathrm{R}[X_1,\ldots,X_k]$, then

(3.16)

$$b(\mathrm{Zer}(\mathcal{P},\mathrm{R}^k),\mathbb{Z}_2) \leq 1+(-1)^{k-\ell+1}+d_1\cdots d_\ell\cdot\left(\sum_{j=0}^{k-\ell}(-1)^{k+j+1}\binom{k}{j+\ell}h_j(d_1,\ldots,d_\ell)\right),$$

if $\ell < k$, *and*

$$b(\mathrm{Zer}(\mathcal{P}, \mathrm{R}^k), \mathbb{Z}_2) \leq d_1 \cdots d_k, \tag{3.17}$$

if $\ell = k$.

In the case $d_1 = \cdots = d_\ell = d$,

$$b(\mathrm{Zer}(\mathcal{P}, \mathrm{R}^k), \mathbb{Z}_2) \leq 1 + (-1)^{k-\ell+1} + \binom{k-1}{\ell-1}(d^k + k - 1), \tag{3.18}$$

if $\ell < k$, *and*

$$b(\mathrm{Zer}(\mathcal{P}, \mathrm{R}^k), \mathbb{Z}_2) \leq d^k, \tag{3.19}$$

if $\ell = k$.

Proof. See Appendix. □

Remark 19. A special case of Proposition 7 will be used later, and we record it here for future use. Let $\mathcal{P} \subset \mathrm{R}[X_1, \ldots, X_k]$ be generic and equal to the disjoint union of $\mathcal{P}_1$ and $\mathcal{P}_2$, with $\mathrm{card}(\mathcal{P}_i) = \ell_i$, and the total degrees of the polynomials in $\mathcal{P}_i$ equal d_i, $i = 1, 2$. Applying Proposition 7 to this special case we obtain that $b(\mathrm{Zer}(\mathcal{P}, \mathrm{R}^k), \mathbb{Z}_2)$ is bounded by

$$\begin{aligned}
& 1 + (-1)^{k-\ell+1} + d_1^{\ell_1} d_2^{\ell_2} \cdot \left(\sum_{j=0}^{k-\ell} (-1)^{k-\ell+j} \binom{k}{j+\ell} h_j(\underbrace{d_1, \ldots, d_1}_{\ell_1}, \underbrace{d_2, \ldots, d_2}_{\ell_2}) \right) \\
= \; & 1 + (-1)^{k-\ell+1} + d_1^{\ell_1} d_2^{\ell_2} \cdot \left(\sum_{j=0}^{k-\ell} \sum_{i=0}^{j} (-1)^{k-\ell+j} \binom{k}{j+\ell} \binom{\ell+i-2}{i} d_1^{j-i} d_2^{i} \right) \\
\leq \; & \binom{k-2}{\ell-2} d_1^{\ell_1} d_2^{k-\ell_1} + O(1)^k d_1^{\ell_1+1} d_2^{k-\ell_1-1}.
\end{aligned}$$

In particular if $\ell_1 = 1$, and $d_1 = 2$, $b(\mathrm{Zer}(\mathcal{P}, \mathrm{R}^k), \mathbb{Z}_2)$ is bounded by

$$\begin{aligned}
& 1 + (-1)^{k-\ell+1} + 2d^{\ell-1} \cdot \left(\sum_{j=0}^{k-\ell} \sum_{i=0}^{j} (-1)^{k-\ell+j} \binom{k}{j+\ell} \binom{\ell+i-2}{i} 2^{j-i} d^{i} \right) \\
\leq \; & 2\binom{k-2}{\ell-2} d^{k-1} + (O(d))^{k-2}.
\end{aligned} \tag{3.20}$$

Proposition 8. *Let* $\mathcal{P} \subset \mathrm{C}[\mathbf{X}^{(1)}, \ldots, \mathbf{X}^{(p)}]$, *for* $1 \leq i \leq p$, $\mathbf{X}^{(i)} = (X_1^{(i)}, \ldots, X_{k_i}^{(i)})$ *and* $\deg_{\mathbf{X}^{(i)}}(P) \leq d_i, P \in \mathcal{P}$, *with* $\ell = \mathrm{card}(\mathcal{P}) > 0$. *Suppose also that the polynomials in* $\mathcal{P}$ *are generic.*

Then,

$$b(\mathrm{Zer}(\mathcal{P}, \mathrm{C}^k), \mathbb{Z}_2) \leq 1 + (-1)^{k-\ell+1} + (k-\ell+2)^2 \binom{k}{\ell-1} \binom{k}{\mathbf{k}}^{-1} \frac{(1+p)^{3k-\ell+1}}{p(p+2)} d_1^{k_1} \cdots d_p^{k_p}. \tag{3.21}$$

In case, $\mathcal{P} \subset \mathrm{R}[\mathbf{X}^{(1)}, \ldots, \mathbf{X}^{(p)}]$, *then*

$$b(\mathrm{Zer}(\mathcal{P}, \mathrm{R}^k), \mathbb{Z}_2) \leq 1 + (-1)^{k-\ell+1} + (k-\ell+2)^2 \binom{k}{\ell-1} \binom{k}{\mathbf{k}}^{-1} \frac{(1+p)^{3k-\ell+1}}{p(p+2)} d_1^{k_1} \cdots d_p^{k_p}. \tag{3.22}$$

Proof. See Appendix. □

In order to investigate the tightness of the inequalities in Proposition 8, it is instructive to consider the special case of Proposition 8 when the block sizes are all equal to one (i.e. $p = k$) and $\ell = 1$.

Proposition 9. *Let P be a generic polynomial in $\mathrm{C}[X_1, \ldots, X_k]$, and $\deg_{X_i}(P) \leq d_i$ with $d_1 \geq d_2 \geq \cdots \geq d_k \geq 0$. We denote by $\bar{d} = (d_1, \ldots, d_k)$, and for $J \subset [1, k]$, we denote $\bar{d}^J = \prod_{i \in J} d_i$. Then,*

$$b(\mathrm{Zer}(P, \mathrm{C}^k), \mathbb{Z}_2) \leq 1 + (-1)^k + \left(\sum_{j=1}^{k} (-1)^{k-j} \sum_{\substack{J \subset [1,k] \\ \mathrm{card}(J) = j \leq k}} j! \bar{d}^J \right), \tag{3.23}$$

and if $P \in \mathrm{R}[X_1, \ldots, X_k]$, then

$$b(\mathrm{Zer}(P, \mathrm{R}^k), \mathbb{Z}_2) \leq 1 + (-1)^k + \left(\sum_{j=1}^{k} (-1)^{k-j} \sum_{\substack{J \subset [1,k] \\ \mathrm{card}(J) = j \leq k}} j! \bar{d}^J \right). \tag{3.24}$$

Proof. See Appendix. □

Remark 20 (Comparison with the Oleĭnik-Petrovskiĭ-Thom-Milnor bound). Theorem 1 gives a bound of $D(2D-1)^{k-1}$ in the context of Proposition 9 with the total degree $D = d_1 + \cdots + d_k$. This bound is in general much worse than the bound in inequality (3.24) in Proposition 9. For example, take $k = 2$, $\bar{d} = (d, d)$. Then the bound from Theorem 1 (i.e. the Oleĭnik-Petrovskiĭ-Thom-Milnor bound) is $2d(4d-1)$, while Proposition 9 yields a bound of

$$1 + 1 - 2d + 2d^2 = 2d^2 - 2d + 2 < 2d(4d-1)$$

for all $d > 0$.

Remark 21. Proposition 9 is tight when $k_1 = k_2 = 1$ and $\bar{d} = (2, 2)$. Then, the bound in Proposition 9 is (using the formula in Remark 20)

$$2 \cdot 2^2 - 2 \cdot 2 + 2 = 6.$$

Consider the polynomial

$$P_\varepsilon = (X_1^2 - 1)(X_2^2 - 1) - \varepsilon,$$

and let $V = \mathrm{Zer}(P_\varepsilon, \mathrm{R}^2)$. Then, for all sufficiently small $\varepsilon > 0$, $b_0(V, \mathbb{Z}_2) = 5$, and $b_1(V, \mathbb{Z}_2) = 1$ (see Figure 1), so that $b(V, \mathbb{Z}_2) = 6$.

Notice that the Oleĭnik-Petrovskiĭ-Thom-Milnor bound of $d(2d-1)^{k-1}$, where d is the total degree, yields in this case $4 \cdot (8-1) = 28$, which is much worse than the bound in Proposition 9.

Proposition 10. *Let $B_1, \ldots, B_\ell \subset \mathbb{R}^k, 1 \leq \ell \leq k$, and for $1 \leq i \leq \ell$, $B_i = [0, d_{i,1}] \times \cdots \times [0, d_{i,k}]$. We denote by $\mathbf{d} = (d_{i,j})_{\substack{1 \leq i \leq \ell \\ 1 \leq j \leq k}}$. Let $\mathcal{P} = \{P_1, \ldots, P_\ell\} \subset$*

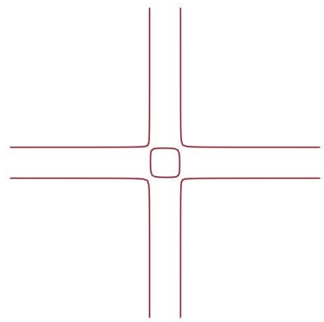


FIGURE 1. The set of real zeros in R^2 of $P_\varepsilon = (X_1^2-1)(X_2^2-1)-\varepsilon$ for small ε.

$\mathrm{C}[X_1,\ldots,X_k$, *with* $\mathrm{supp}(P_i) = B_i$, $\mathcal{P}$ *generic.*

$$b(\mathrm{Zer}(\mathcal{P},\mathrm{C}^k),\mathbb{Z}_2) \leq 1+(-1)^{k-\ell+1} + \left(\sum_{j=\ell}^{k}\sum_{J\in\binom{[1,k]}{j}}(-1)^{k-j}\sum_{\substack{\boldsymbol{\alpha}=(\alpha_1,\ldots,\alpha_\ell)\in\mathbb{Z}_{>0}^\ell\\ \alpha_1+\cdots+\alpha_\ell=j}} N(\mathbf{d}_J,\boldsymbol{\alpha})\right)$$

$$\leq O(\ell)^k\cdot \max_{\substack{\boldsymbol{\alpha}=(\alpha_1,\ldots,\alpha_\ell)\in\mathbb{Z}_{>0}\\ \alpha_1+\cdots+\alpha_\ell=k}} \max_{(J_1,\ldots,J_\ell)\in\binom{[1,k]}{\boldsymbol{\alpha}}}\left(\prod_{\substack{1\leq i\leq \ell\\ j\in J_i}} d_{i,j}\right), \tag{3.25}$$

and if $P\in\mathrm{R}[X_1,\ldots,X_k]$, *then*

$$b(\mathrm{Zer}(\mathcal{P},\mathrm{R}^k),\mathbb{Z}_2) \leq 1+(-1)^{k-\ell+1} + \left(\sum_{j=\ell}^{k}\sum_{J\in\binom{[1,k]}{j}}(-1)^{k-j}\sum_{\substack{\boldsymbol{\alpha}=(\alpha_1,\ldots,\alpha_\ell)\in\mathbb{Z}_{>0}^\ell\\ \alpha_1+\cdots+\alpha_\ell=j}} N(\mathbf{d}_J,\boldsymbol{\alpha})\right)$$

$$\leq O(\ell)^k\cdot \max_{\substack{\boldsymbol{\alpha}=(\alpha_1,\ldots,\alpha_\ell)\in\mathbb{Z}_{>0}\\ \alpha_1+\cdots+\alpha_\ell=k}} \max_{(J_1,\ldots,J_\ell)\in\binom{[1,k]}{\boldsymbol{\alpha}}}\left(\prod_{\substack{1\leq i\leq \ell\\ j\in J_i}} d_{i,j}\right), \tag{3.26}$$

where for $J\subset[1,k]$, $\mathbf{d}_J$ *is the* $\ell\times\mathrm{card}(J)$ *sub-matrix obtained by extracting the columns corresponding to* J *in* $\mathbf{d}$, *and* $N(\mathbf{d}_J,\boldsymbol{\alpha})$ *is defined in* (3.5).

Proof. See Appendix. □

3.5.1. *Quadratic and partially quadratic case.* We now use Theorem 24 to obtain bounds on the Betti numbers of generic intersections of quadratic and partially quadratic polynomials. Since the dependence of the bounds on the different parameters in this case are rather different from the previous cases, we start by explaining the most simple case in detail.

Proposition 11. *Let* P_1, P_2 *be two generic quadratic polynomials in* $\mathrm{C}[X_1,\ldots,X_k]$. *Then,*

$$b(\mathrm{Zer}(\{P_1,P_2\},\mathrm{C}^k),\mathbb{Z}_2) = 2k. \tag{3.27}$$

Proof. First note that $\mathrm{supp}(P_1), \mathrm{supp}(P_2)$ are both equal to the convex hull of $\mathbf{0}, (2,0,\ldots,0), \ldots, (0,\ldots,0,2)$. It follows applying Theorem 24 that for $k \geq 2$,

$$\begin{aligned}
\chi(\mathrm{Zer}(\{P_1,P_2\},\mathrm{C}^k),\mathbb{Z}_2) &= \sum_{j=2}^{k} \binom{k}{j} (-1)^j (j-1) j! \frac{2^j}{j!} \\
&= 1 + \sum_{j=0}^{k} \binom{k}{j} (-1)^j (j-1) 2^j \\
&= 1 + \sum_{j=0}^{k} \binom{k}{j} (-1)^j j 2^j - \sum_{j=0}^{k} \binom{k}{j} (-1)^j 2^j \\
&= 1 + 2\sum_{j=0}^{k} \binom{k}{j} (-1)^j j 2^{j-1} - \sum_{j=0}^{k} \binom{k}{j} (-1)^j 2^j \\
&= 1 + 2k(1-2)^{k-1}(-1) - (1-2)^k \\
&= 1 + (-1)^k 2k - (-1)^k \\
&= 1 + (-1)^k (2k-1).
\end{aligned}$$

This implies (using Eqn. (3.9)) that

$$\begin{aligned}
b(V,\mathbb{Z}_2) &= 1 + (-1)^k (\chi(V_k,\mathbb{Z}_2) - 1) \\
&= 1 + (-1)^{k+1} + (-1)^k \chi(V_k,\mathbb{Z}_2) \\
&= 1 + (-1)^{k+1} + (-1)^k \left(1 + (-1)^k (2k-1)\right) \\
&= 2k.
\end{aligned}$$

□

Remark 22. In particular, when $k = 2$, $b(\mathrm{Zer}(\{P_1,P_2\},\mathrm{C}^k),\mathbb{Z}_2) = 4$, agreeing with the fact that two generic quadratic polynomials in two variables will have 4 points in their intersection.

In the case $k = 3$, notice that intersection, W of two generic quadric surfaces in $\mathbb{P}^3_{\mathrm{C}}$ is topologically a torus $\mathbf{S}^1 \times \mathbf{S}^1$. The intersection of W with the plane at infinity is 4 points. Hence, $\mathrm{Zer}(\{P_1,P_2\},\mathrm{C}^k)$ in this case is homeomorphic to $\mathbf{S}^1 \times \mathbf{S}^1$ minus 4 points. This gives,

$$\begin{aligned}
b_0(\mathrm{Zer}(\{P_1,P_2\},\mathrm{C}^k),\mathbb{Z}_2) &= 1, \\
b_1(\mathrm{Zer}(\{P_1,P_2\},\mathrm{C}^k),\mathbb{Z}_2) &= 5.
\end{aligned}$$

This gives, $\chi(V,\mathbb{Z}_2) = -4$ which agrees with the formula above.

We now consider a more general situation.

Proposition 12. *Let $\mathcal{P} = \{P_1,\ldots,P_\ell\}$ be a finite set of generic polynomials in $\mathrm{C}[X_1,\ldots,X_{k_1},Y_1,\ldots,Y_{k_2}]$ with $0 < \ell \leq k = k_1+k_2$, and $\deg_{\mathbf{X}}(P_i) \leq d, \deg_{\mathbf{Y}}(P_i) \leq 2$.*

Then,

$$b(\mathrm{Zer}(\mathcal{P},\mathrm{C}^k),\mathbb{Z}_2) \leq 2 + (-1)^{k-\ell+1} + \ell 2^\ell (k_1+k_2)^{\ell-1} \left(2d(k_1+k_2)+1\right)^{k_1}. \tag{3.28}$$

If additionally $\mathcal{P} \subset \mathrm{R}[X_1,\ldots,X_{k_1},Y_1,\ldots,Y_{k_2}]$, then

$$b(\mathrm{Zer}(\mathcal{P},\mathrm{R}^k),\mathbb{Z}_2) \leq 2 + (-1)^{k-\ell+1} + \ell 2^\ell (k_1+k_2)^{\ell-1} \left(2d(k_1+k_2)+1\right)^{k_1}. \tag{3.29}$$

Proof. See Appendix. □

3.5.2. *Generic intersections of quadrics in affine and projective spaces.* Since the intersections of quadrics is a very well studied topic [6, 40] we investigate the special case of Proposition 12 where $k_1 = 0$. In particular, we calculate the leading coefficient of the polynomial in k giving the sum of the Betti numbers of the intersection of ℓ generic quadrics in $\mathbb{P}^k_{\mathrm{C}}$ for every fixed ℓ, thus solving a problem posed in [40] (see Eqn. 3.32 below).

Setting $k_1 = 0$ and $k_2 = k$, in the above calculation and keeping the same notation, we obtain that

$$\chi(\mathrm{Zer}(\mathcal{P}, \mathrm{C}^k), \mathbb{Z}_2) = 1 + (-1)^{k+1}\left(\sum_{h=0}^{\ell-1}\binom{k}{h}(-2)^h\right). \tag{3.30}$$

In order to calculate the Euler-Poincaré characteristic of a generic complete intersection of dimension $k-\ell$ in $\mathbb{P}^k_{\mathrm{C}}$ in terms of a fixed degree sequence $(d_1, \ldots, d_\ell)$, it suffices to take the sum of the Euler-Poincaré characteristics of the corresponding affine varieties in $\mathrm{C}^k, \mathrm{C}^{k-1}, \ldots, \mathrm{C}^\ell$ (with the same degree sequence). Applying this in our situation we obtain that if $\mathcal{P} = \{P_1, \ldots, P_\ell\}$ are generic homogeneous quadrics in $\mathrm{C}[X_0, \ldots, X_k]$ then it follows from the above and (3.30) that

$$\begin{aligned}\chi(\mathrm{Zer}(\mathcal{P}, \mathbb{P}^k_{\mathrm{C}}), \mathbb{Z}_2) &= \sum_{j=\ell}^{k}\left(1 + (-1)^{j+1}\left(\sum_{h=0}^{\ell-1}\binom{j}{h}(-2)^h\right)\right)\\ &= \sum_{h=0}^{\ell-1}(-2)^h\left(\sum_{j=\ell}^{k}(-1)^{j+1}\binom{j}{h}\right) + (k-\ell+1), \end{aligned}\tag{3.31}$$

and using Proposition 4

$$b(\mathrm{Zer}(\mathcal{P}, \mathbb{P}^k_{\mathrm{C}}), \mathbb{Z}_2) = (1 + (-1)^{k-\ell+1})(k-\ell+1) + (-1)^{k-\ell}\chi(\mathrm{Zer}(\mathcal{P}, \mathbb{P}^k_{\mathrm{C}}), \mathbb{Z}_2) \tag{3.32}$$

for $\ell < k$.

Remark 23. It is easy to deduce directly from (3.31) and (3.32) that for $\ell \geq 3$

$$b(\mathrm{Zer}(\mathcal{P}, \mathbb{P}^k_{\mathrm{C}}), \mathbb{Z}_2) = \frac{2^{\ell-2}}{(\ell-1)!}k^{\ell-1} + O(k^{\ell-2}), \tag{3.33}$$

for fixed ℓ and k large.

To see this define,

$$B(h, k, \ell) = 2^h\left(\sum_{j=\ell}^{k}(-1)^{j+1}\binom{j}{h}\right).$$

$B(\ell-1, k, \ell)$ is the absolute value of the term corresponding to $h = \ell - 1$ in the expression in (3.31). We have

$$\begin{aligned}B(\ell-1, k, \ell) &= 2^{\ell-1}\left(\binom{k}{\ell-1} - \binom{k-1}{\ell-1} + \cdots + (-1)^{k-\ell-1}\binom{\ell-1}{\ell-1}\right)\\ &= 2^{\ell-1}\left(\binom{k-1}{\ell-2} + \binom{k-3}{\ell-2} + \cdots\right).\end{aligned}\tag{3.34}$$

Lemma 2. *For $p \geq 0$, and all large n,*

$$\sum_{i=0}^{\lfloor (n-p)/2 \rfloor} \binom{n-2i}{p} = \frac{1}{2}\binom{n+1}{p+1} + O(n^p).$$

Proof. Let

$$A(n,p) = \sum_{i=0}^{\lfloor (n-p)/2 \rfloor} \binom{n-2i}{p}.$$

From standard binomial identities we deduce

$$A(n,p) + A(n-1,p) = \sum_{i=0}^{\lfloor (n-p) \rfloor} \binom{n-i}{p} = \binom{n+1}{p+1},$$

and

$$A(n,p) - A(n-1,p) = A(n-1,p-1) \leq A(n,p-1).$$

The lemma now follows by induction on p, the case $p = 0$ being trivial. □

It follows from Lemma 2 and (3.34) that

$$B(\ell-1,k,\ell) = 2^{\ell-2}\binom{k}{\ell-1} + O(k^{\ell-2}). \tag{3.35}$$

Moreover,

$$\sum_{h=0}^{\ell-2} B(h,k,\ell) = O(k^{\ell-2}). \tag{3.36}$$

It follows from (3.35), (3.36), Proposition 4 that

$$b(\mathrm{Zer}(\mathcal{P}, \mathbb{P}^k_{\mathrm{C}}), \mathbb{Z}_2) = 2^{\ell-2}\binom{k}{\ell-1} + O(k^{\ell-2}),$$

which implies inequality (3.33).

This answers a question raised in [40, page 4], where the first few values of the leading coefficients were given, and the problem of calculating it exactly was posed. Notice that this coefficient, $\frac{2^{\ell-2}}{(\ell-1)!}$, goes to zero exponentially fast with ℓ.

Using Theorem 22 (Smith inequality) and (3.33) we obtain the following theorem (cf. [40]).

Theorem 25. *Let $\mathcal{P} \subset \mathrm{R}[X_0, \ldots, X_k]$ be a set of ℓ generic homogeneous polynomials of degree 2. Then for every fixed $\ell \geq 3$,*

$$b(\mathrm{Zer}(\mathcal{P}, \mathbb{P}^k_{\mathrm{R}}), \mathbb{Z}_2) \leq 2^{\ell-2}\binom{k}{\ell-1} + O(k^{\ell-2}).$$

Remark 24. Note that a more naive approach using a bound of $(O(k))^{\ell-1}$ on the Betti numbers of generic intersections of ℓ affine quadrics in C^k, the fact that $\mathbb{P}^k_{\mathrm{C}}$ is the disjoint union of $\mathrm{C}^k, \mathrm{C}^{k-1}, \ldots, \mathrm{C}^0$, and the additivity property of the Euler-Poincaré characteristics, yields a slightly coarser bound of $(O(k))^{\ell}$. The signs thus play an important role in the proof of Theorem 25.

3.5.3. *One quadratic block and multi-degree case.* We now consider the case of generic intersections of polynomials having one block of variables of degree 2, while the other variables are allowed to have different degrees. More precisely, we prove:

Proposition 13. *Let $\mathcal{P} = \{P_1, \ldots, P_\ell\} \subset \mathrm{C}[X_1, \ldots, X_{k_1}, Y_1, \ldots, Y_{k_2}]$ be a finite set of generic polynomials with $\deg_{X_i}(P) \leq d_i$ and $\deg_{\mathbf{Y}}(P) \leq 2$ for all $P \in \mathcal{P}$ with $d_1 \geq d_2 \geq \cdots \geq d_{k_1}$ and $0 < \ell \leq k = k_1 + k_2$. Then,*

(3.37)
$$b(\mathrm{Zer}(\mathcal{P}, \mathrm{C}^k), \mathbb{Z}_2) \leq 2 + (-1)^{k-\ell+1} + \ell 2^\ell k_1!(k_1+k_2)^{\ell-1} \left(2(k_1+k_2)+1\right)^{k_1} d_1 \cdots d_{k_1}.$$

If additionally $\mathcal{P} \subset \mathrm{R}[X_1, \ldots, X_{k_1}, Y_1, \ldots, Y_{k_2}]$, then

(3.38)
$$b(\mathrm{Zer}(\mathcal{P}, \mathrm{R}^k), \mathbb{Z}_2) \leq 2 + (-1)^{k-\ell+1} + \ell 2^\ell k_1!(k_1+k_2)^{\ell-1} \left(2(k_1+k_2)+1\right)^{k_1} d_1 \cdots d_{k_1}.$$

Proof. See Appendix. □

4. Proofs of the main theorems

4.1. **Summary of the methods.** Our main tools are the bounds on the Betti numbers of generic intersections proved in Section 3.5 above, which are all consequences of Theorem 24 and Theorem 22 (Smith inequality), the techniques of infinitesimal perturbations ([17, Chapter 7]), and the inequalities derived from the Mayer-Vietoris exact sequence (Proposition 1). Using the techniques of infinitesimal perturbations, and the inequalities in Proposition 1, we reduce the problem of bounding the Betti numbers of semi-algebraic sets defined by general (non-generic) polynomials $P \in \mathcal{P} \subset \mathrm{R}[X_1, \ldots, X_k]$ with support contained in given Newton polytopes $\Delta_P, P \in \mathcal{P}$ (the tuple $(\Delta_P)_{P \in \mathcal{P}}$ satisfying Property 1), to bounding the Betti numbers of a collection of real affine algebraic varieties defined by generic polynomials with (nearly) the same support. The proofs of Theorems 11, 12, 14, 16, and 20 (i.e. the cases of different classes of algebraic sets) are very similar to each other, differing only in the application of the appropriate generic bounds. Because of this reason we explain only the proof of Theorem 11 in full detail. Similarly, the proofs of Theorems 13, 15, 18, and 21 (the semi-algebraic cases) are all similar in structure to the proof of [17, Theorem 7.30] and [17, Theorem 7.38], again differing only in the application of the appropriate generic bounds. We refer the reader to [17] for any missing detail.

4.2. **Proof of Theorem 11.**

Proof of Theorem 11. We first prove

$$b(V, \mathbb{Z}_2) \leq \frac{1}{2}(1 + (2d-1)^k). \tag{4.1}$$

Let

$$F(X_1, \ldots, X_k) = (Q_1^2 + \cdots + Q_\ell^2)/(r^2 - \|\mathbf{X}\|^2).$$

The set of critical values of F is finite, so there exists $c_0 \in \mathrm{R}, c_0 > 0$ so that $\mathrm{Zer}(\tilde{Q}, \mathrm{R}^k)$ is a non-singular hypersurface in R^k, where

$$\tilde{Q} = Q_1^2 + \cdots + Q_\ell^2 + c(\|\mathbf{X}\|^2 - r^2) = 0, \tag{4.2}$$

for all $c \in (0, c_0)$.

Denote by $\tilde{V} = \mathrm{Zer}(\tilde{Q}, \mathrm{R}^k)$. Since $\deg(\tilde{Q}) \leq 2d$, we have by Proposition 6 that

$$b(\tilde{V}, \mathbb{Z}_2) \leq 1 + (2d-1)^k. \tag{4.3}$$

Notice that the the closed and bounded semi-algebraic set, S defined by $\tilde{Q} \leq 0$, is semi-algebraically homotopy equivalent to V.

Since S is bounded by $\tilde{V}$,

$$b(S, \mathbb{Z}_2) \leq \frac{1}{2} b(\tilde{V}, \mathbb{Z}_2) \tag{4.4}$$

(using for example [17, Proposition 7.27]).

Combining (4.3) and (4.4) we obtain

$$b(V, \mathbb{Z}_2) = b(S, \mathbb{Z}_2) \leq \frac{1}{2} b(\tilde{V}, \mathbb{Z}_2) \leq \frac{1}{2}(1 + (2d-1)^k),$$

which proves (4.1).

We now prove that

$$b(V, \mathbb{Z}_2) \leq \binom{\ell}{k} 2^k d'^k + \sum_{j=1}^{k-1} \binom{\ell}{j} 2^j (F_1(d', k, j) + F_2(d', k, j)) + 3, \tag{4.5}$$

where d' is the least even integer ≥ 0.

Denote for every *even* $d'' \geq 0$, by $H_{d'',k} \subset \mathrm{R}\langle\varepsilon_0\rangle[X_1, \ldots, X_k]$ the subspace of polynomials of degree $\leq d''$, and observe that there is a non-empty, open semi-algebraic subset $U_{d'',k} \subset H_{d'',k}$ such that for every $H \in U_{d'',k}$, H is strictly positive on $\mathrm{R}\langle\varepsilon_0\rangle^k$.

Let

$$\tilde{Q}_0 = \|\mathbf{X}\|^2 - 1/\varepsilon_0^2 - \delta_0 H_0,$$

and for $1 \leq i \leq \ell$, let

$$\begin{aligned} \tilde{Q}_{i,+} &= Q_i + \delta_i H_{2i-1}, \\ \tilde{Q}_{i,-} &= Q_i - \delta_i H_{2i}, \end{aligned}$$

where the polynomials $H_0 \in U_{2,k}$, and $H_1, \ldots, H_{2\ell} \in U_{d',k}$ are chosen to be sufficiently generic.

Also, let $\mathrm{R}' = \mathrm{R}\langle\varepsilon_0, \delta_0, \delta_1, \ldots, \delta_\ell\rangle$.

We need the following lemma.

Lemma 3. *The real algebraic variety* $\mathrm{Ext}(V, \mathrm{R}')$ *(cf. Notation 9) is semi-algebraically homotopy equivalent to the semi-algebraic set* $\tilde{S} \subset \mathrm{R}'^k$ *defined by*

$$(\tilde{Q}_0 < 0) \wedge \bigwedge_{1 \leq i \leq \ell} ((\tilde{Q}_{i,+} > 0) \wedge (\tilde{Q}_{i,-} < 0)).$$

Proof. Follows from [17, Lemma 16.17]. □

Let

$$\tilde{W} = \mathrm{Zer}(\tilde{Q}_0, \mathrm{R}'^k) \cup \bigcup_{\substack{1 \leq i \leq \ell \\ \epsilon \in \{+,-\}}} \mathrm{Zer}(\tilde{Q}_{i,\epsilon}, \mathrm{R}'^k).$$

Note that $\mathrm{R}'^k \setminus \tilde{W}$ is an open semi-algebraic set, and $\tilde{S}$ is the union of a subset of the semi-algebraically connected components of $\mathrm{R}'^k \setminus \tilde{W}$ which are bounded. This implies that

$$b(\tilde{S}, \mathbb{Z}_2) \leq b(\dot{\mathrm{R}}'^k \setminus \tilde{W}, \mathbb{Z}_2),$$

where $\overset{\bullet}{\mathrm{R}'^k}$ is the one-point compactification of R'^k, and is semi-algebraically homeomorphic to the sphere $\mathbf{S}^k$ defined over R'. Further using Alexander duality [50, page 296], and the fact that $\tilde{W}$ is non-empty we obtain

$$b(\tilde{S}, \mathbb{Z}_2) \leq b(\overset{\bullet}{\mathrm{R}'^k} \setminus \tilde{W}, \mathbb{Z}_2) \leq b(\tilde{W}, \mathbb{Z}_2) + 1.$$

We now bound $b(\tilde{W}, \mathbb{Z}_2)$ using Proposition 1 noting that $\tilde{W}$ is the union of $2\ell+1$ real algebraic sets.

Let

$$\tilde{\mathcal{Q}} = \{\tilde{Q}_{i,\epsilon} \mid 1 \leq i \leq \ell, \epsilon \in \{+,-\}\}.$$

Proposition 1 implies that

$$b(\tilde{W}, \mathbb{Z}_2) \leq \sum_{\substack{\tilde{\mathcal{Q}}' \subset \tilde{\mathcal{Q}},\\ \mathrm{card}(\tilde{\mathcal{Q}}') \leq k-1}} b(\mathrm{Zer}(\{\tilde{Q}_0\} \cup \tilde{\mathcal{Q}}', \mathrm{R}'^k), \mathbb{Z}_2) + \sum_{\substack{\tilde{\mathcal{Q}}' \subset \tilde{\mathcal{Q}},\\ \mathrm{card}(\tilde{\mathcal{Q}}') \leq k}} b(\mathrm{Zer}(\tilde{\mathcal{Q}}', \mathrm{R}'^k), \mathbb{Z}_2). \tag{4.6}$$

Notice that for $\tilde{\mathcal{Q}}' \subset \tilde{\mathcal{Q}}$ with $\mathrm{card}(\tilde{\mathcal{Q}}') = k$, we have by Bezout's theorem that

$$b(\mathrm{Zer}(\tilde{\mathcal{Q}}', \mathrm{R}'^k), \mathbb{Z}_2) \leq d'^k. \tag{4.7}$$

Inequalities (4.7) and (4.6) imply

$$b(\tilde{W}, \mathbb{Z}_2) \leq \binom{\ell}{k} 2^k d'^k + \sum_{\substack{\tilde{\mathcal{Q}}' \subset \tilde{\mathcal{Q}},\\ \mathrm{card}(\tilde{\mathcal{Q}}') \leq k-1}} (b(\mathrm{Zer}(\{\tilde{Q}_0\} \cup \tilde{\mathcal{Q}}', \mathrm{R}'^k), \mathbb{Z}_2) + b(\mathrm{Zer}(\tilde{\mathcal{Q}}', \mathrm{R}'^k), \mathbb{Z}_2)). \tag{4.8}$$

Finally for $\tilde{\mathcal{Q}}' \subset \tilde{\mathcal{Q}}$ with $\mathrm{card}(\tilde{\mathcal{Q}}') = j, 1 \leq j \leq \min(\ell, k-1)$, we have,

$$b(\mathrm{Zer}(\{\tilde{Q}\} \cup \tilde{\mathcal{Q}}', \mathrm{R}'^k), \mathbb{Z}_2) \leq F_1(d', k, j), \tag{4.9}$$

using inequality (3.20) in Remark 19, and

$$b(\mathrm{Zer}(\tilde{\mathcal{Q}}', \mathrm{R}'^k), \mathbb{Z}_2)) \leq F_2(d', k, j) \tag{4.10}$$

using inequality (3.18) in Proposition 7. Note that if $j = 0$, then

$$b(\mathrm{Zer}(\{\tilde{Q}_0\}, \mathrm{R}'^k), \mathbb{Z}_2) = 2,$$

and

$$b(\mathrm{Zer}(\tilde{\mathcal{Q}}', \mathrm{R}'^k), \mathbb{Z}_2) = b(\mathrm{Zer}(\emptyset, \mathrm{R}'^k), \mathbb{Z}_2) = b(\mathrm{R}'^k, \mathbb{Z}_2) = 1.$$

Inequality (4.5) now follows from Lemma 3, and inequalities (4.6), (4.8), (4.9), and (4.10).

Finally, inequality (2.1) follows from inequalities (4.1) and (4.5). □

4.3. **Proofs of Theorems 12 and 13.**

Proof of Theorem 12. We first prove

$$b(V, \mathbb{Z}_2) \quad \leq \quad \frac{1}{2} G_{\mathrm{gen}}(2\mathbf{d}, \mathbf{k}, 1). \tag{4.11}$$

The proof is similar to that of Theorem 11, but we note that the polynomial (4.2) $\tilde{Q} = Q_1^2 + \cdots + Q_\ell^2 + c(\|\mathbf{X}\|^2 - r^2) \in \mathrm{R}[\mathbf{X}^{(1)}, \ldots, \mathbf{X}^{(p)}]$ has multi-degree bounded by $2\mathbf{d}$. Therefore, $b(\mathrm{Zer}(\tilde{Q}, \mathrm{R}^k), \mathbb{Z}_2) \leq G_{\mathrm{gen}}(2\mathbf{d}, \mathbf{k}, 1)$ using Proposition 8.

We now prove

$$(4.12)\qquad b(V,\mathbb{Z}_2) \leq 3+\sum_{j=1}^{k}\binom{\ell}{j}2^j(G_{\mathrm{gen}}(\mathbf{d}',\mathbf{k},j)+G_{\mathrm{gen}}(\mathbf{d}',\mathbf{k},j+1)).$$

We proceed in the same manner as in the proof of Theorem 11. We note that the sphere can also be viewed as a polynomial in $\mathrm{R}[\mathbf{X}^{(1)},\ldots,\mathbf{X}^{(p)}]$, where each block has degree equal to 2. Notice that we assume all $d_i \geq 2$, so we can view the sphere as another polynomial with the same block structure and degree bounds as each polynomial in $\mathcal{Q}$. Therefore, we can replace both $F_1(d',k,j)$ (resp. $F_2(d',k,j)$) with $G_{\mathrm{gen}}(\mathbf{d}',\mathbf{k},j+1)$ (resp. $G_{\mathrm{gen}}(\mathbf{d}',\mathbf{k},j)$). The theorem follows from inequalities (4.11) and (4.12). □

Proof of Theorem 13. The proof is similar to those of [17, Theorem 7.30] and [17, Theorem 7.38]. From Proposition 7 with the following modification that instead of using Theorem 1 for bounding the sum of the Betti numbers of various algebraic sets that occur, we use the bound in Theorem 12. □

4.4. Proofs of Theorems 14 and 15.

Proof of Theorem 14. The proof is similar to the proof of Theorem 12 using Proposition 10 instead of Proposition 8. □

Proof of Theorem 15. The proof is similar to the proof of Theorem 13 using Proposition 10 instead of Proposition 8. □

4.5. Proofs of Theorems 16, 17, 18, and 19.

Proof of Theorem 16. The proof is similar to that of Theorem 12. Since we have ℓ partially quadratic polynomials, we use $H_{\mathrm{gen}}(d',k_1,k_2,j)$ in place of $G_{\mathrm{gen}}(\mathbf{d}',\mathbf{k},j)$, where $H_{\mathrm{gen}}(d',k_1,k_2,j)$ is the bound from Proposition 12, noting that we assume $d \geq 2$. □

Proof 0f Theorem 17. The proof is similar to the proof of Theorem 11 above using generic *positive* quadrics to perturb the given polynomials (in lieu of the polynomials H_i in the proof of Theorem 11), noticing that there is no need to add a new polynomial corresponding to the big ball, and finally using the expression in (3.32) in lieu of the generic bound, and Theorem for the asymptotic bound. □

Proof of Theorem 18. The proof is similar to those of [17, Theorem 7.30] and [17, Theorem 7.38] with the modification that instead of using Theorem 1 for bounding the sum of the Betti numbers of various algebraic sets that occur, we use the bound in Theorem 16. □

Proof of Theorem 19. The proof is again similar to the proofs of Theorem 7.30 and Theorem 7.38 in [17] with several modifications. In the proof of Theorem 7.30 in [17], we let $\mathcal{Q}=\{0\}$, and $\mathcal{P}=\mathcal{P}_1\cup\mathcal{P}_2$, and we let $\mathcal{P}_1=\{P_1,\ldots,P_s\}$ and $\mathcal{P}_2=\{P_{s+1},\ldots,P_{s+m}\}$. In Proposition 7.34, for each sign condition $\sigma\in\{0,1,-1\}^{\mathcal{P}}$, we redefine the basic closed semi-algebraic set $\overline{\mathrm{Reali}(\sigma)}\subset \mathrm{R}\langle\delta,\delta_1,\ldots,\delta_{s+m},\varepsilon_1,\ldots,\varepsilon_{s+m}\rangle^k$ in the following way. Without loss of generality assume

$$\sigma(P_h) = 0 \text{ if } h \in I,$$
$$\sigma(P_h) = 1 \text{ if } h \in J,$$
$$\sigma(P_h) = -1 \text{ if } h \in \{1, \ldots, s+m\} \setminus (I \cup J),$$

and denote by $\overline{\mathrm{Reali}(\sigma)}$ the subset of $\mathrm{R}\langle \delta, \delta_1, \ldots, \delta_{s+m}, \varepsilon_1, \ldots, \varepsilon_{s+m}\rangle^k$ defined by

$$\delta^2(|\mathbf{X}|^2 + |\mathbf{Y}|^2) \leq 1,$$
$$-\varepsilon_h \leq P_h \leq \varepsilon_h, \text{ if } h \in I,$$
$$P_h \geq \delta_h, \text{ if } h \in J,$$
$$P_h \leq -\delta_h, \text{ if } h \in \{1, \ldots, s\} \setminus (I \cup J).$$

It is easy to verify that Proposition 7.34 in [17] remains true with this new definition of $\overline{\mathrm{Reali}(\sigma)}$.

Now we observe that since $\mathcal{P}_1 \subset \mathrm{R}[X_1, \ldots, X_{k_1}]$, no more than k_1 polynomials amongst the set $\{P_1 \pm \delta_1, P_1 \pm \varepsilon_1, \ldots, P_s \pm \delta_s, P_s \pm \varepsilon_s\}$ can have a common zero. Each non-empty real algebraic set V defined by some subset of the polynomials $\{P_1 \pm \delta_1, P_1 \pm \varepsilon_1, \ldots, P_s \pm \delta_s, P_s \pm \varepsilon_s\} \cup \{P_0\}$, where $P_0 = \delta(|\mathbf{X}|^2 + |\mathbf{Y}|^2) - 1$, is the set of zeros of two sets of polynomials, namely

$$(P_h + \epsilon_h \eta_h)_{h \in J_1}, \eta_h \in \{\varepsilon_h, \delta_h\}, \epsilon_h \in \{\pm 1, \pm 2\}, J_1 \subset [1, s],$$
$$(P_h + \epsilon_h \eta_h)_{h \in J_2}, \eta_h \in \{\varepsilon_h, \delta_h\}, \epsilon_h \in \{\pm 1, \pm 2\}, J_2 \subset [s+1, s+m],$$

and possibly P_0, with $j_1 = \mathrm{card}(J_1) \leq k_1$, and $j_2 = \mathrm{card}(J_2) \leq \min(m+1, k_1 + k_2 - j_1 - i)$.

We note that V is also defined by the $(\mathrm{card}(J_2)+1)$ or $(\mathrm{card}(J_2)+2)$ (depending on whether P_0 is included or not) polynomials

$$\sum_{h \in J_1} (P_h + \epsilon_h \eta_h)^2, (P_h + \epsilon_h \eta_h)_{h \in J_2}$$

(and possibly P_0). The degrees of these polynomials are at most $2d$ in $\mathbf{X}$, and at most 2 in $\mathbf{Y}$. We can use Theorem 16 to bound $b(V, \mathbb{Z}_2)$ by $H(2d, k_1, k_2, \mathrm{card}(J_2)+1)$ or $H(2d, k_1, k_2, \mathrm{card}(J_2)+2)$ (depending on whether P_0 is included or not) (cf. Eqn. (2.6)).

Moreover, the total number of non-empty real algebraic sets V that occur in the proof is bounded by

$$\sum_{\substack{0 \leq j_1 \leq k_1 \\ 0 \leq j_2 \leq \min(m+1, k_1+k_2-j_1-i)}} \binom{s}{j_1} \binom{m+1}{j_2} 4^{j_1+j_2}.$$

One now obtains inequality (2.11) by following the rest of the argument in the proof of Theorem 7.30. Note that we needed to increase the number of polynomials by one by including the polynomial P_0. This accounts for the $m+1$ in the subscript of the second sum in the bound.

The proof of inequality (2.12) is by a similar modification of the proof of Theorem 7.38 in [17] and is omitted. □

4.6. **Proofs of Theorems 20 and 21.**

Proof of Theorem 20. The proof is similar to that of Theorem 12. Since we have ℓ partially quadratic polynomials with several blocks, we use the bound $M_{\mathrm{gen}}(\mathbf{d}', k_1, k_2, j)$ in place of $G_{\mathrm{gen}}(\mathbf{d}', \mathbf{k}, j)$, where $M_{\mathrm{gen}}(\mathbf{d}', k_1, k_2, j)$ is the bound from Proposition 13, noting that we assume each $d_i \geq 2$. □

Proof of Theorem 21. The proof is similar to that of Theorem 13, except that the bound for the sum of the Betti numbers of a subset of size j is now given by $M(\mathbf{d}', k_1, k_2, j)$. □

5. A few applications

In this section we give a few applications of the results proved in the last section.

5.1. **Bounding Betti numbers of pull-backs and direct images under polynomial maps.**

We discuss a few immediate applications of the multi-degree bounds proved in Section 4.

Theorem 26 (Bound on pull-back)**.** *Let $\mathcal{F} = \{F_1, \ldots, F_m\} \subset \mathrm{R}[X_1, \ldots, X_k]$ and $\mathcal{G} \subset \mathrm{R}[Y_1, \ldots, Y_m]$, with $\deg(F) \leq d, F \in \mathcal{F}$, and $\deg(G) \leq D, G \in \mathcal{G}$, and let $\mathrm{card}(\mathcal{G}) = s$. Let $\mathbf{F} : \mathrm{R}^k \to \mathrm{R}^m$ denote the polynomial map $x \mapsto (F_1(x), \ldots, F_m(x))$, and let $S \subset \mathrm{R}^m$ be a $\mathcal{G}$-closed semi-algebraic set. Then,*

$$
\begin{aligned}
b(\mathbf{F}^{-1}(S), \mathbb{Z}_2) &\leq \sum_{i=0}^{k+m} \sum_{j=1}^{k+m-i} \binom{m+s+1}{j} 6^j G_{\min}(\mathbf{d}, \mathbf{k}, j) \\
&\leq O(1)^{k+m} (m+s)^{k+m} d^k D^m.
\end{aligned}
$$

Proof. Suppose that $\Phi(Y_1, \ldots, Y_m)$ is a $\mathcal{G}$-closed formula defining S. Notice that $\mathbf{F}^{-1}(S)$ is semi-algebraically homeomorphic to the semi-algebraic subset of R^{k+m} defined by the formula

$$
\Psi(\mathbf{X}, \mathbf{Y}) := \bigwedge_{i=1}^{m} (Y_i - F_i = 0) \wedge \Phi(Y_1, \ldots, Y_m).
$$

The number of polynomials appearing in Ψ is bounded by $m + s$. The degrees in $\mathbf{Y}$ of the polynomials appearing in Ψ are bounded by D, while the degrees in $\mathbf{X}$ are bounded by d. Applying Theorem 13 with $p = 2$, $\mathbf{k} = (k, m)$, and $\mathbf{d} = (d, D)$, we obtain

$$
\begin{aligned}
b(\mathrm{Reali}(\Psi, \mathrm{R}^{k+m}), \mathbb{Z}_2) &\leq \sum_{i=0}^{k+m} \sum_{j=1}^{k+m-i} \binom{m+s+1}{j} 6^j G_{\min}(\mathbf{d}, \mathbf{k}, j) \\
&\leq \sum_{i=0}^{k+m} \sum_{j=1}^{k+m-i} \binom{m+s+1}{j} 6^j O(1)^{k+m} d^k D^m \text{ (using (2.4))} \\
&\leq O(1)^{k+m} (m+s)^{k+m} d^k D^m.
\end{aligned}
$$

□

Theorem 27 (Bound on image)**.** *Let $\mathcal{F} = \{F_1, \ldots, F_m\}, \mathcal{G} \subset \mathrm{R}[X_1, \ldots, X_k]$, with $\deg(F) \leq d, F \in \mathcal{F}$, and $\deg(G) \leq D, G \in \mathcal{G}$, and let $\mathrm{card}(\mathcal{G}) = s$. Let $\mathbf{F} : \mathrm{R}^k \to \mathrm{R}^m$ denote the polynomial map $x \mapsto (F_1(x), \ldots, F_m(x))$, and let $T \subset \mathrm{R}^k$ be a bounded $\mathcal{G}$-closed semi-algebraic set. Suppose also that $d \geq D$.*

Then, for $0 \le i \le m$,

$$\begin{aligned} b_i(\mathbf{F}(T),\mathbb{Z}_2) &\le \sum_{j=0}^{i}\sum_{h=0}^{\alpha_j}\sum_{\ell=1}^{\alpha_j-h}\binom{(j+1)(m+s)+1}{\ell}6^\ell G_{\min}(\mathbf{d},\mathbf{k},\ell)\\ &\le O(i)^{\alpha_i}(m+s)^{\alpha_i}d^{(i+1)k}D^m \end{aligned}$$

where $\alpha_i = (i+1)k+m$.

Proof. Using the descent spectral sequence we have that

$$\begin{aligned} b_i(\mathbf{F}(T),\mathbb{Z}_2) &\le \sum_{j=0}^{i} b_{i-j}(\underbrace{T\times_{\mathbf{F}}\cdots\times_{\mathbf{F}}T}_{(j+1)},\mathbb{Z}_2)\\ &\le \sum_{j=0}^{i} b(\underbrace{T\times_{\mathbf{F}}\cdots\times_{\mathbf{F}}T}_{(j+1)},\mathbb{Z}_2). \end{aligned} \tag{5.1}$$

Suppose that T is defined by a $\mathcal{G}$-closed formula Ψ. Notice that for all $j \ge 0$, $\underbrace{T\times_{\mathbf{F}}\cdots\times_{\mathbf{F}}T}_{(j+1)}$ is defined by the formula

$$\Psi^{(j)}(\mathbf{X}^{(0)},\ldots,\mathbf{X}^{(j)},\mathbf{Y}) := \bigwedge_{i=1}^{m}\bigwedge_{h=0}^{j}\Psi(\mathbf{X}^{(h)},\mathbf{Y})\wedge(Y_i - F_i(\mathbf{X}^{(h)})=0),$$

where $\mathbf{Y}=(Y_1,\ldots,Y_m), \mathbf{X}^{(h)}=(X_1^{(h)},\ldots,X_k^{(h)}), 0\le h\le j$.

The cardinality of the set of polynomials appearing in $\Psi^{(j)}$ is $(j+1)(m+s)$, the degree in each block $\mathbf{X}^{(h)}$ is bounded by d, and that in $\mathbf{Y}$ is bounded by D.

Denote by $\alpha_j = (j+1)k+m$. Now apply Theorem 13 with $p=j+2$, $\mathbf{k} = (\underbrace{k,\ldots,k}_{j+1},m)$, $\mathbf{d} = (\underbrace{d,\ldots,d}_{j+1},D)$ to obtain

$$\begin{aligned} b(\underbrace{T\times_{\mathbf{F}}\cdots\times_{\mathbf{F}}T}_{(j+1)},\mathbb{Z}_2) &\le \sum_{h=0}^{\alpha_j}\sum_{\ell=1}^{\alpha_j-h}\binom{(j+1)(m+s)+1}{\ell}6^\ell G_{\min}(\mathbf{d},\mathbf{k},\ell)\\ &\le O(j)^{\alpha_j}(m+s)^{\alpha_j}d^{(j+1)k}D^m \end{aligned} \tag{5.2}$$

The theorem now follows from Eqns. (5.1) and (5.2). ☐

Remark 25. Note that versions of Theorem 26 and Theorem 27 without the distinction between the two degrees d and D were known before (see [27]). The novel aspect of these two theorems is the different dependence of the bounds proved on the two degrees d, D. In certain applications, this distinction is important.

We record the following result similar to that of Theorem 27, which is also useful in practice. The following situation occurs very frequently in semi-algebraic geometry.

Let $S_1 \subset \mathrm{R}^k$, and $S_2 \subset \mathrm{R}^k\times\mathrm{R}^m$ be semi-algebraic subsets, and $\pi_{\mathbf{X}} : \mathrm{R}^k\times\mathrm{R}^m \to \mathrm{R}^k, \pi_{\mathbf{Y}} : \mathrm{R}^k\times\mathrm{R}^m \to \mathrm{R}^m$ be the two projection maps on the first and second factors resp.. Let $T = \pi_{\mathbf{Y}}(\pi_{\mathbf{X}}^{-1}(S_1) \cap S_2)$ (see figure below).

$$\begin{array}{ccc} & \pi_{\mathbf{X}}^{-1}(S_1) \cap S_2 & \\ {}^{\pi_{\mathbf{X}}}\swarrow & & \searrow^{\pi_{\mathbf{Y}}} \\ S_1 & & T. \end{array}$$

Theorem 28 (Set-theoretic Fourier-Mukai transform)**.** *With the same notation as above, let $\mathcal{P}_1 \subset \mathrm{R}[X_1, \ldots, X_k]$ and $\mathcal{P}_2 \subset \mathrm{R}[X_1, \ldots, X_k, Y_1, \ldots, Y_m]$ be finite sets such that $S_1 \subset \mathrm{R}^k$ is a $\mathcal{P}_1$-closed semi-algebraic set and $S_2 \subset \mathrm{R}^{k+m}$ is a bounded $\mathcal{P}_2$-closed semi-algebraic set. Suppose that $\deg_{\mathbf{X}}(\mathcal{P}_1), \deg_{\mathbf{X}}(\mathcal{P}_2) \leq d$ and $\deg_{\mathbf{Y}}(\mathcal{P}_2) \leq D$. Suppose also that $d \geq D$.*

Then for $0 \leq i \leq m$,

$$\begin{aligned} b_i(T, \mathbb{Z}_2) &\leq \sum_{j=0}^{i} \sum_{h=0}^{\alpha_j} \sum_{\ell=1}^{\alpha_j - h} \binom{(j+1)(s_1+s_2)+1}{\ell} 6^{\ell} G_{\min}(\mathbf{d}, \mathbf{k}, \ell) \\ &\leq O(i)^{\alpha_i} (s_1+s_2)^{\alpha_i} d^{(i+1)k} D^m, \end{aligned}$$

where $\alpha_i = (i+1)(k+m)+k$, and $s_1 = \mathrm{card}(\mathcal{P}_1), s_2 = \mathrm{card}(\mathcal{P}_2)$.

Proof. Note that the semi-algebraic set $\pi_{\mathbf{X}}^{-1}(S_1) \cap S_2$ is a bounded $(\mathcal{P}_1 \cup \mathcal{P}_2)$-closed semi-algebraic set, with degrees in $\mathbf{X}$ bounded by d and in $\mathbf{Y}$ bounded by D. Note that if S_1 is defined by the formula $\Phi(X_1, \ldots, X_k)$ and S_2 is defined by the formula $\Psi(X_1, \ldots, X_k, Y_1, \ldots, Y_m)$, then the set $\pi_{\mathbf{X}}^{-1}(S_1) \cap S_2$ is defined by $\Phi(X_1, \ldots, X_k) \wedge \Psi(X_1, \ldots, X_k, Y_1, \ldots, Y_m)$. Note that with the above notation, for all $j \geq 0$, $\underbrace{T \times_{\pi_{\mathbf{Y}}} \cdots \times_{\pi_{\mathbf{Y}}} T}_{(j+1)}$ is defined by the formula

$$\Theta^{(j)}(\mathbf{X}^{(0)}, \ldots, \mathbf{X}^{(j)}, \mathbf{Y}) := \bigwedge_{h=0}^{j} \Phi(\mathbf{X}^{(h)}, \mathbf{Y}) \wedge \Psi(\mathbf{X}^{(h)}, \mathbf{Y}),$$

where $\mathbf{Y} = (Y_1, \ldots, Y_m), \mathbf{X}^{(h)} = (X_1^{(h)}, \ldots, X_k^{(h)}), 0 \leq h \leq j$.

The cardinality of the set of polynomials appearing in $\Theta^{(j)}$ is $(j+1)(s_1+s_2)$, the degree in each block $\mathbf{X}^{(h)}$ is bounded by d, and that in $\mathbf{Y}$ is bounded by D.

Applying Theorem 27 with $\alpha_j = (j+1)k + m$ we get

$$\begin{aligned} b(\underbrace{T \times_{\pi_{\mathbf{Y}}} \cdots \times_{\pi_{\mathbf{Y}}} T}_{(j+1)}, \mathbb{Z}_2) &\leq \sum_{h=0}^{\alpha_j} \sum_{\ell=1}^{\alpha_j - h} \binom{(j+1)(s_1+s_2)+1}{\ell} 6^{\ell} G_{\min}(\mathbf{d}, \mathbf{k}, \ell) \\ &\leq O(j)^{\alpha_j} (s_1+s_2)^{\alpha_j} d^{(j+1)k} D^m. \qquad (5.3) \end{aligned}$$

The theorem follows from the inequality (5.1). □

5.2. **An application to discrete geometry.** The theory of transversals is a very well-studied topic in discrete geometry with many applications. Suppose that $S \subset \mathrm{R}^k$ is a closed and bounded semi-algebraic set. We define the space $\mathrm{Transversal}_{k'}(S) \subset \mathrm{AffGr}_{k,k'}(\mathrm{R})$ to be the set of k'-dimensional affine subspaces ℓ of R^k such that $\ell \cap S \neq \emptyset$ (where we denote by $\mathrm{AffGr}_{k,k'}(\mathrm{R})$ the space (the affine Grassmannian) of k'-dimensional affine subspaces of R^k . Upper bounds on the topology of such spaces of transversals are important in discrete geometry (see for example [30]).

We prove the following theorem which improves the bound that one obtains using previously known methods by exploiting the multi-degree bounds proved in the current paper (see Remark 26).

Theorem 29. *Let $S \subset \mathrm{R}^k$ be a bounded $\mathcal{P}$-closed semi-algebraic set, where $\mathcal{P} \subset \mathrm{R}[X_1, \ldots, X_k]$ with $\deg(P) \leq d, P \in \mathcal{P}$, and $\mathrm{card}(\mathcal{P}) = s$. Then, for all $k', 0 \leq k' \leq k$, $b_i(\mathrm{Transversal}_{k'}(S), \mathbb{Z}_2)$ is bounded by*

$$\begin{aligned} & \sum_{j=0}^{i} \sum_{h=0}^{\alpha_j} \sum_{\ell=1}^{\alpha_j - h} \binom{(j+1)(s+m+2(k+1))+1}{\ell} 6^\ell G_{\min}(\mathbf{d}, \mathbf{k}, \ell) \\ \leq \quad & O(i)^{\alpha_i} (s+m+2(k+1))^{\alpha_i} d^{(i+1)k}, \end{aligned}$$

where $\mathbf{k} = (\underbrace{k, \ldots, k}_{j+1}, m)$, $\mathbf{d} = (\underbrace{d, \ldots, d}_{j+1}, 2)$, *and* $\alpha_i = (i+1)k + m$, *with* $m = (k+1)(k+2)/2 - 1$.

Proof. We first identify $\mathrm{AffGr}_{k,k'}(\mathrm{R})$ with the real Grassmannian $\mathrm{Gr}_{k+1,k'+1}(\mathrm{R})$ of $(k'+1)$-dimensional subspaces of R^{k+1} in the standard way, identifying $\ell \in \mathrm{AffGr}(k, k')$ with the linear hull of $\ell' = \{(x, 1) \mid x \in \ell' \subset \mathrm{R}^k\} \subset \mathrm{R}^{k+1}$. Similarly, let $S_1 = \{(x, 1) \mid x \in S\} \subset \mathrm{R}^k$. The set $\mathrm{Transversal}_{k'}(S)$ can then be identified with the space (which we also denote by $\mathrm{Transversal}_{k'}(S)$)

$$\{\ell' \in \mathrm{Gr}_{k+1,k'+1}(\mathrm{R}) \mid \ell \cap S_1 \neq \emptyset\}.$$

Now $\mathrm{Gr}_{k+1,k'+1}(\mathrm{R})$ is semi-algebraically homeomorphic to the real affine variety defined by

$$\{A \in \mathrm{R}^{(k+1)\times(k+1)} \mid A^t = A, A^2 = A, \mathrm{Tr}(A) = k'+1\}. \tag{5.4}$$

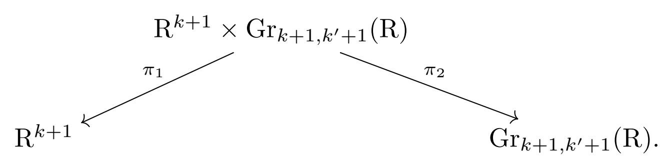
(see for example [21, Theorem 3.4.4]).

We identify $\mathrm{Gr}_{k+1,k'+1}(\mathrm{R}) \subset \mathrm{R}^{(k+1)(k+2)/2-1}$ with the subset of the linear subspace of the space of $(k+1)\times(k+1)$ symmetric matrices with entries in R having trace $k'+1$ (notice that the subspace containing $\mathrm{Gr}_{k+1,k'+1}(\mathrm{R})$ has dimension $(k+1)(k+2)/2-1$ and that the degrees of the polynomials in $(k+1)(k+2)/2-1$ variables defining $\mathrm{Gr}_{k+1,k'+1}$ are all bounded by 2).

Let $S_2 \subset \mathrm{R}^k \times \mathrm{R}^{(k+1)(k+1)/2-1}$ be the semi-algebraic set (the total space of the tautological bundle over $\mathrm{Gr}_{k+1,k'+1}(\mathrm{R})$) defined by

$$S_2 = \{(x, A) \mid x \in \mathrm{R}^k, A \in \mathrm{Gr}_{k+1,k'+1}(\mathrm{R}), Ax' = x', x' = (x, 1)\}. \tag{5.5}$$

Let π_1, π_2 be the projection maps as depicted in the following figure.



Observe that

$$\mathrm{Transversal}_{k'}(S) = \pi_2(\pi_1^{-1}(S_1) \cap S_2).$$

Now apply Theorem 28 noting that the number of polynomial equations (each of degree 2) used to define $\mathrm{Gr}_{k+1,k'+1}(\mathrm{R})$ in Eqn. (5.4) is equal to $k+m+1$, where $m = (k+1)(k+2)/2-1$, and hence the number of equations (each of degree at most 2) used in the definition of S_2 in Eqn. (5.5) is equal to $m+2(k+1)$. □

Remark 26. Note that if we used the more standard Plücker embedding of the Grassmannian $\mathrm{Gr}_{k+1,k'+1}(\mathrm{R})$ in the projective space $\mathbb{P}(\bigwedge^{k'+1}\mathrm{R}^{k+1})$ of dimension $\binom{k+1}{k'+1}-1$, we would obtain a bound which is doubly exponential in k in the worst case. The fact that over a real closed field, the Grassmannians are semi-algebraically homeomorphic to the real affine variety described in Eqn. (5.4) allows us to obtain a much better bound (which is only singly exponential in k). Secondly, if we used the best known prior results on effective quantifier elimination to estimate $b_i(\mathrm{Transversal}_{k'}(S),\mathbb{Z}_2)$ from above, we would obtain a bound of $(O(ksd))^{km}$ which has a much worse dependence on d than the bound proved in Theorem 29.

6. Bound on the Betti numbers of real varieties defined by two polynomials having different degrees

6.1. **Background.** It was mentioned in the introduction that quantitative bounds on the Betti numbers (in particular, on the 0-th Betti number) has proved to be important tools in several areas. More recently, triggered by the development of a new technique in discrete geometry (namely, the *polynomial partitioning* method) it became necessary to prove bounds which has a finer dependence on the degree *sequence* of the polynomials rather than on the maximum degree (as in Theorem 2). The following theorem (conjectured by J. Matoušek [42]) was proved in [9] to meet the needs of discrete geometry and has already found several applications.

Theorem 30. [9] *Let $\mathcal{Q},\mathcal{P}\subset \mathrm{R}[X_1,\ldots,X_k]$ be finite subsets of non-zero polynomials such that $\deg(Q)\le d_1$ for all $Q\in\mathcal{Q}$, $\deg P = d_2$ for all $P\in\mathcal{P}$, and suppose that $d_1\le d_2$. Suppose that the real dimension of $\mathrm{Zer}(\mathcal{Q},\mathrm{R}^k)$ is $k'\le k$, and that $\mathrm{card}(\mathcal{P})=s$.*

Then,

$$\sum_{\sigma\in\{0,1,-1\}^{\mathcal{P}}} b_0(\mathrm{Reali}(\sigma,\mathrm{Zer}(\mathcal{Q},\mathrm{R}^k)),\mathbb{Z}_2)$$

is at most

$$\sum_{j=0}^{k'} 4^j\binom{s+1}{j}\left(\binom{k+1}{k-k'+j+1}(2d_1)^{k-k'}d^j\ \max\{2d_1,d_2\}^{k'-j}+2(k-j+1)\right).$$

In particular,

$$\sum_{\sigma\in\{0,1,-1\}^{\mathcal{P}}} b_0(\mathrm{Reali}(\sigma,\mathrm{Zer}(\mathcal{Q},\mathrm{R}^k)),\mathbb{Z}_2) \quad\le\quad O(1)^k(sd_2)^{k'}d_1^{k-k'}. \tag{6.1}$$

Theorem 30 has proved to be important in incidence questions in discrete geometry [49, 57, 48, 18]. Even though in these applications it is usually a bound on the number of semi-algebraically connected components of semi-algebraic sets defined by polynomials of possibly different degrees that is important, it is a very interesting mathematical question (asked already in [9]) if the inequality (6.1) in Theorem 30 can be extended to a bound on the higher Betti numbers. We formulate below a more precise conjecture.

Conjecture 1. *With the same notation and hypothesis as in Theorem 30, for all $i, 0\le i\le k'$,*

$$\sum_{\sigma\in\{0,1,-1\}^{\mathcal{P}}} b_i(\mathrm{Reali}(\sigma,\mathrm{Zer}(\mathcal{Q},\mathrm{R}^k)),\mathbb{Z}_2) \quad\le\quad O(1)^k s^{k'-i}d_1^{k-k'}d_2^{k'}. \tag{6.2}$$

At present we do not know how to prove Conjecture 1 except in the case $i = 0$, which is Theorem 30, and the techniques used in proving Theorem 30 do not easily extend to the case of $i > 0$. In this paper, we make some progress on this problem by proving Conjecture 1 for all $i \geq 0$, but only in the special case when $k' = k - 1$. In fact we prove the following slightly stronger theorem.

Unlike in the previous sections the bounds stated in this section will be valid for Betti numbers with coefficients in an arbitrary field $\mathbb{F}$ rather than just $\mathbb{Z}_2$. This is because we do not use Smith inequality in our proofs.

Theorem 31. *With the same notation and hypothesis as in Theorem 30, for all $i, 0 \leq i \leq k' < k$, and any field of coefficients $\mathbb{F}$,*

$$\sum_{\sigma \in \{0,1,-1\}^{\mathcal{P}}} b_i(\mathrm{Reali}(\sigma, \mathrm{Zer}(\mathcal{Q}, \mathrm{R}^k)), \mathbb{F})$$

is bounded by

$$\sum_{j=1}^{k'-i} \binom{s}{j} 4^j (F(2d_1, 2d_2, k) + F(2d_1, 2d_2, k-1) + 1) \leq O(1)^k s^{k'-i} d_1 d_2^{k-1}, \tag{6.3}$$

where

$$F(d_1, d_2, k) = \binom{k+1}{2} d_1 \left((d_1 - 1)^{k-1} + \frac{4(k-1)}{3} d_2 (d_2 - 1)^{k-2} \right).$$

The rest of this section is devoted to the proof Theorem 31. We begin as usual with the algebraic case.

6.2. **The algebraic case.** In this section we prove a nearly optimal bound on the sum of the Betti numbers of a real variety $V \subset \mathrm{R}^k$ defined by two polynomials of possibly differing degrees $d_1 \leq d_2$. We prove that

$$b(V, \mathbb{F}) \leq O(1)^k d_1 d_2^{k-1}.$$

The above bound follows from the following more precise theorem.

Theorem 32. *Let $P_1, P_2 \in \mathrm{R}[X_1, \ldots, X_k]$, with $0 < \deg(P_1) \leq d_1, \deg(P_2) \leq d_2, 2 \leq d_1 \leq d_2$, and $V = \mathrm{Zer}(\{P_1, P_2\}, \mathrm{R}^k)$. Then,*

$$b(V, \mathbb{F}) \leq F(d_1, d_2, k) + F(d_1, d_2, k-1) + 1,$$

where

$$F(d_1, d_2, k) = \binom{k+1}{2} d_1 \left((d_1 - 1)^{k-1} + \frac{4(k-1)}{3} d_2 (d_2 - 1)^{k-2} \right).$$

In particular,

$$b(V, \mathbb{Z}_2) \leq 8 \binom{k+1}{3} d_1 d_2 (d_2 - 1)^{k-2}.$$

Remark 27. Notice that direct application of Theorem 11 would yield a bound of $O(d_2)^k$ which is not optimal if $d_1 \ll d_2$.

We now prove Theorem 32. The proof involves several steps and utilizes a few results from stratified Morse theory that we recall first.

6.2.1. *Stratified Morse Theory.* We follow the exposition in [31] (and also [13].

A Whitney stratification of a space X is a decomposition of X into sub-manifolds called strata, which satisfy certain frontier conditions, (see [31] page 37). In particular, given a compact set bounded by a smooth algebraic hypersurface, the boundary and the interior form a Whitney stratification.

Now, let S be a compact Whitney stratified subset of $\mathbb{R}^k$, and f a restriction to S of a smooth function. A *critical point* of f is defined to be a critical point of the restriction of f to any stratum, and a critical value of f is the value of f at a critical point. A function is called a Morse function if it has only non-degenerate critical points when restricted to each stratum, and all its critical values are distinct. (There is an additional non-degeneracy condition which states that the differential of F at a critical point p of a strata S should not annihilate any limit of tangent spaces to a stratum other than S. However, in our situation this will always be true.)

We now assume that $S \subset \mathbb{R}^k$ is a Whitney-stratified set, and suppose that $f : S \to \mathrm{R}$ is a Morse function. We denote S_x (resp. $S_{\leq x}$) denote $S \cap f^{-1}(x)$ (resp. $S \cap f^{-1}((-\infty, x]))$.

The first fundamental result of stratified Morse theory is the following.

Theorem 33. [31] *As c varies in the open interval between two adjacent critical values, the topological type of $S \cap \pi^{-1}((-\infty, c])$ remains constant.*

Stratified Morse theory actually gives a recipe for describing the topological change in $S_{\leq c}$ as c crosses a critical value of f. This is given in terms of *Morse data*, which consists of a pair of topological spaces $(A, B), A \supset B$, with the property that as c crosses the critical value $v = f(p)$, the change in $S_{\leq c}$ can be described by gluing in A along B.

In stratified Morse theory the Morse data is presented as a product of two pairs, called the *tangential Morse data* and the *normal Morse data*. The notion of product of pairs is the standard one in topology, namely

$$(A, B) \times (A', B') = (A \times A', A \times B' \cup B \times A').$$

Definition 2 (Tangential Morse data [31])**.** The tangential Morse data at a critical point p is then given by $(B^\lambda \times B^{k-\lambda}, (\partial B^\lambda) \times B^{k-\lambda})$ where B^k is the closed k-dimensional disk, ∂ is the boundary map, and λ is the index of the Hessian matrix of f (in any local co-ordinate system of the stratum containing p in a neighborhood of p) of f (restricted to the stratum containing p) at p.

Definition 3 (Normal Morse data [31])**.** Let p be a critical point in some k'-dimensional stratum Z of a stratified subset S of R^k.

Let N' be any $(k-k')$-dimensional hyperplane passing through the point p which is transverse to Z which intersects the stratum Z locally at the single point p.

Then, the *normal slice*, $N(p)$ at the point p is defined to be,

$$N(p) = N' \cap S \cap \overline{B_k(p, \delta)},$$

for sufficiently small $\delta > 0$.

Choose $\delta \gg \epsilon > 0$, and let $\ell^- = N(p) \cap f^{-1}(f(p) - \epsilon)$. The normal Morse data has the homotopy type of the pair $(\mathrm{cone}(\ell^-), \ell^-)$.

The following theorem measures the change in topology as we cross a critical value.

Theorem 34. [31, page 69] *Let $[a,b] \subset \mathbb{R}$ an interval which contains no critical values except for an isolated critical value $v \in (a,b)$ which corresponds to a critical point p of f restricted to some stratum Z of S. Let λ be the Morse index of the critical point p, Then, the space $S_{\leq b}$ has the homotopy type of a space which is obtained from $S_{\leq a}$ by attaching the pair $(B^\lambda, \partial B^\lambda) \times (\mathrm{cone}(\ell^-), \ell^-)$.*

We will need to use Theorem 34 in the following particularly simple situation. Let $S \subset \mathrm{R}^k$ be a closed and bounded semi-algebraic set defined by $\bigwedge_{P\in\mathcal{P}}(P = 0) \wedge Q \geq 0$, where $\mathcal{P} \cup \{Q\} \subset \mathrm{R}[X_1,\ldots,X_k]$ such that $\mathrm{Zer}(P,\mathrm{R}^k), \mathrm{Zer}(Q,\mathrm{R}^k), P \in \mathcal{P}$ are non-singular hypersurfaces intersecting transversally. T hen S is Whitney stratified with two strata – namely, $Z = \mathrm{Zer}(\mathcal{P} \cup \{Q\}, \mathrm{R}^k)$ and $Z' = S \setminus Z$. Suppose that f is a Morse function on the stratified set S, and moreover f restricted to $\mathrm{Zer}(\mathcal{P},\mathrm{R}^k)$ has no critical points that belong to S. We prove the following theorem as a consequence of Theorems 33 and 34 above.

Theorem 35. *With the assumptions stated above, $b(S,\mathbb{Z}_2)$ is bounded by the number of critical points of f restricted to $\mathrm{Zer}(\mathcal{P} \cup \{Q\}, \mathrm{R}^k)$.*

Proof. We note first that it suffices to prove the theorem for $\mathrm{R} = \mathbb{R}$. The general case then follows after a standard application of the Tarski-Seidenberg transfer principle. Let $p \in \mathrm{R}^k$ be a critical point of f restricted to $\mathrm{Zer}(\mathcal{P} \cup \{Q\}, \mathrm{R}^k)$ and without loss of generality let $p = \mathbf{0}$. Let $W = T_p\mathrm{Zer}(\mathcal{P},\mathrm{R}^k)$ and $V = T_p(\mathrm{Zer}(\mathcal{P} \cup \{Q\}, R^k)$, and we have V is subspace of W of codimension one. Since p is a non-degenerate critical point of f restricted to $\mathrm{Zer}(\mathcal{P} \cup \{Q\}, \mathrm{R}^k)$, but not of $\mathrm{Zer}(\mathcal{P},\mathrm{R}^k)$, the linear form df vanishes on W, but not on V. Let $\mathbf{u}$ (resp. $\mathbf{v}$) denote the orthogonal projection of $\mathrm{grad}(Q)(p)$ (resp. $\mathrm{grad}(f)(p)$) to V. Note that $\mathbf{u}, \mathbf{v} \neq 0$. There are two cases to consider. We denote by $(\cdot,\cdot)$ the standard inner product in $\mathbb{R}^k$.

(a) $(\mathbf{u},\mathbf{v}) > 0$: In this case following the notation in Definition 3, $\ell^- = \emptyset$, and it follows from Definition 3 that the normal Morse data at p equals $(p,\emptyset)$, and hence the product of the tangential and the normal Morse data equals the tangential Morse data in this case. Thus, in this case the change in $b(S_{\leq c})$ as c crosses $f(p)$ is ± 1 as in ordinary Morse theory.
(b) $(\mathbf{u},\mathbf{v}) < 0$. In this case the normal Morse data is homotopy equivalent to the pair $([0,1],\{0\})$. Since the product $(B^\lambda,\partial B^\lambda) \times ([0,1],\{0\})$ where λ is the index of the critical point of p, the Morse data is homotopy equivalent to $(*,*)$. Thus in this case there is no change in the homotopy type of the sublevel set $S_{\leq c}$ as c crosses the critical value $f(p)$ (using Theorem 34) as the pair that is being added is contractible.

The theorem now follows from Theorems 33 and 34 just as in the case of usual Morse theory. □

6.2.2. *Summary of the ideas behind the proof of Theorem 32.* For simplicity lets assume that $V = \mathrm{Zer}(\{P_1,P_2\},\mathrm{R}^k)$ is bounded. The case of unbounded V introduces an additional complication which we ignore in this informal summary. We replace V by a closed bounded semi-algebraic subset $S \subset \mathrm{R}\langle\varepsilon_1,\varepsilon_2\rangle^k$ defined by $-\varepsilon_i \leq P_i \leq \varepsilon_i, i = 1,2$. Then, S is semi-algebraically homotopy equivalent to V, and moreover S is a topological manifold whose boundary is a union of basic closed semi-algebraic sets, S_1, S_2, where S_1 is defined by $P_1^2 - \varepsilon_1^2 = 0 \wedge -\varepsilon_2 \leq P_2 \leq \varepsilon_2$,

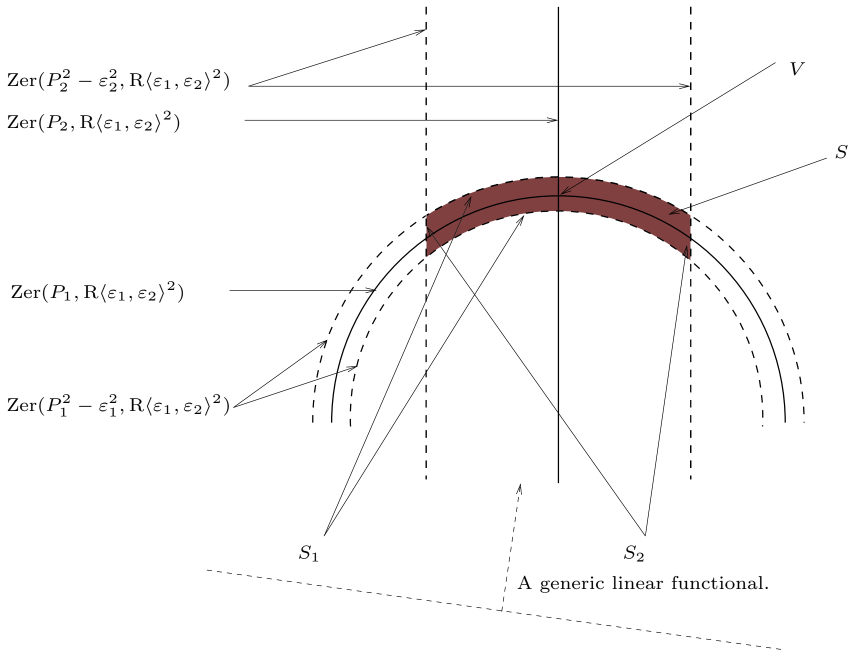


FIGURE 2. An Illustrative figure in the plane.

and S_2 is defined by $P_2^2 - \varepsilon_2^2 = 0 \land -\varepsilon_1 \leq P_1 \leq \varepsilon_1$. Figure 2 gives a schematic diagram of all these sets. Using Alexander duality, in order to bound $b(S, \mathbb{F})$ it suffices to bound $b(\partial S, \mathbb{F})$ (see Lemma 5 below). Now, in order to bound $b(S_1 \cup S_2, \mathbb{F})$ it suffices to bound (using inequality (3.1)) $b(S_1, \mathbb{F}), b(S_2, \mathbb{F})$ as well as $b(S_1 \cap S_2, \mathbb{F})$ (see Lemma 6 below).

The techniques used for bounding each of the above quantities are distinct. We bound $b(S_1, \mathbb{F})$ by first reducing the problem to bounding $b(\partial S_1, \mathbb{F})$ and bounding $b(\mathrm{Zer}(P_1^2 - \varepsilon_1, \mathrm{R}\langle\varepsilon_1, \varepsilon_2\rangle^k), \mathbb{F})$ using inequality (3.1), and then using Corollary 2 to bound these quantities (see Proposition 15).

In order to bound $b(S_2, \mathbb{F})$, we observe that a generic linear functional has no critical points in the relative interior of S_2 (see Lemmas 7 and 8, and 9). Note that this fact is not necessarily true for S_1. This fact allows us to bound $b(S_2, \mathbb{F})$ by counting the critical points of the functional on its boundary strata using Theorem 35 (see Lemma 10). Finally, we bound the number of such critical points using Proposition 5 (see Lemma 11). The number of such critical points also gives an upper bound on $b(S_1 \cap S_2, \mathbb{F})$ (Corollary 2).

6.2.3. *Proof of Theorem 32.* For the rest of this section we keep the same notation as in Theorem 32.

Let $\mathrm{R}_0 = \mathrm{R}\langle\varepsilon_0\rangle, \mathrm{R}_1 = \langle\varepsilon_0, \varepsilon_1\rangle, \mathrm{R}_2 = \mathrm{R}\langle\varepsilon_0, \varepsilon_1, \varepsilon_2\rangle$, and let $W_{k+1} \subset \mathrm{R}_0^{k+1}$ denote the real variety defined by the polynomials P_1, P_2 and $Q_{k+1} = \varepsilon_0 \sum_{i=1}^{k+1} X_i^2 - 1$, and $W_k \subset \mathrm{R}_0^k$ the real variety defined by the polynomials P_1, P_2 and $Q_k = \varepsilon_0 \sum_{i=1}^{k} X_i^2 - 1$.

It follows from [21, Corollary 9.3.7] (Local Conic Structure at infinity of semi-algebraic sets) that, the intersection, W_{k+1}^+ (resp. W_{k+1}^-) of W_{k+1} with the closed half-space defined by $X_{k+1} \geq 0$ (resp. $X_{k+1} \leq 0$) are each semi-algebraically homeomorphic to V.

$$\begin{aligned} W^+_{k+1} \cup W^-_{k+1} &= W_{k+1}, \\ W^+_{k+1} \cap W^-_{k+1} &= W_k. \end{aligned}$$

It now follows from inequality (3.3) that:

Proposition 14.

$$b(V, \mathbb{F}) \leq \frac{1}{2}\left(b(W_k, \mathbb{F}) + b(W_{k+1}, \mathbb{F})\right).$$

We now bound $b(W_{k+1}, \mathbb{F})$ (the proof for the bound on $b(W_k, \mathbb{F})$ is very similar). Consider the closed and bounded semi-algebraic set, $\tilde{W}_{k+1} \subset \mathrm{R}_2^{k+1}$ defined by $Q_{k+1} = 0, -\varepsilon_1 \leq P_1 \leq \varepsilon_1, -\varepsilon_2 \leq P_2 \leq \varepsilon_2$. Let for $i = 1, 2$,

$$\begin{aligned} Z_i^+ &= \mathrm{Zer}(\{Q_{k+1}, P_i + \varepsilon_i\}, \mathrm{R}_i^{k+1}), \\ Z_i^- &= \mathrm{Zer}(\{Q_{k+1}, P_i - \varepsilon_i\}, \mathrm{R}_i^{k+1}), \\ S_1^+ &= \mathrm{Ext}(Z_1^+, \mathrm{R}_2) \cap \tilde{W}_{k+1}, \\ S_1^- &= \mathrm{Ext}(Z_1^-, \mathrm{R}_2) \cap \tilde{W}_{k+1}, \\ S_2^+ &= Z_2^+ \cap \tilde{W}_{k+1}, \\ S_2^- &= Z_2^- \cap \tilde{W}_{k+1}. \end{aligned}$$

Lemma 4. *The semi-algebraic set $\tilde{W}_{k+1}$ is semi-algebraically homotopy equivalent to* $\mathrm{Ext}(W_{k+1}, \mathrm{R}')$. *In particular,*

$$b(\tilde{W}_{k+1}, \mathbb{F}) = b(W_{k+1}, \mathbb{F}).$$

Proof. Clearly $\tilde{W}_{k+1}$ is closed and bounded over R', and $\lim_{\varepsilon_1} \tilde{W}_{k+1}$. The lemma now follows from [17, Lemma 16.17]. □

Lemma 5.

$$b(\tilde{W}_{k+1}, \mathbb{F}) \leq \frac{1}{2}\left(b(\bigcup_{\substack{i=1,2 \\ \epsilon \in \{+,-\}}} S_i^\epsilon, \mathbb{F}) + 1 \right).$$

Proof. Let $\tilde{W}'_{k+1}$ be the closure of the semi-algebraic set $\mathbf{S}^k(0, \varepsilon_0^{-1/2}) \setminus \tilde{W}_{k+1}$. Then,

$$\begin{aligned} \tilde{W}_{k+1} \cup \tilde{W}'_{k+1} &= \mathbf{S}^k(0, \varepsilon_0^{-1/2}), \\ \tilde{W}_{k+1} \cap \tilde{W}'_{k+1} &= \bigcup_{\substack{i=1,2 \\ \epsilon \in \{+,-\}}} S_i^\epsilon. \end{aligned}$$

We also have that $\tilde{W}'_{k+1}$ is semi-algebraically homotopy equivalent to $\mathbf{S}^k(0, \varepsilon_0^{-1/2}) \setminus \tilde{W}_{k+1}$, and hence by [50, page 296] (Alexander duality)

$$b(\tilde{W}_{k+1}, \mathbb{F}) = b(\tilde{W}'_{k+1}, \mathbb{F}) - 1. \tag{6.4}$$

Also, using inequality (3.1) we have

$$b(\tilde{W}_{k+1}, \mathbb{F}) + b(\tilde{W}'_{k+1}, \mathbb{F}) \leq b(\bigcup_{\substack{i=1,2 \\ \epsilon \in \{+,-\}}} S_i^\epsilon, \mathbb{F}) + 2. \tag{6.5}$$

The lemma now follows from (6.4) and (6.5). □

Lemma 6.

$$b(\bigcup_{\substack{i=1,2\\ \epsilon\in\{+,-\}}} S_i^{\epsilon},\mathbb{F}) \leq \sum_{\substack{i=1,2\\ \epsilon\in\{+,-\}}} b(S_i^{\epsilon},\mathbb{F}) + \sum_{\epsilon_1,\epsilon_2\in\{+,-\}} b(S_1^{\epsilon_1}\cap S_2^{\epsilon_2},\mathbb{F}).$$

Proof. Apply inequality (3.1). □

We need separate arguments to bound $b(\bigcup_{\epsilon_1\in\{+,-\}} S_1^{\epsilon},\mathbb{F})$ and $b(\bigcup_{\epsilon_2\in\{+,-\}} S_2^{\epsilon_2},\mathbb{F})$. We first bound $b(\bigcup_{\epsilon_1\in\{+,-\}} S_1^{\epsilon_1},\mathbb{F})$.

Proposition 15.

$$b(\bigcup_{\epsilon_1\in\{+,-\}} S_1^{\epsilon_1},\mathbb{F}) \quad\leq\quad 4\binom{k+1}{2}d_1\left(\frac{2(k-1)}{3}d_2(d_2-1)^{k-2}+(d_1-1)^{k-1}\right).$$

Proof. Using inequality (3.1) and Corollary 2 we have

$$\begin{aligned} b(\bigcup_{\epsilon_1\in\{+,-\}} S_1^{\epsilon_1},\mathbb{F}) &\leq \sum_{\epsilon_1\in\{+,-\}} (b(\partial S_1^{\epsilon_1},\mathbb{F})+b(Z_1^{\epsilon_1},\mathbb{F}))\\ &= \sum_{\epsilon_1,\epsilon_2\in\{+,-\}} b(S_2^{\epsilon_2}\cap S_1^{\epsilon_1},\mathbb{F}) + \sum_{\epsilon_1\in\{+,-\}} b(Z_1^{\epsilon_1},\mathbb{F})\\ &\leq 8\binom{k+1}{3}d_1d_2(d_2-1)^{k-2}+4\binom{k+1}{2}d_1(d_1-1)^{k-1}\\ &= 4\binom{k+1}{2}d_1\left(\frac{2(k-1)}{3}d_2(d_2-1)^{k-2}+(d_1-1)^{k-1}\right). \end{aligned}$$

□

Next we bound $b(\bigcup_{\epsilon_2\in\{+,-\}} S_2^{\epsilon_2},\mathbb{F})$ as follows.

Lemma 7. *There exists a linear functional $F:\mathrm{R}_1^{k+1}\to\mathrm{R}_1$, such that the set of critical points of F restricted to $Z_1^{\pm}$ has an empty intersection with* $\mathrm{Zer}(P_2,\mathrm{R}_1^{k+1})$.

Proof. The semi-algebraic subset $T\subset \mathrm{Gr}_{k+1,k}(\mathrm{R})$ defined by,

$$T := \cup_{x\in Z_1^{\pm}\cap\mathrm{Zer}(P_2,\mathrm{R}''^{k+1})}\{H\in\mathrm{Gr}_{k+1,k}(\mathrm{R})\mid H\supset T_xZ_1^{\pm}\}$$

is of co-dimension at least 1 in $\mathrm{Gr}_{k+1,k}(\mathrm{R})$. Thus, the complement of T in $\mathrm{Gr}_{k+1,k}(\mathrm{R})$ contains an open dense set. □

Lemma 8. *There exists an open dense subset of linear functionals $F:\mathrm{R}_2^{k+1}\to\mathrm{R}_2$, such that the set of critical points of F restricted to* $\mathrm{Ext}(Z_1^{\pm},\mathrm{R}_2)$ *has an empty intersection with $S_1^{\pm}$.*

Proof. Follows from Lemma 7 and the fact that $S_1^{\pm}$ is bounded over R_1, and ε_2 is infinitesimal with respect to R_1. □

Lemma 9. *There exists an open dense subset of linear functionals $F:\mathrm{R}_2^{k+1}\to\mathrm{R}_2$, such that the critical points of F restricted to* $\mathrm{Ext}(Z_1^{\pm},\mathrm{R}_2)\cap Z_2^{\pm}$ *are non-degenerate.*

Proof. The lemma can be deduced as a special case of [7, Theorem 2]. □

Lemma 10. *Let F be a linear functional satisfying the hypothesis of Lemmas 8 and 9. Then, for $\epsilon\in\{+,-\}$, $b(S_1^{\epsilon},\mathbb{Z}_2)$ is bounded by the number of critical points of F restricted to* $\mathrm{Ext}(Z_1^{\epsilon},\mathrm{R}_2)\cap Z_2^{\pm}$.

Proof. Follows from Theorem 35. ☐

Lemma 11. *Let F be a linear functional satisfying the hypothesis of Lemmas 8 and 9 below. Then, for $\epsilon \in \{+,-\}$, $b(S_1^\epsilon, \mathbb{Z}_2)$ the number of critical points of F restricted to $\mathrm{Ext}(Z_1^\epsilon, \mathrm{R}_2) \cap Z_2^\pm$ is bounded by*

$$8\binom{k+1}{3} d_1 d_2 (d_2-1)^{k-2}.$$

Proof. For $d, k \geq 0$, let $\Delta_{d,k} \subset \mathbb{Q}^k$ denote the simplex defined as the convex hull of $(d,0,\ldots,0),\ldots,(0,\ldots,0,d),\mathbf{0}$. For $\epsilon_1, \epsilon_2 \in \{+,-\}$, the set of critical points of F restricted to $\mathrm{Ext}(Z_1^\epsilon, \mathrm{R}_2) \cap Z_2^\pm$ satisfies the following system of equations:

$$\begin{aligned} Q_{k+1} &= 0, \\ P_1 - \epsilon_1 \varepsilon_1 &= 0, \\ P_2 - \epsilon_2 \varepsilon_2 &= 0, \\ \frac{\partial Q_{k+1}}{\partial X_1} + \lambda_1 \frac{\partial Q_{k+1}}{\partial X_1} + \lambda_2 \frac{\partial P_1}{\partial X_1} + \lambda_3 \frac{\partial P_2}{\partial X_1} &= 0, \\ \vdots \quad & \vdots \quad \vdots \\ \frac{\partial Q_{k+1}}{\partial X_{k+1}} + \lambda_1 \frac{\partial Q_{k+1}}{\partial X_{k+1}} + \lambda_2 \frac{\partial P_1}{\partial X_{k+1}} + \lambda_3 \frac{\partial P_2}{\partial X_{k+1}} &= 0. \end{aligned} \tag{6.6}$$

Using Proposition 5 we obtain that the number of solutions of the system (6.6) is bounded by

$$\mathrm{MV}(\Delta_{2,k+1}, \Delta_{d_1,k+1}, \Delta_{d_2,k+1}, \underbrace{\Delta_{d_2-1,k+1} + \Delta_{1,3}, \ldots, \Delta_{d_2-1,k+1} + \Delta_{1,3}}_{k+1})$$

$$\leq \binom{k+1}{3} \mathrm{MV}(\Delta_{2,k+1}, \Delta_{d_1,k+1}, \Delta_{d_2,k+1}, \underbrace{\Delta_{d_2-1,k+1}, \ldots, \Delta_{d_2-1,k+1}}_{k-2}, \underbrace{\Delta_{1,3}, \ldots, \Delta_{1,3}}_{3})$$

$$= \binom{k+1}{3} 2 d_1 d_2 (d_2-1)^{k-2}.$$

Hence, the number of critical points of F restricted to $\mathrm{Ext}(Z_1^\epsilon, \mathrm{R}_2) \cap Z_2^\pm$ is bounded by

$$4\binom{k+1}{3} 2 d_1 d_2 (d_2-1)^{k-2} = 8\binom{k+1}{3} d_1 d_2 (d_2-1)^{k-2}.$$

☐

In particular, we obtain as an immediate corollary that

Corollary 2.

$$\sum_{\epsilon_1,\epsilon_2 \in \{+,-\}} b(S_1^{\epsilon_1} \cap S_2^{\epsilon_2}, \mathbb{Z}_2) \leq 8\binom{k+1}{3} d_1 d_2 (d_2-1)^{k-2}.$$

Proposition 16.

$$b(\bigcup_{\epsilon_2 \in \{+,-\}} S_2^{\epsilon_2}, \mathbb{F}) \leq 8\binom{k+1}{3} d_1 d_2 (d_2-1)^{k-2}.$$

Proof. Follows from Lemmas 7, 8, 9, 10 and 11. ☐

Proposition 17.

$$b(W_{k+1},\mathbb{F}) \leq 2\binom{k+1}{2} d_1 \left((d_1-1)^{k-1} + \frac{4(k-1)}{3} d_2 (d_2-1)^{k-2}\right) + 1,$$

$$b(W_k,\mathbb{Z}_2) \leq 2\binom{k}{2} d_1 \left((d_1-1)^{k-2} + \frac{4(k-2)}{3} d_2 (d_2-1)^{k-3}\right) + 1.$$

Proof. The inequality for $b(W_{k+1},\mathbb{F})$ follows from Lemmas 5, 6, Corollary 2, and Propositions 15 and 16. The proof of the inequality involving $b(W_k,\mathbb{F})$ follows the same steps as in the proof for $b(W_{k+1},\mathbb{F})$ replacing k by $k-1$. We omit the steps. □

Proof of Theorem 32. The theorem follows from Propositions 14 and 17. □

6.3. **The semi-algebraic case: Proof of Theorem 31.**

Proof of Theorem 31. The proof is by following the proof of Proposition 7.30 in [17] using Theorem 32 to bound the Betti numbers of the algebraic sets that arise instead of Theorem 1. □

7. Multi-degree polynomial partitioning

In this section we describe an application of multi-degree bounds of the kind stated in Section 2 to *polynomial partitioning* – a very important construction in discrete geometry.

7.1. **Background.**

Efficient methods to partition finite subsets of R^k using semi-algebraic subsets has been an important and extremely well-studied technique in discrete geometry (see [43]). Still recently, the most efficient and useful method was that of "semi-cylindrical" decomposition introduced in [23], and its variants. More recently, a different method, called *polynomial partitioning* was introduced by Guth and Katz which have had important new applications. The polynomial partitioning theorem proved by Guth and Katz [33] and its various generalizations ([37, 36, 57, 48, 18]) have become very important tools for studying incidence problems in discrete geometry. They also have algorithmic applications, for example in the problem of range searching with semi-algebraic sets [2].

Polynomial partitioning results in the literature are of the following form. Given a finite set $S \subset \mathrm{R}^k$, and a parameter $r > 0$, these partitioning theorems give a partition of S, into finite subsets $(S_i)_{i\in I}, S \setminus \cup_{i\in I} S_i$, such that:

1. $\mathrm{card}(S_i) \leq \mathrm{card}(S)/r$ for each $i \in I$;
2. For any real variety W defined by a polynomial of degree d and of dimension $k' \leq k$, $\mathrm{card}(\{i \in I \mid S_i \cap W \neq \emptyset\})$ is bounded from above by some function $F_1(r,d,k,k')$ of r,d,k,k';
3. the residual set $S \setminus \bigcup_{i\in I} S_i$ is contained in a real variety V (which we will refer to as the residual variety of the partition) defined by a polynomial of degree bounded by some function $F_2(r,k)$. Note that no upper bound on the cardinality of the residual set is required, and so in principle it could be as large as S itself.

Definition 4. We will call the pair $((S_i)_{i\in I}, V)$ a *partition of S*.

Remark 28. Notice that formulated as above, Definition 4 extends to finite subsets of $\mathbb{F}^k$ where $\mathbb{F}$ is an arbitrary (not necessarily real closed) field.

Partitions in the above sense have been associated so far with partitioning polynomials – a fundamental notion introduced by Guth and Katz in [33]. We recall here the notion of an r-partitioning polynomial. We will use the following notation.

Notation 13. For any semi-algebraic set $S \subset \mathrm{R}^k$, we denote by $\mathcal{C}(S)$ the finite set of semi-algebraically connected components of S.

Definition 5 (r-partitioning polynomial)**.** Given a finite subset $S \subset \mathrm{R}^k$, we say that $P \in \mathrm{R}[X_1, \ldots, X_k]$ is an *r-partitioning polynomial for S* if for every $C \in \mathcal{C}(\mathrm{R}^k \setminus \mathrm{Zer}(P, \mathrm{R}^k))$, $\mathrm{card}(C \cap S) \leq \frac{\mathrm{card}(S)}{r}$.

Observation 1. Notice that the existence of an r-partitioning polynomial of degree bounded by r, for any finite subset $S \subset \mathrm{R}^k$ is obvious. Just choose a generic hyperplane $v \in \mathrm{R}^k$, and a well-chosen set, $H_1, \ldots, H_r$ hyperplanes perpendicular to v, such that for each connected component C of $\mathrm{R}^k \setminus \bigcup_{i=1}^r H_i$, $\mathrm{card}(S \cap C) \leq n/r$. It is also clear, that we can choose the set $(H_i)_{1 \leq i \leq r}$ to have the additional property that $S \cap \bigcup_{i=1}^r H_i = \emptyset$. Let P be the product of the polynomials each of degree 1, defining the hyperplanes H_i. Clearly, $\deg(P) = r$, and it is an r-partitioning polynomial for S. Moreover, $S \cap \mathrm{Zer}(P, \mathrm{R}^k) = \emptyset$.

The following theorem due to Guth and Katz, guarantees the existence of an r-partitioning polynomial of much smaller degree than r, and was the starting of applications of polynomial partitioning in incidence geometry.

Theorem 36 (Polynomial partitioning theorem [33])**.** *For every $k > 0$ there exists a constant $c_k > 0$, such that for every finite subset $S \subset \mathrm{R}^k$ and $r > 0$, there exists a r-partitioning polynomial P of S, with $\deg(P) \leq c_k \cdot r^{1/k}$.*

Remark 29. Notice that even though the degree bound on the r-partitioning polynomial obtained in Theorem 36 is much better than the trivial one described in Observation 1, namely an r-partitioning polynomial of degree bounded by r, the price that we pay for this improvement in the degree is that unlike in the case of Observation 1, in Theorem 36 many (or even all) of the points in S, could belong to the variety $\mathrm{Zer}(P, \mathrm{R}^k)$. In Observation 1 it is possible to choose the r-partitioning polynomial such that the intersection of S with the variety defined by the r-partitioning polynomial is empty.

In view of Remark 29 it is natural to ask if it is possible to interpolate between Theorem 36 and the existence of the trivial partition described in Observation 1. Such an interpolation should balance the "efficiency" of the partition measured by the "complexity" of the sets $C \cap S$ in the partition having cardinality bounded by $\mathrm{card}(S)/r$ (i.e. the function $F_1(r, d, k, k')$ in 2 above), against the "complexity" of the "residual variety" V (measured by the degree of the real polynomial defining V) which contains the subset of S not covered by the sets C (i.e. the function $F_2(r, k)$ in 3 above).

To our knowledge, a formal study of the space of "partitions" in the sense of Definition 4 in which different competing costs balance each other, has not been undertaken until now. Such a study might have implications both in computational and discrete geometry, and we initiate it by defining more precisely four parameters controlling the "complexity" of partitions of finite subsets of R^k.

Depending on applications, one or more of these parameters could be more important than the others. We stress that in applications of the polynomial partitioning techniques to incidence problems (such as in [33, 37, 36, 57, 48, 18]), only these parameters and no other finer detail of the partitions involved enter into the arguments.

Till now we have restricted our discussions to polynomial partitions of finite subsets of R^k. It might be useful to consider such partitions over more general fields (especially, the field of complex numbers). To our knowledge polynomial partitions have not yet been defined over general fields till date, because the current definition (as in the formulation of Theorem 36) seemed to use the properties of real closed fields very strongly. However, from the point of view of this paper, taking into account the "residual variety" we can formulate the notion of polynomial partitions over arbitrary fields, which we proceed to do below. This notion by itself might have important applications (see Remark 31 below).

Notation 14. Let $\mathbb{F}$ be any field. We extend Notation 1 and denote for any finite subset $\mathcal{P} \subset \mathbb{F}[X_1, \ldots, X_k]$ the set of common zeros of $\mathcal{P}$ in $\mathbb{F}^k$ by $\mathrm{Zer}(\mathcal{P}, \mathbb{F}^k)$.

Definition 6 (Density, size and complexity of a partition)**.** Let $\mathbb{F}$ be any field, and let $S \subset \mathbb{F}^k$ a finite subset, and let $(\mathcal{S} = (S_i \subset S)_{i\in I}, V = \mathrm{Zer}(P, \mathbb{F}^k))$ be a partition of S (cf. Definition 4 and Remark 28). We define:

(1)
$$\mathrm{density}(\mathcal{S}; S) = \max_{i\in I} \frac{\mathrm{card}(S_i)}{\mathrm{card}(S)};$$

(2)
$$\mathrm{size}(\mathcal{S}) = \mathrm{card}(I);$$

(3) for $d \geq 0, 0 \leq k' \leq k$,
$$\mathrm{complexity}(\mathcal{S}, d, k') = \max_{\substack{\mathcal{P}\subset\mathbb{F}[X_1,\ldots,X_k],\\ \mathrm{card}(\mathcal{P})<\infty, \deg(P)\leq d, P\in\mathcal{P}\\ \dim_{\mathbb{F}}(\mathrm{Zer}(\mathcal{P},\mathbb{F}^k))=k'}} \mathrm{card}(\{i \in I \mid S_i \cap \mathrm{Zer}(P, \mathbb{F}^k) \neq \emptyset\}).$$

Remark 30. While items (1) and (2) in Definition 6 are self-explanatory, item (3) needs a remark. Using [9, Theorem 1,1], we deduce that for any $d \leq r^{1/k}$, the number of sets in the partition given in Theorem 36 which are met by a real variety of real dimension $k' \leq k$, and defined by a polynomial of degree $\leq d$, is bounded by $O(1)^k d^{k-k'} r^{k'/k}$. Similarly, the number of sets in the partition given in Observation 1 which are met by a real variety of real dimension $k' \leq k$, and defined by a polynomial of degree $\leq d$, is bounded by $O(1)^k d^{k-k'} r^{k'}$. Clearly, a bound on this number – i.e. the number of sets in the partition which are met by a real variety of real dimension $k' \leq k$, and defined by a polynomial of degree $\leq d$ – as a function of r, d, k, and k' is an important measure of the quality of the partition. One would like this number to be as small (i.e. as close to the Guth-Katz bound, $O(1)^k d^{k-k'} r^{k'/k}$) as possible.

Notice also that an inspection of the proofs of the theorems that use the polynomial partitioning technique (see for example [33, 37, 36, 57, 48, 18]) reveals that the *density*, *size*, *complexity*, and the *degree of the residual variety* V, as defined in Definition 6, are the only properties of the partition that are used in these proofs.

We now define for any field $\mathbb{F}$:

Definition 7 ((r, r', e, e')-partition of a finite subset $S \subset \mathbb{F}^k$)**.** Let $S \subset \mathbb{F}^k$ be a finite set of points. For any $r, r', e, e' > 0$, we say that a pair $(\mathcal{S} = (S_i)_{i\in I}, V = \mathrm{Zer}(P, \mathbb{F}^k))$ is a (r, r', e, e')*-partition of* S, if the following conditions hold.

1.
$$\text{density}(\mathcal{S}; S) \leq \frac{1}{r};$$
2.
$$\text{size}(\mathcal{S}) \leq r';$$
3. For all $d \geq 0, 0 \leq k' \leq k$,
$$\text{complexity}(\mathcal{S}, d, k') \leq d^{k-k'} \max(d, e)^{k'};$$
4.
$$S \setminus \bigcup_{i \in I} S_i \subset V;$$
and
5.
$$\deg(P) \leq e'.$$

Using the notation introduced in Definition 7 the following theorem is an immediate consequence of Theorem 36, explicit bounds on the number of semi-algebraically connected components of sign conditions of a family of polynomials on a variety [9, Theorem 1.1], and Observation 1.

Theorem 37. *There exist constants $c_1, c_2, c_3, c_4 > 0$ such that for every finite subset $S \subset \mathrm{R}^k$, $r > 0$, there exist:*
(1) a $(c_1 r, c_2 r, c_3 r, c_4)$-partition, and
(2) a $(c_1 r, c_2 r, c_3 r^{1/k}, c_4 r^{1/k})$-partition
of S.

Its reasonable to ask whether there exists an interpolation between the two partitions 1 and 2 in Theorem 37. In this paper we give an answer to the question in the case $k = 2$, leaving open the case of general k. However, in order to prove our result (Theorem 38 below) we prove certain intermediate results valid for general k, including a *multi-degree partitioning theorem* (Theorem 40 below) that could be of interest, independent of the interpolating theorem mentioned above.

Remark 31. Also, observe that it is reasonable to ask if Theorem 37 holds over arbitrary fields, in particular over the field of complex numbers. Note that if this conjecture is true then it would provide a relatively easy proof of the Szemerédi-Trotter theorem [52] over C (see [54, 56]).

Conjecture 2. *Let $\mathbb{F}$ be a field. There exist constants $c_1, c_2, c_3, c_4 > 0$ such that for every finite subset $S \subset \mathbb{F}^k$, $r > 0$, there exist:*
(1) a $(c_1 r, c_2 r, c_3 r, c_4)$-partition, and
(2) a $(c_1 r, c_2 r, c_3 r^{1/k}, c_4 r^{1/k})$-partition
of S.

7.2. **Interpolating theorem for polynomial partitions of the plane.** We prove the following theorem which (almost) achieves the goal of interpolating between the two cases in Theorem 37 (in the case $k = 2$). Note that there is a loss of a factor of $\log(e)$ in the bound below, making the result slightly weaker than a perfect interpolation.

Theorem 38. *There exist constants* $c_1, c_2, c_3, c_4 > 0$ *such that for all* r, n *with* $0 < r < n^{1/2}$, $1 \leq e \leq r^{1/2}$, *and for every finite subset* $S \subset \mathrm{R}^2$, *with* $\mathrm{card}(S) = n$, *there exists a* $(c_1 r, c_2(1 + \log(e))r, c_3 e r^{1/2}, c_4 e^{-1} r^{1/2})$*-partition of* S.

Remark 32. Generalizing Theorem 38 to all values of k (with an appropriate notion of a residual variety) is an interesting problem. The methods used in our proof do not extend easily to higher dimensions.

We can also ask whether Theorem 38 is true over an arbitrary field $\mathbb{F}$.

Conjecture 3. *Let* $\mathbb{F}$ *be any field. There exist constants* $c_1, c_2, c_3, c_4 > 0$ *such that for all* r, n *with* $0 < r < n^{1/2}$, $1 \leq e \leq r^{1/2}$, *and for every finite subset* $S \subset \mathbb{F}^2$, *with* $\mathrm{card}(S) = n$, *there exists a* $(c_1 r, c_2(1 + \log(e))r, c_3 e r^{1/2}, c_4 e^{-1} r^{1/2})$*-partition of* S.

7.3. **Proof of Theorem 38.** The proof of Theorem 38 will require the following slight generalizations of the polynomial ham-sandwich theorem and of the polynomial partitioning theorem. They are stated and proved for all dimensions $k \geq 2$, and might be of independent interest.

7.3.1. *Multi-degree polynomial partition.*

Definition 8 (Bisector)**.** We say that a polynomial $P \in \mathrm{R}[X_1, \ldots, X_k]$ *bisects* a finite set $S \subset \mathrm{R}[X_1, \ldots, X_k]$, if the cardinalities of the subsets $S^+ = \{x \in S \mid P(x) > 0\}$ and $S^- = \{x \in S \mid P(x) < 0\}$ are both bounded by $\frac{\mathrm{card}(S)}{2}$.

Theorem 39 (Multi-degree ham-sandwich)**.** *Let* $b_1, \ldots, b_k \in \mathrm{R}_{>0}$, $p = \lceil b_1 \rceil \times \cdots \times \lceil b_k \rceil$, *and* $S_1, \ldots, S_p$ *finite subsets of* R^k. *Then, there exists a polynomial* $P \in \mathrm{R}[X_1, \ldots, X_k]$ *such that* $\mathrm{supp}(P) \subset [0, \lceil b_1 \rceil] \times \cdots \times [0, \lceil b_k \rceil]$ *(cf. Notation 11), and with the property that* P *simultaneously bisects* $S_1, \ldots, S_p$.

Proof. The proof is the same as that of the ordinary polynomial ham-sandwich theorem [51], except that instead of using the Veronese embedding , we use the toric embedding $\phi_{\mathcal{A}} : \mathrm{R}^k \to \mathrm{R}^{\mathcal{A}}$ where $\mathcal{A} = [0, \lceil b_1 \rceil] \times \cdots \times [0, \lceil b_k \rceil]$ is defined by $\mathbf{x} \mapsto (\cdots, \mathbf{x}^{\boldsymbol{\alpha}}, \cdots)_{\boldsymbol{\alpha} \in \mathcal{A}}$. We apply the ham-sandwich theorem to the set of points $\phi_{\mathcal{A}}(S) \subset \mathrm{R}^{\mathcal{A}}$ in $\mathrm{R}^{\mathcal{A}}$. Notice that $p \leq \mathrm{card}(\mathcal{A})$, and thus by the ordinary ham-sandwich theorem there exists a hyperplane H defined by a polynomial L of degree 1 in $\mathrm{R}^{\mathcal{A}}$ that simultaneously bisects $\phi_{\mathcal{A}}(S_1), \ldots, \phi_{\mathcal{A}}(S_p)$. The pull-back of $P = \phi^*_{\mathcal{A}}(L)$ is a non-zero polynomial with $\mathrm{supp}(P) \subset [0, \lceil b_1 \rceil] \times \cdots \times [0, \lceil b_k \rceil]$, and P simultaneously bisects $S_1, \ldots, S_p$. □

Theorem 40 (Multi-degree polynomial partitioning)**.** *Let* $B = [0, b_1] \times \cdots \times [0, b_k] \subset \mathrm{R}^k$, *where* $b_1, \ldots, b_k \in \mathrm{R}_{>0}$. *Then, for any* $r > 0$, *and any finite set* $S \subset \mathrm{R}^k$, *there exists an* r*-partitioning polynomial* $P \in \mathrm{R}[X_1, \ldots, X_k]$ *of* S *such that* $\mathrm{supp}(P) \subset \lambda \cdot B$, *with*

$$\lambda \leq 1 + \log(r) + \frac{2^{1/k}}{2^{1/k} - 1} \cdot \left(\frac{r}{b_1 \cdots b_k} \right)^{1/k}.$$

Proof. We first assume that r is power of 2, and for $c_1, \ldots, c_k \in \mathrm{R}_{>0}$, and $C = [0, c_1] \times \cdots \times [0, c_k]$, we will denote by $\lceil C \rceil = [0, \lceil c_1 \rceil] \times \cdots \times [0, \lceil c_k \rceil]$. The polynomial P will be the product of polynomials $P_0, P_1, P_2, \ldots, P_{\log(r)}$, where each P_i is defined as follows. Let $P_0 = 1$ and $\mathcal{S}_0 = \{S\}$. Now suppose that we have a partition $\mathcal{S}_i = \{S_1, \ldots, S_{2^i}\}$, where for each $j, 1 \leq j \leq 2^i$, $S_j \subset S$, $\mathrm{card}(S_j) \leq \frac{\mathrm{card}(S)}{2^i}$. Let P_i be a polynomial with $\mathrm{supp}(P_i) \subset \lceil \lambda_i \cdot B \rceil \subset \lceil \lambda_i \rceil \lceil B \rceil$ and with the property that P_i

simultaneously bisects $S_1, \ldots, S_{2^{i-1}}$, where $\lambda_i = \left(\frac{2^{i-1}}{b_1 \cdots b_k}\right)^{1/k}$. Such a polynomial exists by Theorem 39. Clearly, $P = P_0 \cdots P_{\log(r)}$ is an r-partitioning polynomial, and it follows from the construction that $\mathrm{supp}(P) \subset \lambda \cdot \lceil B \rceil$, where

$$
\begin{aligned}
\lambda &= \lceil \lambda_1 \rceil + \cdots + \lceil \lambda_{\log(r)} \rceil \\
&\leq \sum_{i=1}^{\log(r)} \left\lceil \left(\frac{2^{i-1}}{b_1 \cdots b_k}\right)^{1/k} \right\rceil \\
&\leq \log(r) + \sum_{i=1}^{\log(r)} \left(\frac{2^{i-1}}{b_1 \cdots b_k}\right)^{1/k} \\
&\leq \log(r) + (2^{1/k} - 1)^{-1} \left(\frac{r}{b_1 \cdots b_k}\right)^{1/k}.
\end{aligned}
$$

Finally, it is clear that by replacing r by the closest power of 2 which is $\geq r$, we can choose

$$
\begin{aligned}
\lambda &\leq 1 + \log(r) + (2^{1/k} - 1)^{-1} \left(\frac{2r}{b_1 \cdots b_k}\right)^{1/k} \\
&\leq 1 + \log(r) + \frac{2^{1/k}}{2^{1/k} - 1} \cdot \left(\frac{r}{b_1 \cdots b_k}\right)^{1/k}.
\end{aligned}
$$

□

We introduce the following technical notation that will be convenient to use in the proof of Theorem 38.

Notation 15. We will denote by $\preceq$ the lexicographical order on $\{(0,0)\} \cup \mathbb{Z}_{\geq 0} \times \{1,2\}$. For $(i,j), (i',j') \in \{(0,0)\} \cup \mathbb{Z}_{\geq 0} \times \{1,2\})$, we denote $(i',j') \prec (i,j)$ if $(i',j') \preceq (i,j)$ and $(i',j') \neq (i,j)$. We will also denote by $\mathrm{succ}(i,j)$ the successor $(i,j) \in \mathbb{Z} \times \{1,2\}$, and by $\mathrm{pred}(i,j)$ the predecessor of $(i,j) \in \{(0,0)\} \cup \mathbb{Z}_{\geq 0} \times \{1,2\}$ (if it exists), with respect to the order $\preceq$.

Proof of Theorem 38. Define for $0 \leq i \leq \log(r)$, and $1 \leq j \leq 2$ (see Figure 3)

$$
\begin{aligned}
B_{i,j} &= [0, 2^i r^{1/2}] \times [0, 2^{-i} r^{1/2}] \text{ if } j = 1, \\
&= [0, 2^{-i} r^{1/2}] \times [0, 2^i r^{1/2}] \text{ if } j = 2, \\
C_{i,j} &= \bigcap_{(i',j') \preceq (i,j)} B_{i,j} \\
&= [0, 2^{-i+1} r^{1/2}] \times [0, 2^{-i} r^{1/2}] \text{ if } j = 1, \qquad (7.1)\\
&= [0, 2^{-i} r^{1/2}] \times [0, 2^{-i} r^{1/2}] \text{ if } j = 2.
\end{aligned}
$$

Notice that

$$
\begin{aligned}
\mathrm{vol}_2(B_{i,j}) &= r, \\
\mathrm{vol}_2(C_{i,j}) &= 2^{-(2i+j)} r.
\end{aligned}
$$

We now define for $(i,j) \in \{(0,0)\} \cup \mathbb{Z}_{\geq 0} \times \{1,2\}$, polynomials $Q_{i,j}, P_{i,j}$ as follows. Applying Theorem 40 with

$$
\begin{aligned}
B &= \lambda_i \cdot ([0,1] \times [0, 2^{2i}]) \text{ if } j = 1, \\
\mathrm{supp}(P_{\mathrm{succ}(i,j)}) &\subset \lambda_i \cdot ([0, 2^{2(i+1)}] \times [0,1]) \text{ if } j = 2,
\end{aligned}
$$

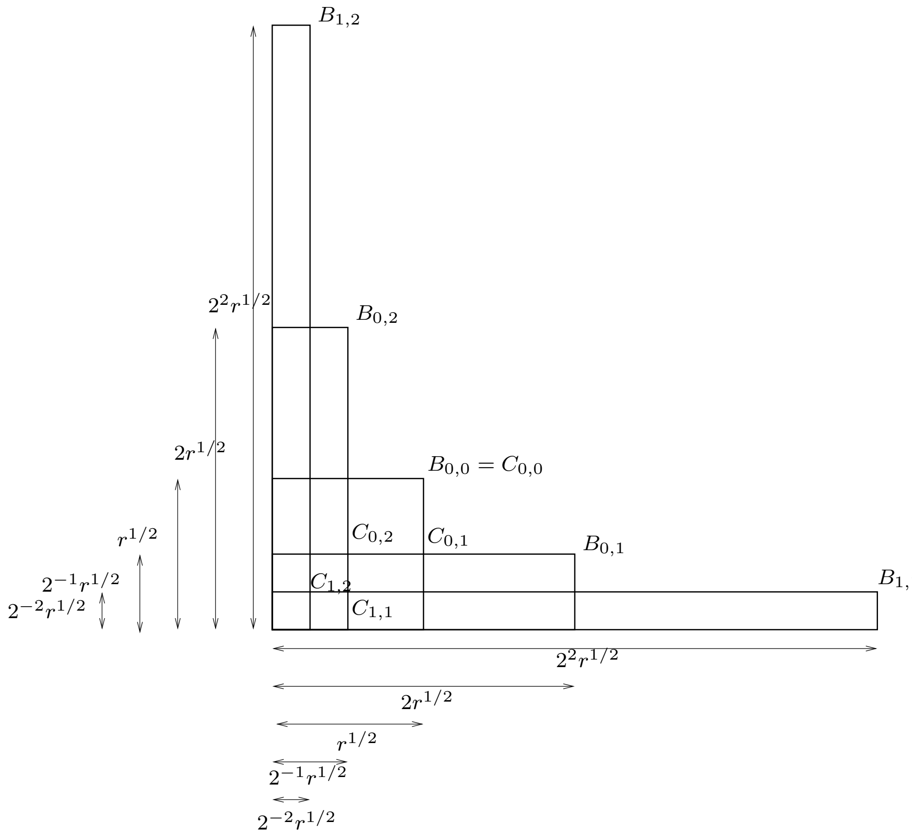


FIGURE 3. $B_{i,j}$ and $C_{i,j}$, for $i = 0, 1$ and $j = 1, 2$.

obtain an r-partitioning polynomial $P_{\mathrm{succ}(i,j)}$ of $S \cap \mathrm{Zer}(Q_{i,j}, \mathrm{R}^2)$ with

$$
\begin{aligned}
\mathrm{supp}(P_{\mathrm{succ}(i,j)}) &\subset \lambda_i \cdot ([0,1] \times [0, 2^{2i}]) \text{ if } j = 1, \\
\mathrm{supp}(P_{\mathrm{succ}(i,j)}) &\subset \lambda_i \cdot ([0, 2^{2(i+1)}] \times [0,1]) \text{ if } j = 2,
\end{aligned}
$$

where

$$
\begin{aligned}
\lambda_i &\leq 1 + \log(r) + \frac{2^{1/2}}{2^{1/2}-1} \cdot \left(\frac{r}{2^{2(i+j-1)}}\right)^{1/2} \\
&\leq 1 + \log(r) + \frac{2^{1/2}}{2^{1/2}-1} \cdot 2^{-(i+j-1)} r^{1/2} \\
&\leq c \cdot 2^{-(i+j-1)} r^{1/2}
\end{aligned}
$$

for some constant $c > 0$. Notice that

$$
\begin{aligned}
B_{i,j} &= \mu_i \cdot ([0,1] \times [0, 2^{2i}]) \text{ if } j = 1, \\
&= \mu_i \cdot ([0, 2^{2(i+1)}] \times [0,1]) \text{ if } j = 2,
\end{aligned}
$$

with

$$
\mu_i = 2^{-i-j+1} r^{1/2},
$$

and thus,

$$\mathrm{supp}(P_{\mathrm{succ}(i,j)}) \subset c \cdot B_{i,j}.$$

Set

$$Q_{\mathrm{succ}(i,j)} = \gcd(Q_{i,j}, P_{\mathrm{succ}(i,j)}).$$

We now prove that there exists constants $c_1, c_2, c_3, c_4 > 0$ such that the following statement holds for each $(i, j) \in \mathbb{Z}_{\geq 0} \times \{1, 2\}$. This implies the theorem.

Statement 1. (1) For each $(i', j') \preceq (i, j)$, $Q_{i,,j}$ divides $Q_{i',j'}$ and $P_{i',j'}$.
(2) $\mathrm{supp}(Q_{i,j}) \subset c \cdot C_{i,j}$;
(3) $\mathrm{supp}(P_{i,j}) \subset c \cdot B_{i,j}$;
(4) the pair $(\mathcal{S}_{i,j}, V_{i,j} = \mathrm{Zer}(Q_{i,j}, \mathrm{R}^2))$, where

$$\mathcal{S}_{i,j} = \big(S \cap D \cap (\mathrm{Zer}(Q_{\mathrm{pred}(i',j')}, \mathrm{R}^2) \setminus \mathrm{Zer}(Q_{i',j'}, \mathrm{R}^2))\big)_{\substack{D \in \mathcal{C}(\mathrm{R}^2 \setminus \mathrm{Zer}(P^2_{i',j'} - \varepsilon, \mathrm{R}^2)) \\ (i',j') \preceq (i,j)}},$$

is a $(c_1 r, c_2(i+1)r, c_3 2^i r^{1/2}, c_4 2^{-i} r^{1/2})$-partition of S, for all $\varepsilon > 0$ and small enough.

Parts (1), (2) and (3) of Statement 1 are immediate from the definitions of the polynomials, $Q_{i,',j'}, P_{i',j'}, (i', j') \preceq (i, j)$.

In order to prove Part (4) first observe that if $D \in \mathcal{C}(\mathrm{R}^2 \setminus \mathrm{Zer}(P^2_{i',j'} - \varepsilon, \mathrm{R}^2))$, then either D is a semi-algebraically connected component of the set defined by $P_{i'j'} - \varepsilon > 0$, or if D is a semi-algebraically connected component of the set defined by $P_{i'j'} - \varepsilon < 0$. We have thus two cases:

(a) Suppose that D is a semi-algebraically connected component of the set defined by $P_{i'j'} - \varepsilon > 0$. In this case $D \subset D'$ for some $D' \in \mathcal{C}(P_{i',j'})$, and using the definition of the polynomial $P_{i',j'}$, $\mathrm{card}(S \cap D') \leq n/r$. This implies that

$$\mathrm{card}(S \cap D \cap (\mathrm{Zer}(Q_{\mathrm{pred}(i',j')}, \mathrm{R}^2) \setminus \mathrm{Zer}(Q_{i',j'}, \mathrm{R}^2))) \leq \mathrm{card}(S \cap D) \leq \mathrm{card}(S \cap D') \leq n/r.$$

(b) Now suppose that D is a semi-algebraically connected component of the set defined by $P_{i'j'} - \varepsilon < 0$. Then for all small enough $\varepsilon > 0$,

$$\mathrm{card}(S \cap D \cap (\mathrm{Zer}(Q_{\mathrm{pred}(i',j')}, \mathrm{R}^2) \setminus \mathrm{Zer}(Q_{i',j'}, \mathrm{R}^2)))$$

is bounded by the number of isolated points of the intersection, $\mathrm{Zer}(P_{i,j'}, \mathrm{R}^2) \cap \mathrm{Zer}(Q_{\mathrm{pred}(i',j')}, \mathrm{R}^2)$, which is bounded by $b_0(\mathrm{Zer}(\{P_{i',j'}, Q_{\mathrm{pred}(i',j')}\}, \mathrm{R}^2) \leq c'r \leq c'n/r$ for some constant $c' > 0$ (using Theorem 1), and the fact that $r \leq n^{1/2}$ by assumption. Note that any semi-algebraically connected component of $\mathrm{Zer}(\{P_{i',j'}, Q_{\mathrm{pred}(i',j')}\}, \mathrm{R}^2)$ dimension one is contained in $\mathrm{Zer}(Q_{i',j'}, \mathrm{R}^2)$.

Thus we have for every $S_{i,j} \in \mathcal{S}_{i,j}$,

$$\mathrm{card}(S_{i,j}) \quad \leq \quad c_1 n/r, \tag{7.2}$$

for a constant $c_1 > 0$.

Moreover, using Theorem 13 we obtain that for all $(i', j') \preceq (i, j)$, $\mathrm{card}(\mathcal{C}(\mathrm{R}^2 \setminus \mathrm{Zer}(P^2_{i',j'} - \varepsilon, \mathrm{R}^2))) \leq c'_2 r$ for some constant $c'_2 > 0$. This implies that

$$\mathrm{card}(\mathcal{S}_{i,j}) \quad \leq \quad c_2(i+1)r \tag{7.3}$$

for some constant $c_2 > 0$. Also, it follows from [9, Theorem 1,1] that for any algebraic curve defined by a polynomial $P \in \mathrm{R}[X_1, X_2]$ of total degree at most d,

$\mathrm{card}(\mathcal{C}(\mathrm{Zer}(P, \mathrm{R}^2) \setminus \mathrm{Zer}(P^2_{i',j'} - \varepsilon, \mathrm{R}^2)) \leq c'_3 d \max(d, 2^{i'} r^{1/2})$ for some $c'_3 > 0$, which implies in turn that

$$\mathrm{card}(\{S_{i,j} \in \mathcal{D}_{i,j} \mid D \cap \mathrm{Zer}(P, \mathrm{R}^2) \neq \emptyset\}) \quad \leq \quad c_3 d \max(d, 2^i r^{1/2}) \tag{7.4}$$

for some constant $c_3 > 0$. This implies Part (4) of Statement 1. □

7.3.2. *Open problems and future directions.* In this paper we have initiated the study of the *space* of "polynomial partitions". We proved an interpolating theorem (namely, Theorem 38) in a very special case (when $\mathbb{F}$ is a real closed field and $k = 2$). It would be interesting to generalize this result to arbitrary fields (Conjecture 3 above), and also to higher dimensions. Moreover, it would be interesting to let the other two parameters measuring the quality of polynomial partitions (namely, r, r' in Definition 7 above) to vary, and obtain more general partitioning results, which might have applications in the quantitative study of incidences.

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8. Appendix

Proof of Proposition 3. It's sufficient to prove this theorem in the special case when $C = \mathbb{C}$. The general theorem follows using a standard application of the Tarski-Seidenberg transfer principle. In the following proof we assume $\mathrm{C} = \mathbb{C}$.

The proposition is clearly true if V is 0-dimensional, so we can assume that $k - \ell > 0$. Let $\bar{V} \subset \mathbb{P}^k_{\mathrm{C}}$ be the projective closure of V, and $W = \bar{V} \setminus V$ the sub-variety of $\bar{V}$ contained in the hyperplane at infinity. Note that W can be singular. Applying Lefschetz hyperplane section theorem (see [55, Theorem 1.1]) to the projective variety $\bar{V}$, noting that $\bar{V} \setminus W = V$ is non-singular by assumption, and denoting by $j : W \hookrightarrow \bar{V}$ the inclusion map, we have that

$$j_{*,i} : \mathrm{H}_i(W, \mathbb{Z}_2) \to \mathrm{H}_i(\bar{V}, \mathbb{Z}_2), \tag{8.1}$$

is an isomorphism for $0 \leq i < k - \ell - 1$, and a surjection for $i = k - \ell - 1$.

Now since V is a complex manifold, the compact pair $(\bar{V}, W)$ is a topological relative $(k-\ell)$-manifold (see [50, page 297] for definition), and by Lefschetz duality theorem (see for example [50, page 297] we have that

$$\mathrm{H}_i(V, \mathbb{Z}_2) \cong \tilde{\mathrm{H}}^{k-\ell-i}(\bar{V}, W, \mathbb{Z}_2), \tag{8.2}$$

where $\tilde{\mathrm{H}}^*(\bar{V}, W, \mathbb{Z}_2)$ denotes the *reduced* cohomology groups of the pair $(\bar{V}, W)$.

Finally, since V is a (complex) $(k - \ell)$-dimensional Stein manifold,

$$\mathrm{H}_i(V, \mathbb{Z}_2) = 0 \text{ for } i > k - \ell \tag{8.3}$$

(see [45]).

Now consider the homology exact sequence of the pair $(\bar{V}, W)$:

$$\cdots \to \mathrm{H}_p(W, \mathbb{Z}_2) \xrightarrow{j_{*,p}} \mathrm{H}_p(\bar{V}, \mathbb{Z}_2) \to \mathrm{H}_p(\bar{V}, W, \mathbb{Z}_2) \to \mathrm{H}_{p-1}(W, \mathbb{Z}_2) \xrightarrow{j_{*,p-1}} \cdots$$

For $0 < p < k-\ell$, the fact that in the above exact sequence, the homomorphism $j_{*,p} : \mathrm{H}_p(W, \mathbb{Z}_2) \to \mathrm{H}_p(\bar{V}, \mathbb{Z}_2)$ is a surjection, and the the homomorphism $j_{*,p-1}$: $\mathrm{H}_{p-1}(W, \mathbb{Z}_2) \to \mathrm{H}_{p-1}(\bar{V}, \mathbb{Z}_2)$ is an injection (in fact, an isomorphism) together implies that

$$\mathrm{H}_p(V, W, \mathbb{Z}_2) \cong \tilde{\mathrm{H}}_p(\bar{V}, W, \mathbb{Z}_2) = 0.$$

It then follows from the isomorphism (8.1) that,

$$\mathrm{H}_{k-\ell-p}(V, \mathbb{Z}_2) = 0.$$

This proves that

$$\mathrm{H}_i(V, \mathbb{Z}_2) = 0 \text{ for } 0 < i < k - \ell. \tag{8.4}$$

Finally, we observe that since V is connected,

$$b_0(V, \mathbb{Z}_2) = 1. \tag{8.5}$$

It follows from (8.3), (8.4), and (8.5) that

$$\chi(V, \mathbb{Z}_2) = 1 + (-1)^{k-\ell} b_{k-\ell}(V, \mathbb{Z}_2),$$

and the proposition follows immediately. □

Proof of Theorem 22. The only difficulty in applying inequality (3.4) is that in general an affine sub-variety of C^k will not bounded. In order to apply inequality (3.4) we need to reduce to the closed and bounded case which we do as follows.

Let for $r > 0$, $B_{\mathrm{C}}(r) \subset \mathrm{C}^k$ be defined by

$$B_{\mathrm{C}}(r) = \{(x_1 + iy_1, \ldots, x_k + iy_k) \in \mathrm{C}^k \mid |x_i|, |y_i| \leq r, 1 \leq i \leq k\},$$

and denote by $B_{\mathrm{R}}(r) = B_{\mathrm{C}}(r) \cap \mathrm{R}^k$.

Then, $\mathrm{Zer}(\mathcal{Q}, \mathrm{C}^k) \cap B_{\mathrm{C}}(r)$ is closed and bounded, and using [21, Corollary 9.3.7] (Local Conic Structure at infinity of semi-algebraic sets), we have that for all $r > 0$ and large enough, $\mathrm{Zer}(\mathcal{Q}, \mathrm{C}^k) \cap B_{\mathrm{C}}(r)$ is semi-algebraically homeomorphic to $\mathrm{Zer}(\mathcal{Q}, \mathrm{C}^k)$, and $\mathrm{Zer}(\mathcal{Q}, \mathrm{R}^k) \cap B_{\mathrm{R}}(r)$ is semi-algebraically homeomorphic to $\mathrm{Zer}(\mathcal{Q}, \mathrm{R}^k)$.

The complex conjugation restricts to an involution of $B_{\mathrm{C}}(r)$ with fixed points $B_{\mathrm{R}}(r)$. Now apply inequality (3.4). □

Proof of Proposition 6. First observe that $\mathrm{Zer}(P, \mathrm{C}^k)$ is either 0-dimensional, or is smooth and connected in case $k > 1$ (since $\mathrm{Zer}(P, \mathrm{C}^k)$ is a non-singular projective hypersurface of dimension $k-1$ minus a sub-variety of strictly smaller dimension). Using Theorem 24 we obtain

$$\begin{aligned}
\chi(\mathrm{Zer}(P, \mathrm{C}^k), \mathbb{Z}_2) &= \sum_{j=1}^{k} \binom{k}{j} (-1)^{j+1} j! \frac{d^j}{j!} \\
&= 1 + \sum_{j=0}^{k} \binom{k}{j} (-1)^{j+1} j! \frac{d^j}{j!} \\
&= 1 - (1-d)^k \\
&= 1 + (-1)^{k-1} (d-1)^k.
\end{aligned}$$

This implies using Proposition 3 and the fact that $\mathrm{Zer}(P, \mathrm{C}^k)$ is either 0-dimensional or non-singular and connected that

$$\begin{aligned}
b(\mathrm{Zer}(P, \mathrm{C}^k), \mathbb{Z}_2) &= 1 + (-1)^{k-1} (\chi(\mathrm{Zer}(P, \mathrm{C}^k), \mathbb{Z}_2) - 1) \\
&= 1 + (-1)^k + (-1)^{k-1} \chi(\mathrm{Zer}(P, \mathrm{C}^k), \mathbb{Z}_2) \\
&= 1 + (d-1)^k.
\end{aligned}$$

This proves Eqn.(3.10).

Finally, inequality (3.11) follows from Eqn. (3.10) and Theorem 22 (Smith inequality). □

Proof of Proposition 7. First observe that either $\mathrm{Zer}(\mathcal{P}, \mathrm{C}^k)$ is 0-dimensional (in case $k = \ell$) or is non-singular and connected (if $k > \ell$) since in the latter case $\mathrm{Zer}(\mathcal{P}, \mathrm{C}^k)$ is equal to a non-singular complete intersection variety in a product of projective varieties minus a sub-variety of strictly smaller dimension.

Using Theorem 24 we have

$$\begin{aligned}
\chi(\mathrm{Zer}(\mathcal{P}, \mathrm{C}^k), \mathbb{Z}_2) &= \sum_{j=\ell}^{k} (-1)^{j+\ell} \binom{k}{j} d_1 \cdots d_\ell \sum_{\substack{j_1,\ldots,j_\ell \geq 0 \\ j_1+\cdots+j_\ell = j-\ell}} d_1^{j_1} \cdots d_\ell^{j_\ell} \\
&= d_1 \cdots d_\ell \cdot \left(\sum_{j=\ell}^{k} (-1)^{j+\ell} \binom{k}{j} h_{j-\ell}(d_1, \ldots, d_\ell) \right) \\
&= d_1 \cdots d_\ell \cdot \left(\sum_{j=0}^{k-\ell} (-1)^{j} \binom{k}{j+\ell} h_{j}(d_1, \ldots, d_\ell) \right). \qquad (8.6)
\end{aligned}$$

Eqns. (8.6) and Proposition 3 imply

$$\begin{aligned}
b(\mathrm{Zer}(\mathcal{P}, \mathrm{C}^k), \mathbb{Z}_2) &= 1 + (-1)^{k-\ell} (\chi(\mathrm{Zer}(\mathcal{P}, \mathrm{C}^k), \mathbb{Z}_2) - 1) \\
&= 1 + (-1)^{k-\ell+1} + (-1)^{k-\ell} \chi(\mathrm{Zer}(\mathcal{P}, \mathrm{C}^k), \mathbb{Z}_2) \\
&= 1 + (-1)^{k-\ell+1} + d_1 \cdots d_\ell \cdot \left(\sum_{j=0}^{k-\ell} (-1)^{k-\ell+j} \binom{k}{j+\ell} h_{j}(d_1, \ldots, d_\ell) \right),
\end{aligned}$$

Now assume that $d_1 = \cdots = d_\ell = d$. It follows from (8.6) that

$$\begin{aligned}
\chi(\mathrm{Zer}(\mathcal{P}, \mathrm{C}^k), \mathbb{Z}_2) &= \sum_{j=\ell}^{k} (-1)^{j+\ell} \binom{k}{j} \binom{j-1}{\ell-1} d^j \\
&= \sum_{j=\ell}^{k} (-1)^{j+\ell} \frac{k!}{j!(k-j)!} \frac{(j-1)!}{(j-\ell)!(\ell-1)!} d^j \\
&= \ell \binom{k}{\ell} \sum_{j=\ell}^{k} (-1)^{j+\ell} \binom{k-\ell}{j-\ell} \frac{d^j}{j} \\
&= \ell \binom{k}{\ell} \sum_{j=0}^{k-\ell} (-1)^{j} \binom{k-\ell}{j} \frac{d^{j+\ell}}{j+\ell} \\
&= \ell \binom{k}{\ell} \sum_{j=0}^{k-\ell} (-1)^{j} \binom{k-\ell}{j} \frac{d^{j+\ell}}{j+\ell} \\
&= \ell \binom{k}{\ell} \int_0^d x^{\ell-1} (1-x)^{k-\ell} \mathrm{d}x.
\end{aligned}$$

This implies

$$\begin{aligned}
|\chi(\mathrm{Zer}(\mathcal{P}, \mathrm{C}^k), \mathbb{Z}_2)| &\leq \ell\binom{k}{\ell}\left(\int_0^1 x^{\ell-1}(1-x)^{k-\ell}\mathrm{d}x + \int_1^d x^{\ell-1}(x-1)^{k-\ell}\mathrm{d}x\right)\\
&\leq \ell\binom{k}{\ell}\left(1 + \int_1^d x^{k-1}\mathrm{d}x\right)\\
&= \ell\binom{k}{\ell}\left(1 + \frac{d^k}{k} - \frac{1}{k}\right)\\
&= \binom{k-1}{\ell-1}(d^k + k - 1),
\end{aligned}$$

whence using Proposition 3

$$\begin{aligned}
b(\mathrm{Zer}(\mathcal{P}, \mathrm{C}^k), \mathbb{Z}_2) &= 1 + (-1)^{k-\ell}(\chi(V_k, \mathbb{Z}_2) - 1)\\
&= 1 + (-1)^{k-\ell+1} + (-1)^{k-\ell}\chi(V_k, \mathbb{Z}_2)\\
&\leq 1 + (-1)^{k-\ell+1} + \binom{k-1}{\ell-1}(d^k + k - 1).
\end{aligned}$$

This proves Eqn. (3.12) and inequality (3.14). Inequalities (3.16) and (3.18), follow from Eqn. (3.12), inequality (3.14), and Theorem 22 (Smith inequality). □

Proof of Proposition 8. It follows from Theorem 24 that

$$\begin{aligned}
\chi(V, \mathbb{Z}_2) &= \sum_{j=\ell}^{k}(-1)^{j+\ell} \sum_{\substack{j_1,\ldots,j_p\\ 0\leq j_i\leq k_i, 1\leq i\leq p\\ j_1+\cdots+j_p=j}} \binom{j-1}{\ell-1}\binom{j}{j_1,\ldots,j_p}\left(\prod_{i=1}^{p}\binom{k_i}{j_i}d_i^{j_i}\right)\\
&= \frac{k_1!\cdots k_p!}{(k-\ell)!(\ell-1)!}F(\mathbf{k}, \mathbf{j}, \mathbf{d}, \ell),
\end{aligned}$$

where $\mathbf{k} = (k_1, \ldots, k_p)$, $\mathbf{j} = (j_1, \ldots, j_p)$, $\mathbf{d} = (d_1, \ldots, d_p)$, and $F(\mathbf{k}, \mathbf{j}, \mathbf{d}, \ell)$ is defined as

$$\sum_{j=\ell}^{k}\frac{(-1)^{j+\ell}}{j} \sum_{\substack{j_1,\ldots,j_p\\ 0\leq j_i\leq k_i, 1\leq i\leq p\\ j_1+\cdots+j_p=j}} \binom{j}{j_1,\ldots,j_p}^2 \binom{k-\ell}{k_1-j_1,\ldots,k_p-j_p, j-\ell} d_1^{j_1}\cdots d_p^{j_p}.$$

We now bound $|F(\mathbf{k}, \mathbf{j}, \mathbf{d}, \ell)|$ as follows. $|F(\mathbf{k}, \mathbf{j}, \mathbf{d}, \ell)|$ is bounded by

$$\begin{aligned}
& \left(\prod_{i=1}^{p} d_i^{k_i}\right) \sum_{j=\ell}^{k} (1+p)^{2j} \sum_{\substack{j_1,\ldots,j_p \\ 0\le j_i\le k_i \\ 1\le i\le p}} \binom{k-\ell}{k_1-j_1,\ldots,k_p-j_p,j-\ell} \prod_{i=1}^{p} d_i^{-(k_i-j_i)} \\
= & \left(\prod_{i=1}^{p} d_i^{k_i}\right) \sum_{j=\ell}^{k} (1+p)^{2j} \left(1+\frac{1}{d_1}+\cdots+\frac{1}{d_p}\right)^{k-\ell} \\
\le & \left(\prod_{i=1}^{p} d_i^{k_i}\right) (1+p)^{k+\ell} \sum_{j=0}^{k-\ell} (1+p)^{2j} \\
= & \left(\prod_{i=1}^{p} d_i^{k_i}\right) (1+p)^{k+\ell} \frac{(1+p)^{2(k-\ell+1)}-1}{(1+p)^2-1} \\
\le & \frac{(1+p)^{3k-\ell+1}}{p(p+2)} d_1^{k_1}\cdots d_p^{k_p}.
\end{aligned}$$

This implies that

$$\begin{aligned}
|\chi(V,\mathbb{Z}_2)| & \le \ell(k-\ell+1)\binom{k}{\ell}\binom{k}{\mathbf{k}}^{-1} \frac{(1+p)^{3k-\ell+1}}{p(p+2)} d_1^{k_1}\cdots d_p^{k_p} \\
& \le (k-\ell+2)^2\binom{k}{\ell-1}\binom{k}{\mathbf{k}}^{-1} \frac{(1+p)^{3k-\ell+1}}{p(p+2)} d_1^{k_1}\cdots d_p^{k_p},
\end{aligned}$$

and also (using Eqn. (3.9))

$$b(V,\mathbb{Z}_2) \le 1+(-1)^{k-\ell+1}+(k-\ell+2)^2\binom{k}{\ell-1}\binom{k}{\mathbf{k}}^{-1} \frac{(1+p)^{3k-\ell+1}}{p(p+2)} d_1^{k_1}\cdots d_p^{k_p}.$$

This proves inequality (3.21). Inequality (3.22) follows from inequality (3.21) and Theorem 22 (Smith inequality). □

Proof of Proposition 9. First observe that $\mathrm{Zer}(P, \mathrm{C}^k)$ is non-singular and connected for $k > 1$, and is 0-dimensional if $k = 1$. Using Theorem 24

$$\chi(\mathrm{Zer}(P,\mathrm{C}^k),\mathbb{Z}_2) = \sum_{j=1}^{k}(-1)^{j+1} \sum_{\substack{J\subset[1,k] \\ \mathrm{card}(J)=j\le k}} j!\bar{d}^J.$$

Now using Eqn. (3.9) we get

$$\begin{aligned}
b(\mathrm{Zer}(P,\mathrm{C}^k),\mathbb{Z}_2) & = 1+(-1)^{k-1}(\chi(V_k,\mathbb{Z}_2)-1) \\
& = 1+(-1)^k+(-1)^{k-1}\chi(V_k,\mathbb{Z}_2) \\
& = 1+(-1)^k+(-1)^{k-1}\left(\sum_{j=1}^{k}(-1)^{j+1} \sum_{\substack{J\subset[1,k] \\ \mathrm{card}(J)=j\le k}} j!\bar{d}^J\right) \\
& = 1+(-1)^k+\left(\sum_{j=1}^{k}(-1)^{k-j} \sum_{\substack{J\subset[1,k] \\ \mathrm{card}(J)=j\le k}} j!\bar{d}^J\right).
\end{aligned}$$

This proves inequality (3.23). Inequality (3.24) now follows from inequality (3.23) and Theorem 22 (Smith inequality). □

Proof of Proposition 10. It follows from Theorem 24 that $|\chi(\mathrm{Zer}(\mathcal{P},\mathrm{C}^k),\mathbb{Z}_2)|$ is

$$\leq \sum_{j=\ell}^{k} \sum_{J\in\binom{[1,k]}{j}} \binom{j+\ell-1}{\ell-1} \max_{\substack{\boldsymbol{\alpha}=(\alpha_1,\ldots,\alpha_\ell)\in\mathbb{Z}_{>0}^\ell\\ \alpha_1+\cdots+\alpha_\ell=k}} \max_{(J_1,\ldots,J_\ell)\in\binom{[1,k]}{\boldsymbol{\alpha}}} \left(\prod_{\substack{1\leq i\leq \ell\\ j\in J_i}} d_{i,j}\right),$$

So we have that

$$|\chi(\mathrm{Zer}(\mathcal{P},\mathrm{C}^k),\mathbb{Z}_2)| \leq O(\ell)^k \max_{\substack{\boldsymbol{\alpha}=(\alpha_1,\ldots,\alpha_\ell)\in\mathbb{Z}_{>0}^\ell\\ \alpha_1+\cdots+\alpha_\ell=k}} \max_{(J_1,\ldots,J_\ell)\in\binom{[1,k]}{\boldsymbol{\alpha}}} \left(\prod_{\substack{1\leq i\leq \ell\\ j\in J_i}} d_{i,j}\right). \tag{8.7}$$

Therefore, combining inequality (8.7) and Proposition 3, we have that

$$\begin{aligned} b(\mathrm{Zer}(\mathcal{P},\mathrm{C}^k),\mathbb{Z}_2) &= 1+(-1)^{k-\ell+1}+(-1)^{k-\ell}\chi(\mathrm{Zer}(\mathcal{P},\mathrm{C}^k),\mathbb{Z}_2)\\ &\leq O(\ell)^k \max_{\substack{\boldsymbol{\alpha}=(\alpha_1,\ldots,\alpha_\ell)\in\mathbb{Z}_{>0}^\ell\\ \alpha_1+\cdots+\alpha_\ell=k}} \max_{(J_1,\ldots,J_\ell)\in\binom{[1,k]}{\boldsymbol{\alpha}}} \left(\prod_{\substack{1\leq i\leq \ell\\ j\in J_i}} d_{i,j}\right). \end{aligned} \tag{8.8}$$

This proves inequality (3.25). Inequality (3.26) follows from inequality (3.25) and Theorem 22 (Smith inequality). □

Proof of Proposition 12. Using Theorem 24 we obtain that $\chi(\mathrm{Zer}(\mathcal{P},\mathrm{C}^k),\mathbb{Z}_2)$ equals

$$\begin{aligned} &\sum_{j=0}^{k}(-1)^{j+\ell} \sum_{\substack{0\leq j_1\leq k_1, 0\leq j_2\leq k_2\\ j_1+j_2=j}} \binom{k_1}{j_1}\binom{k_2}{j_2}\binom{j-1}{\ell-1}\binom{j}{j_1} d^{j_1}2^{j_2}+1\\ =\ &\sum_{j_1=0}^{k_1}(-1)^{j_1+\ell}\binom{k_1}{j_1}d^{j_1}\left(\sum_{j_2=0}^{k_2}\binom{j_1+j_2}{j_2}\binom{j_1+j_2-1}{\ell-1}\binom{k_2}{j_2}(-2)^{j_2}\right)+1. \end{aligned}$$

We now bound from above the quantity $|F(j_1,k_2)|$, where $F(j_1,k_2)$ is defined by

$$F(j_1,k_2) := \sum_{j_2=0}^{k_2}\binom{j_1+j_2}{j_2}\binom{j_1+j_2-1}{\ell-1}\binom{k_2}{j_2}(-2)^{j_2}.$$

First notice that $F(j_1,k_2)$ equals

$$\begin{aligned}
& \frac{1}{j_1!(\ell-1)!}\sum_{j_2=0}^{k_2}(j_1+j_2)^{\underline{j_1}}(j_1+j_2-1)^{\underline{\ell-1}}\binom{k_2}{j_2}(-2)^{j_2}\\
= \quad & \frac{1}{j_1!(\ell-1)!}\left[\frac{\mathrm{d}^{j_1}}{\mathrm{d}x^{j_1}}x^{\ell}\left(\frac{\mathrm{d}^{\ell-1}}{\mathrm{d}x^{\ell-1}}(x^{j_1-1}(1+x)^{k_2})\right)\right]_{x=-2}\\
= \quad & \frac{1}{j_1!(\ell-1)!}\left[\sum_{i=0}^{j_1}\binom{j_1}{i}\ell^{\underline{i}}x^{\ell-i}\left(\frac{\mathrm{d}^{\ell-1+j_1-i}}{\mathrm{d}x^{\ell-1+j_1-i}}(x^{j_1-1}(1+x)^{k_2})\right)\right]_{x=-2}\\
= \quad & \frac{1}{j_1!(\ell-1)!}\left[\sum_{i=0}^{j_1}\binom{j_1}{i}\ell^{\underline{i}}x^{\ell-i}\left(\sum_{h=0}^{\alpha}\binom{\alpha}{h}(j_1-1)^{\underline{h}}k_2^{\underline{\alpha-h}}x^{j_1-1-h}(1+x)^{k_2-(\alpha-h)}\right)\right]_{x=-2}
\end{aligned}$$

where $\alpha=\alpha(\ell,j_1,i)=\ell-1+j_1-i$, and we have used the "falling factorial" notation

$$t^{\underline{n}}:=t(t-1)\cdots(t-n+1),$$

for all real t and integer n.

Continuing, we have $F(j_1,k_2)$ equals

$$\frac{1}{j_1!(\ell-1)!}\left[\sum_{i=0}^{j_1}\binom{j_1}{i}\binom{\ell}{i}\alpha!i!x^{\alpha}(1+x)^{k_2-\alpha}\left(\sum_{h=0}^{\alpha}\binom{k_2}{\alpha-h}\binom{j_1-1}{h}(\omega(x))^h\right)\right]_{x=-2},$$

where $\omega(x)=1+\frac{1}{x}$.

This implies that

$$\begin{aligned}
|F(j_1,k_2)| \quad &\leq \quad \frac{1}{j_1!(\ell-1)!}\sum_{i=0}^{j_1}\binom{j_1}{i}\binom{\ell}{i}\alpha!i!2^{\alpha}\left(\sum_{h=0}^{\alpha}\binom{k_2}{\alpha-h}\binom{j_1-1}{h}\right)\\
&\leq \quad \frac{1}{j_1!(\ell-1)!}\sum_{i=0}^{j_1}\binom{j_1}{i}\binom{\ell}{i}\alpha!i!2^{\alpha}\binom{k_2+j_1-1}{\alpha}.
\end{aligned}$$

We obtain

$$
\begin{aligned}
|\chi(V\mathrm{Zer}(\mathcal{P},\mathrm{C}^k),\mathbb{Z}_2)| &\leq 1+\sum_{j_1=0}^{k_1}\binom{k_1}{j_1}d^{j_1}|F(j_1,k_2)| \\
&\leq 1+\sum_{j_1=0}^{k_1}\sum_{i=0}^{j_1}\binom{k_1}{j_1}d^{j_1}\frac{1}{j_1!(\ell-1)!}\binom{j_1}{i}\binom{\ell}{i}\alpha!i!2^{\alpha}\binom{k_2+j_1-1}{\alpha} \\
&\leq 1+\ell\sum_{j_1=0}^{k_1}\sum_{i=0}^{j_1}\binom{k_1}{j_1}d^{j_1}\alpha!2^{\alpha}\binom{k_2+j_1-1}{\alpha} \\
&\leq 1+\ell\sum_{j_1=0}^{k_1}\sum_{i=0}^{j_1}\binom{k_1}{j_1}d^{j_1}2^{\alpha}(k_2+j_1)^{\alpha} \\
&\leq 1+2\ell\sum_{j_1=0}^{k_1}\binom{k_1}{j_1}d^{j_1}\left(2(k_2+k_1)\right)^{\ell-1+j_1} \\
&= 1+\ell 2^{\ell}(k_1+k_2)^{\ell-1}\sum_{j_1=0}^{k_1}\binom{k_1}{j_1}d^{j_1}\left(2(k_1+k_2)\right)^{j_1} \\
&= 1+\ell 2^{\ell}(k_1+k_2)^{\ell-1}\left(2d(k_1+k_2)+1\right)^{k_1}.
\end{aligned}
$$

Now using Eqn. (3.9) we get

$$
\begin{aligned}
b(\mathrm{Zer}(\mathcal{P},\mathrm{C}^k),\mathbb{Z}_2) &= 2+(-1)^{k-\ell}(\chi(V_k,\mathbb{Z}_2)-1) \\
&\leq 2+(-1)^{k-\ell+1}+\ell 2^{\ell}(k_1+k_2)^{\ell-1}\left(2d(k_1+k_2)+1\right)^{k_1}.
\end{aligned}
$$

which proves inequality (3.28). Inequality (3.29) now follows from inequality (3.28) and Theorem 22 (Smith inequality). □

Proof of Proposition **??**. First observe that either $\mathrm{Zer}(\mathcal{P},\mathrm{C}^k)$ is 0 dimensional (in case $k=\ell$), or $\mathrm{Zer}(\mathcal{P},\mathrm{C}^k)$ is non-singular and connected (in case $k>\ell$). We denote by $\overline{d}=\prod_{j=1}^{k_1}d_j$ and for a subset $J_1\subset[1,k_1]$, $\overline{d}^{J_1}=\prod_{j\in J_1}d_j$. Then, using Theorem 24, $\chi(\mathrm{Zer}(\mathcal{P},\mathrm{C}^k),\mathbb{Z}_2)$ equals

$$
\begin{aligned}
&\sum_{j=\ell}^{k}(-1)^{j+\ell}\sum_{\substack{J=J_1\sqcup J_2\\ \mathrm{card}(J_1)=j_1\leq k_1,\mathrm{card}(J_2)=j_2\leq k_2\\ j=j_1+j_2}}\binom{j-1}{\ell-1}\frac{j!}{j_2!}\overline{d}^{J_1}2^{j_2} \\
=\;&\sum_{j_1=0}^{k_1}(-1)^{j_1+\ell}\sum_{\substack{J_1\subset[1,k_1]\\ \mathrm{card}(J_1)=j_1}}\overline{d}^{J_1}\left(\sum_{\substack{J_2\subset[1,k_2]\\ \mathrm{card}(J_2)=j_2}}(-1)^{j_2}\binom{j_1+j_2-1}{\ell-1}\frac{(j_1+j_2)!}{j_2!}2^{j_2}\right)+1 \\
=\;&\sum_{j_1=0}(-1)^{j_1+\ell}\sum_{\substack{J_1\subset[1,k_1]\\ \mathrm{card}(J_1)=j_1}}j_1!\overline{d}^{J_1}\left(\sum_{j_2=0}^{k_2}(-1)^{j_2}\binom{j_1+j_2-1}{\ell-1}\binom{j_1+j_2}{j_2}\binom{k_2}{j_2}2^{j_2}\right)+1.
\end{aligned}
$$

Note that the last sum is the same function $F(j_1, k_2)$ as in Proposition 12. Applying the same bound, we have $|\chi(\mathrm{Zer}(\mathcal{P}, \mathrm{C}^k), \mathbb{Z}_2)|$ is bounded by

$$\begin{aligned}
& 1 + \sum_{j_1=0}^{k_1} \sum_{\substack{J_1 \subset [1,k_1] \\ \mathrm{card}(J_1)=j_1}} j_1! \overline{d}^{J_1} |F(j_1, k_2)| \\
\leq \quad & 1 + \ell 2^{\ell} (k_1 + k_2)^{\ell-1} \sum_{j_1=0}^{k_1} j_1! (2(k_1+k_2))^{j_1} \sum_{\substack{J_1 \subset [1,k_1] \\ \mathrm{card}(J_1)=j_1}} \overline{d}^{J_1} \\
\leq \quad & 1 + \ell 2^{\ell} (k_1 + k_2)^{\ell-1} \sum_{j_1=0}^{k_1} j_1! (2(k_1+k_2))^{j_1} \binom{k_1}{j_1} d_1 \cdots d_{k_1} \\
\leq \quad & 1 + \ell 2^{\ell} k_1! (k_1 + k_2)^{\ell-1} d_1 \cdots d_{k_1} \sum_{j_1=0}^{k_1} \binom{k_1}{j_1} (2(k_1+k_2))^{j_1} \\
= \quad & 1 + \ell 2^{\ell} k_1! (k_1 + k_2)^{\ell-1} ((2(k_1+k_2)+1)^{k_1} d_1 \cdots d_{k_1}.
\end{aligned}$$

Therefore, using Proposition 3, we have $b(\mathrm{Zer}(\mathcal{P}, \mathrm{C}^k), \mathbb{Z}_2)$ equals

$$\begin{aligned}
& 1 + (-1)^{k-\ell} (\chi(\mathrm{Zer}(\mathcal{P}, \mathrm{C}^k), \mathbb{Z}_2) - 1) \\
\leq \quad & 2 + (-1)^{k-\ell+1} + \ell 2^{\ell} k_1! (k_1 + k_2)^{\ell-1} ((2(k_1+k_2)+1)^{k_1} d_1 \cdots d_{k_1}.
\end{aligned}$$

This proves inequality (3.37). Inequality (3.38) follows from inequality (3.38) and Theorem 22 (Smith inequality). □

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