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Efficient spectral–Galerkin methods for fractional partial differential equations with variable coefficients

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ABSTRACT

Efficient spectral–Galerkin algorithms are developed to solve multi-dimensional fractional elliptic equations with variable coefficients in conserved form as well as non-conserved form. These algorithms are extensions of the spectral–Galerkin algorithms for usual elliptic PDEs developed in [24]. More precisely, for separable FPDEs, we construct a direct method by using a matrix diagonalization approach, while for non-separable FPDEs, we employ a preconditioned BICGSTAB method with a suitable separable FPDE with constant-coefficients as preconditioner. The cost of these algorithms is of $O(N^{d+1})$ flops where d is the space dimension. We derive rigorous weighted error estimates which provide more precise convergence rate for problems with singularities at boundaries. We also present ample numerical results to validate the algorithms and error estimates.

1. Introduction

Fractional partial differential equations (FPDEs) appear in the investigation of transport dynamics in complex systems which are governed by the anomalous diffusion and non-exponential relaxation patterns [17]. Anomalous, or non-Fickian, dispersion has been an active area of research in the physics community since the introduction of continuous time random walks (CTRW) by Montroll and Weiss [18]. They have attracted considerable attention recently due to their ability to model certain processes which cannot be adequately described by usual partial differential equations.

The classical 1-D diffusion equation

$$\partial_t p(x,t) - \partial_x[D(x)\partial_x p(x,t)] = 0, \quad x \in (a,b), \tag{1.1}$$

is derived from the conservation of mass

$$\partial_t p(x,t) + \partial_x F = 0, \tag{1.2}$$

and the Fick's first law

$$F = -D(x)\partial_x p(x,t), \tag{1.3}$$

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where p is the density of the diffusing material at location x, F is the flux of the diffusing material, and $D(x)$ is the collective diffusion coefficient for density. Recently, scientists found that the classical Fick's law (1.3) is not adequate to describe anomalous diffusion which occurred in field and laboratory studies, such as transport of a solute in heterogeneous porous media. Instead, a fractional Fick's law is proposed [22]

$$F = -D(x)\Big(\frac{1}{2}(1+\kappa)\,{}_a\partial_x^\nu p(x,t) - \frac{1}{2}(1-\kappa)\,{}_x\partial_b^\nu p(x,t)\Big) \tag{1.4}$$

via Eulerian derivation, where $1/2 < \nu < 1$, and $-1 \le \kappa \le 1$ is a parameter describing the relative probabilities of particle travel ahead or behind the mean velocity, ${}_a\partial_x^\nu p(\cdot,t)$ and ${}_x\partial_b^\nu p(\cdot,t)$ are fractional derivatives which will be defined in Section 2. Combining the above with mass conservation (1.2), we arrive at the following fractional diffusion equation (FDE):

$$\partial_t p(x,t) - \partial_x\Big[D(x)\Big(\frac{1}{2}(1+\kappa)\,{}_a\partial_x^\nu p(x,t) - \frac{1}{2}(1-\kappa)\,{}_x\partial_b^\nu p(x,t)\Big)\Big] = 0. \tag{1.5}$$

On the other hand, we can also consider the following fractional mass conservation for a fluid in porous media [30]:

$$\partial_t p(x,t) + {}_a\partial_x^\mu F = 0, \tag{1.6}$$

where $1/2 < \mu < 1$. Combining the above with (1.4), we arrive at another kind of FDE:

$$\partial_t p(x,t) - {}_a\partial_x^\mu\Big[D(x)\Big(\frac{1}{2}(1+\kappa)_a\partial_x^\nu p(x,t) - \frac{1}{2}(1-\kappa)_x\partial_b^\nu p(x,t)\Big)\Big] = 0. \tag{1.7}$$

If $\kappa = -1$, i.e. solute disperses preferentially at velocities ahead of the mean velocity, and let $\mu = \nu$, we have

$$\partial_t p(x,t) + {}_a\partial_x^\nu\Big[D(x)_x\partial_b^\nu p(x,t))\Big] = 0. \tag{1.8}$$

In the above, only one-sided fractional derivatives are considered. More generally, we can consider the following FDE with fractional derivatives from both sides:

$$\partial_t p(x,t) + {}_a\partial_x^\nu\Big[d_+(x)\,{}_x\partial_b^\nu p(x,t)\Big] + {}_x\partial_b^\nu\Big[d_-(x)\,{}_a\partial_x^\nu p(x,t)\Big] = 0. \tag{1.9}$$

Note that in [21] and [4], the authors considered another class of conserved FDEs which is different from the above but shares similar mathematical properties. FPDEs in various forms have been numerically studied extensively in the last decade. An unconditionally stable first order finite difference method (FDM) is developed in [16], and a second order FDM is proposed in [26]. Some other FDMs have been studied in [14,19]. Pioneer works with finite element analysis are carried out in [21] and [4]. More refined analyses which take into account the singular behavior of FPDEs have been established recently in [10,9]. Moreover, some fast solvers for FD/FE approximations to FPDEs have been developed in [27–29] and [11,20] by exploiting their Toepliz structures. On the other hand, spectral methods for some fractional PDEs have been proposed in [12,13] where the well-posedness of some FPDEs and their spectral approximations have been established. Recently, some efficient spectral/spectral-element DG methods for a class of one-dimensional FPDEs with constant-coefficients and one-sided fractional derivatives have been proposed in [31,33] by using eigenfunctions of fractional Sturm–Liouville problems as basis functions. Related spectral algorithms and their rigorous error analyses have been established in [3]. However, these results cannot extended to more general FPDEs with two-sided fractional derivatives and variable coefficients.

Due to the non-local feature of the fractional derivatives, local methods such as FDM and FEM loss a big advantage that they enjoy for usual PDEs. On the other hand, the main disadvantage of global methods such as spectral methods is no longer an issue for fractional PDEs. Moreover, for fractional PDEs with constant coefficients, spectral methods with suitable basis functions can even result in sparse linear systems [31,3]. The main purpose of this paper is to develop some efficient spectral algorithms to solve a class of fractional elliptic equations. In particular, we shall extend most of the solution techniques in [24] for usual PDEs to fractional PDEs. More precisely, for separable fractional PDEs, we shall apply the matrix diagonalization methods (i.e., discrete separation of variables) [15,8,24] to reduce a (discrete) multi-dimensional problem to a sequence of (discrete) one-dimensional problems or to a diagonal system with a total cost of just a few N^{d+1} flops (d is the space dimension); for non-separable fractional PDEs, we shall apply a preconditioned BICGSTAB iterative method using (i) a related fractional separable problem with constant-coefficients as preconditioned, and (ii) a fast matrix-free algorithm for the matrix–vector multiplication, so that the total cost is still $O(N^{d+1})$. Thus, the cost of a spectral method for fractional PDEs is essentially the same as that for usual PDEs.

The rest of the paper is organized as follows. We describe some basic notations and properties for fractional derivatives in Section 2. We present the Legendre Spectral–Galerkin method, derive an error estimate in weighted norms, and carry out numerical tests for 1D FDE in Section 3 and for 2-D in Section 4. Then, we discuss about some extensions of the algorithms to other situations in Section 5, and conclude with some remarks in the final section.

2. Preliminaries

We first introduce some notations which will be used hereafter.

To fix the idea, we shall restrict ourselves in this paper to the finite interval $\Lambda = (-1, 1)$. Let $\omega(x) > 0$ $(x \in \Lambda)$ be a weight function, we denote by $L^2_\omega(\Lambda)$ the usual weighted Hilbert space with the inner product and norm defined by

$$(u, v)_{\Lambda,\omega} = \int_\Lambda uv\,\omega dx, \quad \|u\|_{\omega,\Lambda} = (u, u)^{\frac{1}{2}}_{\omega,\Lambda}, \ \forall u, v \in L^2_\omega(\Lambda). \tag{2.1}$$

We denote by $H^s_\omega(\Lambda)$ and $H^s_{0,\omega}(\Lambda)$ (with $s \geq 0$) the usual weighted Sobolev spaces with norm $\|\cdot\|_{s,\omega,\Lambda}$ and semi-norm $|\cdot|_{s,\omega,\Lambda}$.

Given a Banach space X with norm $\|\cdot\|_X$ and $s > 0$. We define

$$H^s_\omega(\Lambda; X) := \big\{v \mid \|v(\cdot, x)\|_X \in H^s_\omega(\Lambda)\big\}, \tag{2.2}$$

endowed with the norm:

$$\|v\|_{H^s_\omega(\Lambda;X)} = \big\|\|v(\cdot, x)\|_X\big\|_{s,\omega,\Lambda}.$$

Let $\Omega = \Lambda \times \Lambda$. We also define

$$H^{s,\gamma}_\omega(\Omega) := H^s_\omega(\Lambda; L^2_\omega(\Lambda)) \cap L^2_\omega(\Lambda; H^\gamma_\omega(\Lambda)), \tag{2.3}$$

and for $s, \gamma > 1/2$,

$$H^{s,\gamma}_{0,\omega}(\Omega) := \{v \in H^{s,\gamma}_\omega(\Omega), \ v\,|_{\partial\Omega} = 0\}, \tag{2.4}$$

endowed with the norm:

$$\|v\|_{H^{s,\gamma}_\omega(\Omega)} := \big(\|v\|^2_{H^s_\omega(\Lambda;L^2_\omega(\Lambda))} + \|v\|^2_{L^2_\omega(\Lambda;H^\gamma_\omega(\Lambda))}\big)^{\frac{1}{2}}.$$

When $\omega \equiv 1$, we will drop ω from the above notations, and we will also drop Λ and/or Ω from the notations if no confusion arises.

Let c be a generic positive constant independent of any functions and of any discretization parameters. We use the expression $A \lesssim B$ (respectively $A \gtrsim B$) to mean that $A \leqslant cB$ (respectively $A \geqslant cB$), and use the expression $A \cong B$ to mean that $A \lesssim B \lesssim A$.

We recall below some notations and properties fractional derivatives.

Definition 1. Let $s \in [n-1, n)$ with $n \in \mathbb{N}^+$. The left-sided and right-sided Riemann–Liouville fractional integrals ${}_{-1}I^s_x$ and ${}_xI^s_1$ of order s are defined as

$${}_{-1}I^s_x v(x) := \frac{1}{\Gamma(s)} \int_{-1}^{x} \frac{v(\tau)}{(x-\tau)^{1-s}} d\tau, \quad \forall x \in \Lambda, \tag{2.5}$$

and

$${}_xI^s_1 v(x) := \frac{1}{\Gamma(s)} \int_{x}^{1} \frac{v(\tau)}{(\tau-x)^{1-s}} d\tau, \quad \forall x \in \Lambda, \tag{2.6}$$

respectively, where $\Gamma(\cdot)$ is the Gamma function.

Definition 2. Let $s \in [n-1, n)$ with $n \in \mathbb{N}^+$. The left-sided and right-sided Riemann–Liouville fractional derivatives ${}_{-1}D^s_x$ and ${}_xD^s_1$ of order s are defined as

$${}_{-1}D^s_x v(x) := \frac{1}{\Gamma(n-s)} \frac{d^n}{dx^n} \int_{-1}^{x} \frac{v(\tau)}{(x-\tau)^{s-n+1}} d\tau, \quad \forall x \in \Lambda, \tag{2.7}$$

and

$${}_xD^s_1 v(x) := \frac{(-1)^n}{\Gamma(n-s)} \frac{d^n}{dx^n} \int_{x}^{1} \frac{v(\tau)}{(\tau-x)^{s-n+1}} d\tau, \quad \forall x \in \Lambda, \tag{2.8}$$

respectively.

One easily verifies that

$$({}_{-1}D_x^s\,{}_{-1}I_x^s f)(x) = f(x) \text{ and } ({}_xD_1^s\,{}_xI_1^s f)(x) = f(x). \tag{2.9}$$

In addition to the Riemann–Liouville fractional derivatives, the following Caputo fractional derivatives are also commonly used.

Definition 3. Let $s \in [n-1, n)$ with $n \in \mathbb{N}^+$. The left-sided and right-sided Caputo fractional derivatives ${}^C_{-1}D_x^s$ and ${}^C_xD_1^s$ of order s are defined as

$${}^C_{-1}D_x^s v(x) := \frac{1}{\Gamma(n-s)} \int_{-1}^{x} \frac{v^{(n)}(\tau)}{(x-\tau)^{s-n+1}} d\tau, \quad \forall x \in \Lambda, \tag{2.10}$$

and

$${}^C_xD_1^s v(x) := \frac{(-1)^n}{\Gamma(n-s)} \int_{x}^{1} \frac{v^{(n)}(\tau)}{(\tau-x)^{s-n+1}} d\tau, \quad \forall x \in \Lambda, \tag{2.11}$$

respectively.

The following lemma exhibits the relationship between Riemann–Liouville fractional derivatives and Caputo fractional derivatives.

Lemma 1. *Let $s \in [n-1, n)$ with $n \in \mathbb{N}^+$. Then,*

$${}_{-1}D_x^s f(x) = {}^C_{-1}D_x^s f(x) + \sum_{j=0}^{n-1} \frac{f^{(j)}(-1)}{\Gamma(1+j-s)} (1+x)^{j-s}; \tag{2.12}$$

$${}_xD_1^s f(x) = {}^C_xD_1^s f(x) + \sum_{j=0}^{n-1} \frac{(-1)^j f^{(j)}(1)}{\Gamma(1+j-s)} (1-x)^{j-s}. \tag{2.13}$$

The following two lemmas (cf. [5,13]) play fundamental roles for the analysis of well-posedness for fractional elliptic PDEs.

Lemma 2. *Let $1 < s < 2$, $s \neq 1$. We have*

$$\left\langle {}_{-1}D_x^s w, v \right\rangle_\Lambda = \left({}_{-1}D_x^{\frac{s}{2}} w, {}_xD_1^{\frac{s}{2}} v \right)_\Lambda, \quad \forall w,\ v \in H_0^{\frac{s}{2}}(\Lambda); \tag{2.14}$$

$$\left\langle {}_xD_1^s w, v \right\rangle_\Lambda = \left({}_xD_1^{\frac{s}{2}} w, {}_{-1}D_x^{\frac{s}{2}} v \right)_\Lambda, \quad \forall w,\ v \in H_0^{\frac{s}{2}}(\Lambda). \tag{2.15}$$

Lemma 3. *Let $s > \frac{1}{2}$, $s \neq n - \frac{1}{2}$, $n \in \mathbb{N}^+$. We have*

$$\| {}_{-1}D_x^s v \|_{L^2(\Lambda)} \cong \| v \|_{H^s(\Lambda)}, \quad \forall v \in H_0^s(\Lambda); \tag{2.16}$$

$$\| {}_xD_1^s v \|_{L^2(\Lambda)} \cong \| v \|_{H^s(\Lambda)}, \quad \forall v \in H_0^s(\Lambda). \tag{2.17}$$

3. One-dimensional case

After a suitable time discretization of (1.9) and assume a homogeneous Dirichlet boundary condition, we are led to solve, at each time step, a fractional elliptic problem of the following kind: $(1 < \alpha < 2)$:

$$\rho(x)u + {}_x\partial_1^{\frac{\alpha}{2}} \left[d_+(x)\, {}_{-1}\partial_x^{\frac{\alpha}{2}} u \right] + {}_{-1}\partial_x^{\frac{\alpha}{2}} \left[d_-(x)\, {}_x\partial_1^{\frac{\alpha}{2}} u \right] = f, \quad u(\pm 1) = 0, \tag{3.1}$$

with $\rho,\ d_\pm \in C(\Lambda)$ satisfying

$$\bar{\rho} \geq \rho(x) \geq 0,\ \bar{d} \geq d_\pm(x) \geq \underline{d} > 0. \tag{3.2}$$

3.1. Spectral–Galerkin method and error estimates

Thanks to Lemma 2, its weak formulation is: find $u \in H_0^{\frac{\alpha}{2}}(\Lambda)$, such that

$$A(u,v) := (\rho(x)\,u, v) + (d_+(x)\,{}_{-1}\partial_x^{\frac{\alpha}{2}} u, {}_{-1}\partial_x^{\frac{\alpha}{2}} v) + (d_-(x)\,{}_x\partial_1^{\frac{\alpha}{2}} u, {}_x\partial_1^{\frac{\alpha}{2}} v) = (f,v), \quad \forall v \in H_0^{\frac{\alpha}{2}}(\Lambda). \tag{3.3}$$

Lemma 3 immediately implies that $A(\cdot,\cdot)$ is continuous and coercive in $H_0^{\frac{\alpha}{2}}(\Lambda) \times H_0^{\frac{\alpha}{2}}(\Lambda)$, i.e.,

$$A(u,v) \lesssim \|u\|_{H^{\alpha/2}} \|v\|_{H^{\alpha/2}}; \quad \|u\|^2_{H^{\alpha/2}} \lesssim A(u,u), \quad \forall u, v \in H_0^{\frac{\alpha}{2}}(\Lambda). \tag{3.4}$$

Hence, the problem (3.3) admits a unique solution satisfying

$$\|u\|_{H^{\alpha/2}} \lesssim \|f\|_{(H_0^{\alpha/2})'}, \tag{3.5}$$

where $(H_0^{\alpha/2})'$ is the dual space of $H_0^{\alpha/2}$.

Let P_N be the space of polynomials of degree less than or equal to N, and let $X_N = \{v \in P_N : v(\pm 1) = 0\}$. The spectral–Galerkin approximation for (3.1) is: Find $u_N \in X_N$ such that

$$A(u_N, v_N) := (\rho u_N, v_N) + (d_+(x)\,{}_{-1}\partial_x^{\frac{\alpha}{2}} u_N, {}_{-1}\partial_x^{\frac{\alpha}{2}} v_N) + (d_-(x)\,{}_x\partial_1^{\frac{\alpha}{2}} u_N, {}_x\partial_1^{\frac{\alpha}{2}} v_N) = (f, v_N), \quad \forall v_N \in X_N. \tag{3.6}$$

Since $X_N \subset H_0^s$ for any $s > 1/2$, it is clear that the problem (3.6) admits a unique solution satisfying $\|u_N\|_{H^{\alpha/2}} \lesssim \|f\|_{(H_0^{\alpha/2})'}$. Furthermore, we have from (3.3) and (3.6) that

$$A(u - u_N, v_N) = 0 \quad \forall v_N \in X_N.$$

We derive immediately from (3.4) and Lemma 3 that we have the following error estimate

$$\|u - u_N\|_{H^{\alpha/2}} \lesssim \inf_{v_N \in X_N} \|u - v_N\|_{H^{\alpha/2}}. \tag{3.7}$$

Hence, we only have to estimate the best approximation for u in X_N.

Let $\omega^{a,b}(x) = (1-x)^a(1+x)^b$. We define the non-uniformly weighted Sobolev spaces

$$H^m_{\omega,*}(\Lambda) := \{u \in H_0^1(\Lambda) : \partial_x^k u \in L^2_{\omega^{k-1,k-1}}(\Lambda),\ 1 \le k \le m\}. \tag{3.8}$$

Let $\Pi_N^{1,0} : H_0^1(\Lambda) \to X_N$ be defined by

$$(\partial_x(u - \Pi_N^{1,0} u), \partial_x v_N) = 0, \quad \forall v_N \in X_N. \tag{3.9}$$

Then, it is well-known (cf. Thm. 3.39 in [23]) that the following estimate holds

$$\|u - \Pi_N^{1,0} u\|_{H^s} \lesssim N^{s-m} \|\partial_x^m u\|_{\omega^{m-1,m-1}}, \quad \forall u \in H^m_{\omega,*}(\Lambda),\ s = 0, 1. \tag{3.10}$$

Then by using a standard argument of space interpolation [1], it can be shown that the above estimate holds as well for all $s \in (0,1)$. Combining the above result with (3.7), we arrive at the following:

Theorem 1. *Let u and u_N be the solution of problem (3.3) and (3.6) respectively, and we assume that $u \in H^m_{\omega,*}(\Lambda)$. Then we have*

$$\|u - u_N\|_{H^{\frac{\alpha}{2}}} \lesssim N^{\frac{\alpha}{2} - m} \|\partial_x^m u\|_{\omega^{m-1,m-1}}. \tag{3.11}$$

Remark 1. Unlike the cases with one-sided fractional derivatives considered in [3], it is not clear how the regularity of u for the problem (3.3) depends on the data. Hence, we provide error estimates by assuming u in some non-uniformly weighted Sobolev spaces, namely, $u \in H^m_{\omega,*}(\Lambda)$. This type of error estimates provides more precise convergence rates, than the estimates with usual Sobolev spaces $H^m(\Lambda)$, for solutions with singularities at the endpoints, such as the problem (3.3). This is very important since this type of FDEs usually exhibit singularities at the endpoints. See Example 2 below.

3.2. Efficient implementation and numerical results

3.2.1. Efficient implementation

A main difficulty in implementing a variational based numerical scheme for fractional PDEs, particularly for those with variable coefficients, is how to compute the mass and stiffness matrices efficiently and accurately. We show below how to do this using various properties of Jacobi polynomials.

Setting $\phi_k(x) = L_k(x) - L_{k+2}(x)$, then, we have

$$X_N = \text{span}\{\phi_k(x) : k = 0, 1, \cdots, N-2\}.$$

Denote

$$u_N = \sum_{k=0}^{N-2} \tilde{u}_k \phi_k(x), \quad \bar{u} = (\tilde{u}_0, \tilde{u}_1, \cdots, \tilde{u}_{N-2})^T;$$

$$(s_+^\alpha)_{kj} = \int_\Lambda d_+(x)\, {}_{-1}D_x^{\frac{\alpha}{2}} \phi_j(x)\, {}_{-1}D_x^{\frac{\alpha}{2}} \phi_k(x) dx, \quad S_+^\alpha = \left((s_+^\alpha)_{kj}\right)_{k,j=0}^{N-2};$$

$$(s_-^\alpha)_{kj} = \int_\Lambda d_-(x)\, {}_xD_1^{\frac{\alpha}{2}} \phi_j(x)\, {}_xD_1^{\frac{\alpha}{2}} \phi_k(x) dx, \quad S_-^\alpha = \left((s_-^\alpha)_{kj}\right)_{k,j=0}^{N-2};$$

$$m_{kj} = \int_\Lambda \rho(x)\phi_k(x)\phi_j(x) dx, \quad M = (m_{kj})_{k,j=0}^{N-2};$$

$$f_k = (I_N f, \phi_k(x)), \quad \bar{f} = (f_0, f_1, \cdots, f_{N-2})^T,$$

where $I_N : C(\Lambda) \to P_N$ is the interpolation operator based on Legendre–Gauss–Lobatto points. Then, (3.6) reduces to the matrix system

$$(M + S_+^\alpha + S_-^\alpha)\bar{u} = \bar{f}. \tag{3.12}$$

Next, we shall show how to compute S_+^α and S_-^α efficiently. We shall first prove the following lemma.

Lemma 4. *Let $0 < \mu < 2$. We have*

$${}_{-1}D_x^\mu \hat{L}_n(x) = \frac{\Gamma(n+2)}{\Gamma(n-\mu+2)}(1+x)^{1-\mu} J_n^{\mu,1-\mu}(x); \tag{3.13}$$

$${}_xD_1^\mu \tilde{L}_n(x) = \frac{\Gamma(n+2)}{\Gamma(n-\mu+2)}(1-x)^{1-\mu} J_n^{1-\mu,\mu}(x), \tag{3.14}$$

where

$$\hat{L}_n(x) = L_n(x) + L_{n+1}(x), \quad \tilde{L}_n(x) = L_n(x) - L_{n+1}(x). \tag{3.15}$$

Proof. First by the property (A.8), we have

$$\begin{aligned} \hat{L}_n(x) &= L_n(x) + L_{n+1}(x) \\ &= (1+x) J_n^{0,1}(x), \end{aligned} \tag{3.16}$$

and

$$\begin{aligned} \tilde{L}_n(x) &= L_n(x) - L_{n+1}(x) \\ &= (1-x) J_n^{1,0}(x). \end{aligned} \tag{3.17}$$

Then by the definition of left-sided Riemann–Liouville fractional integral (2.5) and Jacobi Property (A.7), Equation (A.10) can be rewritten as:

$${}_{-1}I_x^\mu \left\{(1+x)^\beta J_n^{\alpha,\beta}(x)\right\} = \frac{\Gamma(n+\beta+1)}{\Gamma(n+\beta+\mu+1)}(1+x)^{\beta+\mu} J_n^{\alpha-\mu,\beta+\mu}(x). \tag{3.18}$$

Setting $\alpha = \mu$ and $\beta = 1-\mu$, equation (3.18) becomes

$${}_{-1}I_x^\mu \left\{(1+x)^{1-\mu} J_n^{\mu,1-\mu}(x)\right\} = \frac{\Gamma(n-\mu+2)}{\Gamma(n+2)}(1+x) J_n^{0,1}(x). \tag{3.19}$$

Applying the left-sided Riemann–Liouville fractional derivative on both sides of the equation (3.19) gives (3.13).

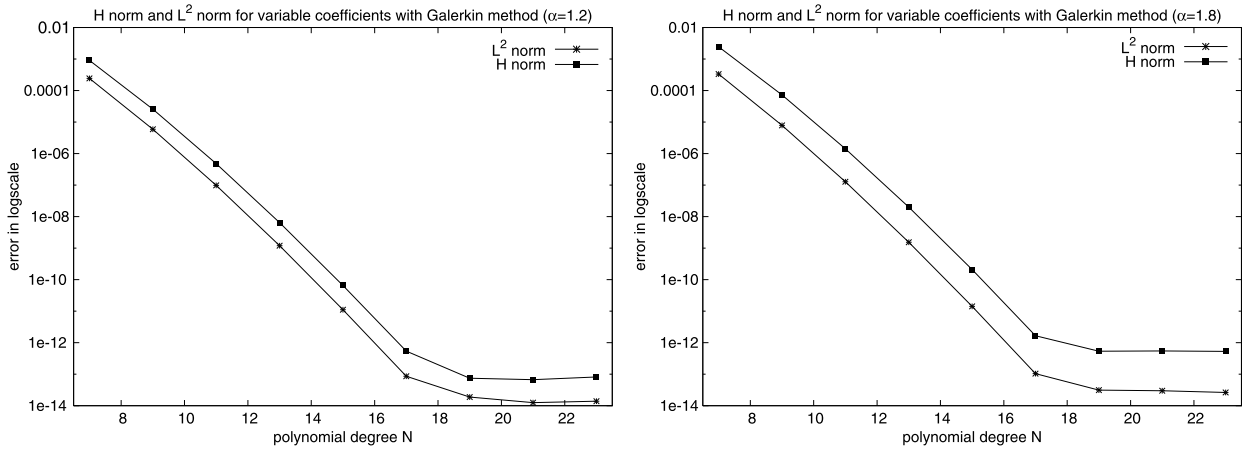

Fig. 1. $H^{\frac{\alpha}{2}}$ and L^2 errors with Galerkin method for variable coefficients (left: $\alpha = 1.2$, right: $\alpha = 1.8$).

By the definition of right-sided Riemann–Liouville integral (2.6) and (A.7), equation (A.11) can be written as

$$ {}_xI_1^{\mu}\left\{(1-x)^{\alpha} J_n^{\alpha,\beta}(x)\right\} = \frac{\Gamma(n+\alpha+1)}{\Gamma(n+\alpha+\mu+1)}(1-x)^{\alpha+\mu} J_n^{\alpha+\mu,\beta-\mu}(x). \tag{3.20} $$

Similarly, the second equation of Lemma 5 gives

$$ {}_xI_1^{\mu}\left\{(1-x)^{1-\mu} J_n^{1-\mu,\mu}(x)\right\} = \frac{\Gamma(n-\mu+2)}{\Gamma(n+2)}(1-x) J_n^{1,0}(x), \tag{3.21} $$

when $\alpha = 1-\mu$ and $\beta = \mu$. Applying the right-sided Riemann–Liouville fractional derivative on both sides of the equation (3.21) gives the right-sided Riemann–Liouville fractional derivative (3.14). □

Using equations (3.13) and (3.14), we obtain

$$ \begin{aligned} (s_+^{\alpha})_{kj} &= \int_{\Lambda} d_+(x) {}_{-1}D_x^{\frac{\alpha}{2}} \phi_j(x) {}_{-1}D_x^{\frac{\alpha}{2}} \phi_k(x) dx \\ &= \int_{-1}^{1} (1+x)^{2-\alpha} d_+(x) \hat{\phi}_j(x) \hat{\phi}_k(x) dx, \end{aligned} \tag{3.22} $$

$$ \begin{aligned} (s_-^{\alpha})_{kj} &= \int_{\Lambda} d_-(x) {}_xD_1^{\frac{\alpha}{2}} \phi_j(x) {}_xD_1^{\frac{\alpha}{2}} \phi_k(x) dx \\ &= \int_{-1}^{1} (1-x)^{2-\alpha} d_-(x) \tilde{\phi}_j(x) \tilde{\phi}_k(x) dx, \end{aligned} \tag{3.23} $$

where

$$ \hat{\phi}_j(x) = a_j J_j^{\frac{\alpha}{2}, 1-\frac{\alpha}{2}}(x) - a_{j+1} J_{j+1}^{\frac{\alpha}{2}, 1-\frac{\alpha}{2}}(x), \tag{3.24} $$

$$ \tilde{\phi}_j(x) = a_j J_j^{1-\frac{\alpha}{2}, \frac{\alpha}{2}}(x) + a_{j+1} J_{j+1}^{1-\frac{\alpha}{2}, \frac{\alpha}{2}}(x), \tag{3.25} $$

with $a_j = \frac{\Gamma(j+2)}{\Gamma(j-\frac{\alpha}{2}+2)}$. Hence, (3.22) and (3.23) can be efficiently computed by Jacobi–Gauss quadrature formula [25,23].

3.2.2. Numerical results

Example 1 *(Smooth solution).* We take $\rho \equiv 1$, $d_+(x) = d_-(x) = 1 + \sin^2 x$ and set the exact solution to be $u = \sin(\pi x)$.

The results are presented in Fig. 1. We observe that all errors decay exponentially, as is expected for spectral approximations to smooth functions.

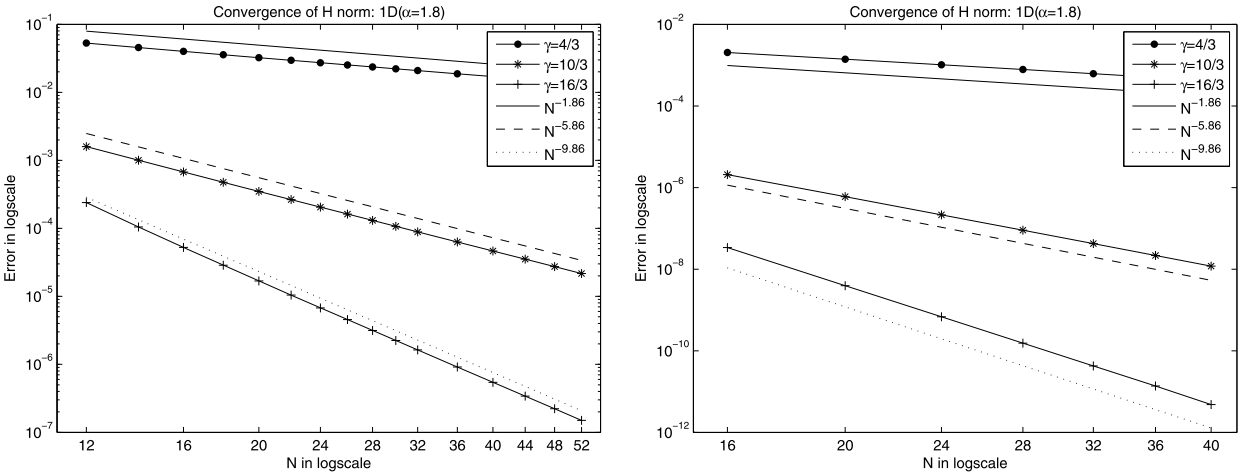

Fig. 2. Convergent rates for singular solutions: Left, $u(t) = x^{\gamma}(1-x^2)$; Right, $u(t) = (1+x)^{\gamma}(1-x)$.

Example 2 *(Non-smooth solutions).* We take $\rho \equiv 0$, $d_+(x) = d_-(x) \equiv 1$ with the exact solution to be $(1+x)^{\gamma}(1-x)$ or $x^{\gamma}(1-x^2)$.

Consider first $u(x) = x^{\gamma}(1-x^2)$ with $\gamma > 0$ not an integer. The solution has a singularity at $x = 0$ and we have $u \in H^{\gamma+\frac{1}{2}-\epsilon}$ for any small $\epsilon > 0$. Hence, the error estimate (3.11) indicates that the error in $H^{\alpha/2}$ should converge with a rate $\gamma + \frac{1}{2} - \frac{\alpha}{2}$. The results with $\alpha = 1.8$, for different γ are plotted in the left figure of Fig. 2. We observe that the convergence rate is roughly 0.93, 2.93 and 4.93 for $\gamma = \frac{4}{3}$, $\gamma = \frac{10}{3}$ and $\gamma = \frac{16}{3}$, respectively. These results agree well with (3.11).

Now, consider $u(x) = (1+x)^{\gamma}(1-x)$. Of course, we still have $u \in H^{\gamma+\frac{1}{2}-\epsilon}$, but by a direct computation, we also have $u \in H^k_{\omega^{k-1,k-1}}$ with $k < 2\gamma - \frac{1}{2}$. Hence, the result in (3.11) indicates that we can expect a convergence rate of $2\gamma - \frac{1}{2}$. Hence, the weighted error estimate in (3.11) provides much improved convergence rate for solutions with singularities at the endpoints. The results for this case are presented in the right of Fig. 2. We observe that the observed convergence rate matches well with the predicted one by (3.11).

4. Two-dimensional case

In this section, we consider the two dimensional generalization of (3.1):

$$\begin{aligned}&\rho(x,y)u(x,y) + {}_x\partial_1^{\frac{\alpha}{2}}\Big[d_+(x,y)\,{}_{-1}\partial_x^{\frac{\alpha}{2}}u(x,y)\Big] + {}_{-1}\partial_x^{\frac{\alpha}{2}}\Big[d_-(x,y)\,{}_x\partial_1^{\frac{\alpha}{2}}u(x,y)\Big]\\ &\quad + {}_y\partial_1^{\frac{\beta}{2}}\Big[e_+(x,y)\,{}_{-1}\partial_y^{\frac{\beta}{2}}u(x,y)\Big] + {}_{-1}\partial_y^{\frac{\beta}{2}}\Big[e_-(x,y)\,{}_y\partial_1^{\frac{\beta}{2}}u(x,y)\Big] = f(x,y), \quad (x,y)\in\Omega,\end{aligned} \tag{4.1}$$

where $1 < \alpha, \beta < 2$, $\Omega = (-1,1)\times(-1,1)$, $0 \le \rho(x,y) \le \bar{\rho}$, $0 < \underline{d} \le d_+(x,y), d_-(x,y) \le \bar{d}$ and $0 < \underline{e} \le e_+(x,y), e_-(x,y) \le \bar{e}$, subjected to

$$u|_{\partial\Omega} = 0. \tag{4.2}$$

4.1. Spectral–Galerkin method and error estimates

We define the following bilinear form $H_0^{\frac{\alpha}{2},\frac{\beta}{2}}(\Omega)\times H_0^{\frac{\alpha}{2},\frac{\beta}{2}}(\Omega)\to\mathbb{R}$:

$$\begin{aligned}\mathcal{A}(u,v) := &(\rho(x,y)u,v) + (d_+(x,y)\,{}_{-1}\partial_x^{\frac{\alpha}{2}}u, {}_{-1}\partial_x^{\frac{\alpha}{2}}v) + (d_-(x,y)\,{}_x\partial_1^{\frac{\alpha}{2}}u, {}_x\partial_1^{\frac{\alpha}{2}}v)\\ &+ (e_+(x,y)\,{}_{-1}\partial_y^{\frac{\beta}{2}}u, {}_{-1}\partial_y^{\frac{\beta}{2}}v) + (e_-(x,y)\,{}_y\partial_1^{\frac{\beta}{2}}u, {}_y\partial_1^{\frac{\beta}{2}}v).\end{aligned} \tag{4.3}$$

Then, a weak formulation for (4.1) with (4.2) is:
Given $f \in H_0^{\frac{\alpha}{2},\frac{\beta}{2}}(\Omega)'$, find $u \in H_0^{\frac{\alpha}{2},\frac{\beta}{2}}(\Omega)$, such that

$$\mathcal{A}(u,v) = <f,v>_{\Omega}, \quad \forall v \in H_0^{\frac{\alpha}{2},\frac{\beta}{2}}(\Omega), \tag{4.4}$$

where $\langle\cdot,\cdot\rangle_{\Omega}$ stands for the duality between $H_0^{\frac{\alpha}{2},\frac{\beta}{2}}(\Omega)'$ and $H_0^{\frac{\alpha}{2},\frac{\beta}{2}}(\Omega)$.

Thanks to the Lemma 3, we can prove the following:

Theorem 2. *The problem (4.4) admits a unique solution satisfying*

$$\|u\|_{H^{\frac{\alpha}{2},\frac{\beta}{2}}(\Omega)} \lesssim \|f\|_{H^{\frac{\alpha}{2},\frac{\beta}{2}}(\Omega)'}. \tag{4.5}$$

Proof. First of all, it is clear from (4.3) that the bilinear form $\mathcal{A}(\cdot,\cdot)$ is self-adjoint, i.e., $\mathcal{A}(u,v)=\mathcal{A}(v,u)$. Next, we show that it is continuous and coercive in $H_0^{\frac{\alpha}{2},\frac{\beta}{2}}(\Omega)\times H_0^{\frac{\alpha}{2},\frac{\beta}{2}}(\Omega)$.

Apply Lemma 3, we find that $\forall u,\ v\in H_0^{\frac{\alpha}{2},\frac{\beta}{2}}(\Omega)$, we have

$$\begin{aligned}
|\mathcal{A}(u,v)| &\le \bar{\rho}\|u\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} + \bar{d}\|{}_{-1}D_x^{\frac{\alpha}{2}}u\|_{L^2(\Omega)}\|{}_{-1}D_x^{\frac{\alpha}{2}}v\|_{L^2(\Omega)}\\
&\quad + \bar{d}\|{}_xD_1^{\frac{\alpha}{2}}u\|_{L^2(\Omega)}\|{}_xD_1^{\frac{\alpha}{2}}v\|_{L^2(\Omega)} + \bar{e}\|{}_{-1}D_y^{\frac{\beta}{2}}u\|_{L^2(\Omega)}\|{}_{-1}D_x^{\frac{\beta}{2}}v\|_{L^2(\Omega)}\\
&\quad + \bar{e}\|{}_yD_1^{\frac{\beta}{2}}u\|_{L^2(\Omega)}\|{}_yD_1^{\frac{\beta}{2}}v\|_{L^2(\Omega)}\\
&\lesssim \|u\|_{H^{\frac{\alpha}{2}}(\Lambda;L^2(\Lambda))}\|v\|_{H^{\frac{\alpha}{2}}(\Lambda;L^2(\Lambda))} + \|u\|_{L^2(\Lambda;H^{\frac{\beta}{2}})}\|v\|_{L^2(\Lambda;H^{\frac{\beta}{2}})}\\
&\lesssim \|u\|_{H^{\frac{\alpha}{2},\frac{\beta}{2}}(\Omega)}\|v\|_{H^{\frac{\alpha}{2},\frac{\beta}{2}}(\Omega)}.
\end{aligned} \tag{4.6}$$

This gives the continuity of $\mathcal{A}$.

For the coercivity, we use Lemma 3 to derive that for all $u\in H_0^{\frac{\alpha}{2},\frac{\beta}{2}}(\Omega)$,

$$\begin{aligned}
\mathcal{A}(u,u) &= (\rho(x,y)u,u) + (d_+(x,y)\,{}_{-1}\partial_x^{\frac{\alpha}{2}}u,\,{}_{-1}\partial_x^{\frac{\alpha}{2}}u) + (d_-(x,y)\,{}_x\partial_1^{\frac{\alpha}{2}}u,\,{}_x\partial_1^{\frac{\alpha}{2}}u)\\
&\quad + (e_+(x,y)\,{}_{-1}\partial_y^{\frac{\beta}{2}}u,\,{}_{-1}\partial_y^{\frac{\beta}{2}}u) + (e_-(x,y)\,{}_y\partial_1^{\frac{\beta}{2}}u,\,{}_y\partial_1^{\frac{\beta}{2}}u))\\
&\gtrsim \underline{d}\|{}_{-1}D_x^{\frac{\alpha}{2}}u\|_{L^2(\Omega)}^2 + \underline{d}\|{}_xD_1^{\frac{\alpha}{2}}u\|_{L^2(\Omega)} + \underline{e}\|{}_{-1}D_y^{\frac{\beta}{2}}u\|_{L^2(\Omega)}^2 + \underline{e}\|{}_yD_1^{\frac{\beta}{2}}u\|_{L^2(\Omega)}\\
&\gtrsim \|u\|_{H^{\frac{\alpha}{2}}(\Lambda;L^2(\Lambda))}^2 + \|u\|_{L^2(\Lambda;H^{\frac{\beta}{2}})}^2\\
&\gtrsim \|u\|_{H^{\frac{\alpha}{2},\frac{\beta}{2}}(\Omega)}^2.
\end{aligned} \tag{4.7}$$

Thanks to the Lax–Milgram lemma, the problem (4.4) admits a unique solution. Finally, taking $v=u$ in (4.3) and using (4.7), we obtain

$$\|u\|_{H^{\frac{\alpha}{2},\frac{\beta}{2}}(\Omega)}^2 \lesssim \langle f,u\rangle_\Omega \lesssim \|f\|_{H^{\frac{\alpha}{2},\frac{\beta}{2}}(\Omega)'}\|u\|_{H^{\frac{\alpha}{2},\frac{\beta}{2}}(\Omega)}. \qquad \square \tag{4.8}$$

To simplify the presentation, we take the same number of modes in each direction. Then, the spectral–Galerkin approximation to (4.4) is: find $u_N\in\mathbf{X}_N=X_N\times X_N$, such that

$$\mathcal{A}(u_N,v_N)=(f,v_N),\quad \forall v_N\in\mathbf{X}_N. \tag{4.9}$$

Since $\mathbf{X}_N\subset H_0^{\frac{\alpha}{2},\frac{\beta}{2}}(\Omega)$, the well-posedness of (4.9) can be established as in Theorem 2, and one derives immediately from Theorem 2 that we have the following error estimate:

$$\|u-u_N\|_{H^{\frac{\alpha}{2},\frac{\beta}{2}}(\Omega)} \lesssim \inf_{v_N\in\mathbf{X}_N}\|u-v_N\|_{H^{\frac{\alpha}{2},\frac{\beta}{2}}(\Omega)}. \tag{4.10}$$

To describe the error in a more precise form, we define the non-uniformly weighted Sobolev spaces (cf. (8.4.30) in [23] with $\alpha=\beta=0$ and $d=2$):

$$\hat{B}^r(\Omega) := \{u\in L^2(\Omega) : \partial_x^{r_1}\partial_x^{r_2}u\in L^2_{\omega_x^{\max(r_1,1)-1}\omega_y^{\max(r_2,1)-1}}(\Omega),\quad \forall\, 0\le r_1,r_2\le r\}, \tag{4.11}$$

where $\omega_x^a=(1-x^2)^a$ and $\omega_y^b=(1-y^2)^b$, with the semi-norm and norm defined by

$$\begin{aligned}
|u|_{\hat{B}^r}^2 &= \sum_{0\le r_1,r_2\le r}\|\partial_x^{r_1}\partial_x^{r_2}u\|^2_{L^2_{\omega_x^{\max(r_1,1)-1}\omega_y^{\max(r_2,1)-1}}},\\
\|u\|_{\hat{B}^r}^2 &= \|u\|_{L^2}^2 + |u|_{\hat{B}^r}^2.
\end{aligned} \tag{4.12}$$

Let $\Pi_N^{1,0}: H_0^1(\Omega) \to \mathbf{X}_N$ be defined by

$$(\nabla(u - \Pi_N^{1,0}u), \nabla v_N) = 0, \quad \forall v_N \in \mathbf{X}_N. \tag{4.13}$$

Then, it is shown (cf. Thms. 8.2 & 8.3 in [23]) that the following estimate holds:

$$\|u - \Pi_N^{1,0}u\|_{H^s(\Omega)} \lesssim N^{s-m}|u|_{\hat{B}^m}, \quad \forall u \in \hat{B}^m(\Omega) \cap H_0^1(\Omega),\ s = 0, 1. \tag{4.14}$$

Then by using a standard argument of space interpolation [1], it can be shown that the above estimate holds as well for all $s \in [0, 1]$. Combining the above result with (4.10), we arrive at the following:

Theorem 3. *Let u and u_N be the solution of problem (4.4) and (4.9) respectively, and we assume that $u \in \hat{B}^m(\Lambda) \cap H_0^1(\Omega)$. Then we have*

$$\|u - u_N\|_{H^{\frac{\alpha}{2},\frac{\beta}{2}}} \lesssim N^{\min(\frac{\alpha}{2},\frac{\beta}{2})-m}|u|_{\hat{B}^m}. \tag{4.15}$$

4.2. Efficient implementation

4.2.1. Separable case

We consider first the separable case. More precisely, we assume in (4.1) that

$$\begin{aligned} &\rho(x, y) = \rho_1(x)\rho_2(y),\ d_+(x, y) = d_+(x),\ d_-(x, y) = d_-(x),\\ &e_+(x, y) = e_+(y),\ e_-(x, y) = e_-(y). \end{aligned} \tag{4.16}$$

Let $\phi_k(\cdot) = L_k(\cdot) - L_{k+2}(\cdot)$ and denote

$$\begin{aligned} u_N &= \sum_{k,j=0}^{N-2} \tilde{u}_{kj}\phi_k(x)\phi_j(y), \quad U = (\tilde{u}_{kj})_{k,j=0}^{N-2};\\ f_{kj} &= (\mathbb{I}_N f, \phi_i(x)\phi_j(y)), \quad F = (f_{kj})_{k,j=0}^{N-2},\\ s_{kj}^x &= \int_\Lambda d_+(x)\ {}_{-1}D_x^{\frac{\alpha}{2}}\phi_j(x)\,{}_{-1}D_x^{\frac{\alpha}{2}}\phi_k(x) + d_-(x)\ {}_xD_1^{\frac{\alpha}{2}}\phi_k(x)\,{}_xD_1^{\frac{\alpha}{2}}\phi_j(x)dx;\\ s_{kj}^y &= \int_\Lambda e_+(y)\ {}_{-1}D_y^{\frac{\beta}{2}}\phi_j(y)\,{}_{-1}D_y^{\frac{\beta}{2}}\phi_k(y) + e_-(y)\ {}_yD_1^{\frac{\beta}{2}}\phi_k(y)\,{}_yD_1^{\frac{\beta}{2}}\phi_j(y)dy;\\ m_{kj}^x &= \int_\Lambda \rho_1(x)\phi_k(x)\phi_j(x)dx, \quad m_{kj}^y = \int_\Lambda \rho_2(y)\phi_k(y)\phi_j(y)dy;\\ M^x &= (m_{kj}^x)_{k,j=0}^{N-2},\ M^y = (m_{kj}^y)_{k,j=0}^{N-2},\ S^x = \left(s_{kj}^x\right)_{k,j=0}^{N-2},\ S^y = \left(s_{kj}^y\right)_{k,j=0}^{N-2}, \end{aligned} \tag{4.17}$$

where $\mathbb{I}_N: C(\Omega) \to P_N \times P_N$ is the interpolation operator associated with tensor product of Legendre–Gauss–Lobatto points. Since S^x, S^y, M^x, M^y are all symmetric, we find that (4.9) with (4.16) is equivalent to the following linear system:

$$M^x U M^y + S^x U M^y + M^x U S^y = F. \tag{4.18}$$

Even when $d_+(x)$, $d_-(x)$, $e_+(x)$ and $e_-(x)$ are constants, the stiffness matrices S^x and S^y are full, so a direct inversion of the above system would be very expensive. However, this cost can be significantly reduced by using a discrete version of "separation of variables" — the matrix decomposition/diagonalization method [15,8,24]. To this end, we consider the following generalized eigenvalue problems:

$$M^x\overline{v} = \lambda^x S^x\overline{v}, \tag{4.19}$$

$$M^y\overline{w} = \lambda^y S^y\overline{w}. \tag{4.20}$$

Let Λ^x (respectively Λ^y) be the diagonal matrix whose diagonal entries λ^x (respectively λ^y) are the eigenvalues of (4.19) (respectively (4.20)), and let E^x (respectively E^y) be the matrix whose columns are the corresponding eigenvectors of (4.19) (respectively (4.20)), then we have

$$M^x E^x = S^x E^x \Lambda^x, \tag{4.21}$$

$$M^y E^y = S^y E^y \Lambda^y. \tag{4.22}$$

Since S^x, S^y, M^x, M^y are all symmetric and positive definite, the eigenvalues, λ^x and λ^y, are all real and positive, and $(E^x)^{-1} = (E^x)^T$ and $(E^y)^{-1} = (E^y)^T$.

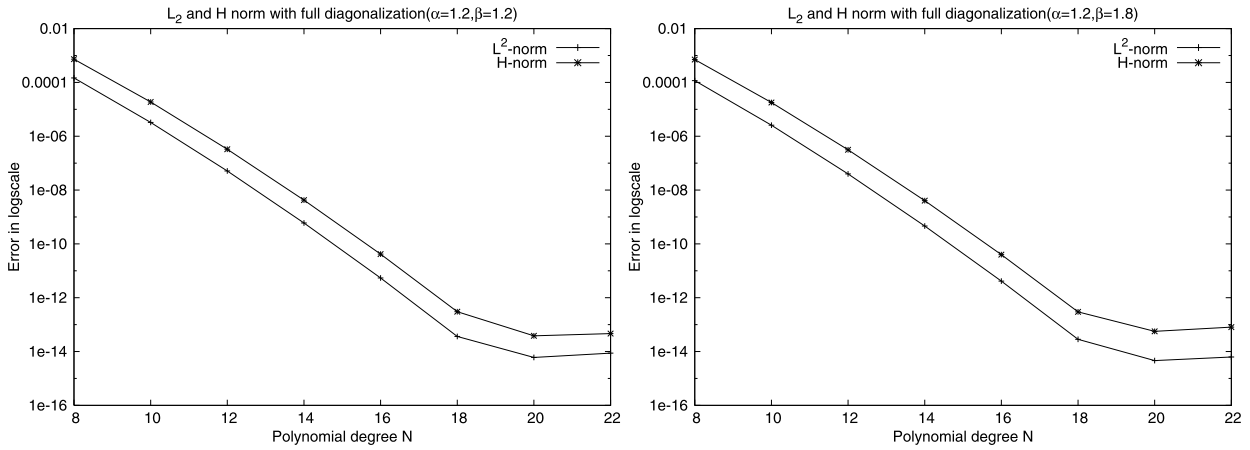

Fig. 3. $H^{\frac{\alpha}{2},\frac{\beta}{2}}$ and L^2 errors with spectral decomposition method for separable variable coefficients. Left: $\alpha = 1.2$, $\beta = 1.2$; right: $\alpha = 1.2$, $\beta = 1.8$.

Setting $U = E^x V (E^x)^T$ in (4.18), we get

$$M^x E^x V (E^y)^T M^y + S^x E^x V (E^y)^T M^y + M^x E^x V (E^y)^T S^y = F. \tag{4.23}$$

Multiplying the left (resp. right) of the above equation by $(E^x S^x)^{-1}$ (resp. $(E^y S^y)^{-T}$), then using (4.21) and (4.22), we arrive at

$$\rho \Lambda^x V \Lambda^y + V \Lambda^y + \Lambda^x V = (E^x S^x)^{-1} F (E^y S^y)^{-T} := H, \tag{4.24}$$

which is equivalent to

$$(\rho \lambda_i^x \lambda_j^y + \lambda_j^y + \lambda_i^x) v_{ij} = h_{ij}, \quad 0 \le i, j \le N-2, \tag{4.25}$$

where v_{ij} and h_{ij} are entries of V and H.

In summary, the matrix diagonalization method for solving (4.18) consists of the following steps:

1. Pre-processing: compute the eigenvalues and eigenvectors of the generalized eigenvalue problem (4.19), (4.20) and compute $(E^x S^x)^{-1}$, $(E^y S^y)^{-T}$;
2. Compute the expansion coefficients of $I_N f$ (backward Legendre transform);
3. Compute $F = (f_{ij})$ with $f_{ij} = (I_N f, \phi_i(x)\phi_j(y))$;
4. Compute $G = (E^x S^x)^{-1} F (E^y S^y)^{-T}$;
5. Obtain V by solving (4.25);
6. Set $U = E^x V (E^y)^T$;
7. Compute the values of u_N at Legendre Gauss–Lobatto points (forward Legendre transform).

The above procedure consists of several matrix–matrix multiplications so the total cost is a small multiple of N^3.

We now present some numerical results.

Example 3 *(Smooth solution).* We take $\rho(x, y) = 1$, $d_+(x) = 1 + \sin^2(\pi x)$, $d_-(x) = 1 + \cos^2(\pi x)$ and $e_+(y) = 1 + \sin^2(\pi y)$, $e_-(y) = 1 + \cos^2(\pi y)$, and consider the exact analytical solution

$$u(x, y) = \sin(\pi x) \sin(\pi y).$$

In Fig. 3, we plot the error in L^2-norms and $H^{\frac{\alpha}{2},\frac{\beta}{2}}$-norms vs. N in semi-log scale. We observe that the errors converge exponentially.

Example 4 *(Non-smooth solutions).* We take $\rho \equiv 0$, $d_+(x) = d_-(x) = e_+(x) = e_-(x) \equiv 1$ with the exact solution to be $(1 + x)^\gamma (1 - x) \sin y$ or $x^\gamma (1 - x^2) \sin y$.

The left figure of Fig. 4 shows that the convergence rate is $N^{-5.23}$ for $\alpha = \beta = 1.2$ and $N^{-4.93}$ for $\alpha = \beta = 1.8$ with $\gamma = 16/3$ which is coincide with the estimate (4.15). We also present the convergence rate for $u(x) = (1+x)^\gamma (1 - x)$ in the right figure of Fig. 4. We observe that the convergence rates also agree with our error estimates.

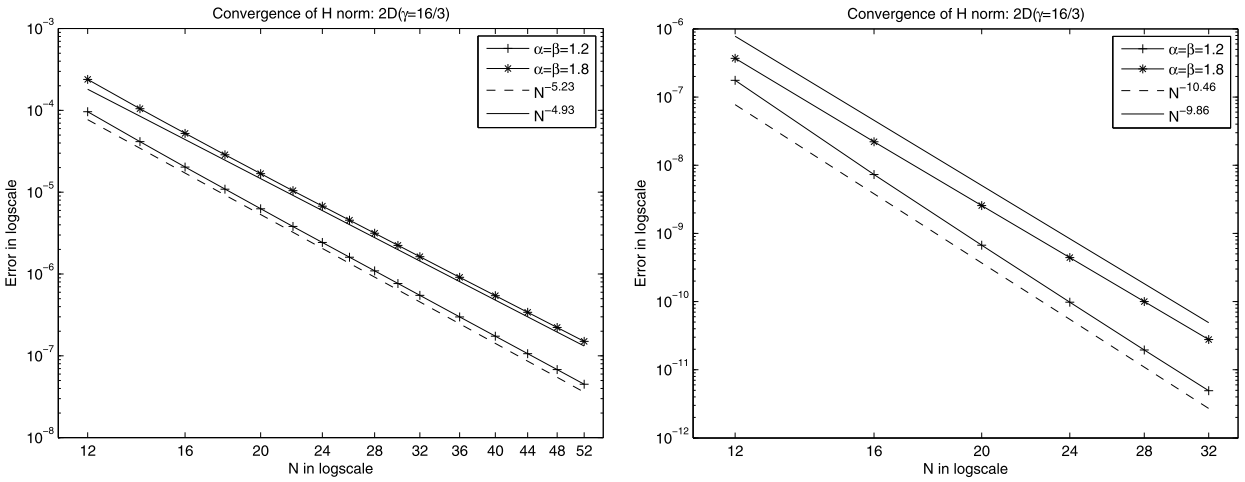

Fig. 4. Convergent rates for singular solutions: Left: $u(x) = x^{16/3}(1-x^2)\sin y$; Right: $u(x) = (1+x)^{16/3}(1-x)\sin y$.

4.2.2. Non-separable case

Now let us consider the non-separable problem (4.1)–(4.2). Since the problem is non-separable, we can no longer use the matrix diagonalization method presented above.

Let $u_N = \sum_{k,j=0}^{N-2} \tilde{u}_{kj}\phi_k(x)\phi_j(y) \in \mathbf{X}_N$ be the solution of (4.9), and set $\bar{u}$ to be the vector consisting $\{\tilde{u}_{kj}\}_{k,j=0}^{N-2}$. Taking $v_N = \phi_k(x)\phi_j(y)$ in (4.9), we can rewrite (4.9) as a linear system:

$$\mathbb{A}\bar{u} = \bar{f}. \tag{4.26}$$

However, it is too expensive to form the matrix $\mathbb{A}$ explicitly and solve the above linear system by a direct method. Hence, we shall use the same iterative approach for the regular PDEs with a suitable separable problem as a preconditioner [24]. The key for the efficiency of this approach is to compute $\bar{u} \to \mathbb{A}\bar{u}$ efficiently. A direct approach for computing $\bar{u} \to \mathbb{A}\bar{u}$ will cost at least $O(N^4)$ which is too expensive. A direct application of the usual transform approach [7,24] does not lead to much savings due to the fact that $\partial_x^{\frac{\alpha}{2}}\phi_j$ is no longer a polynomial.

Below, we describe a procedure to compute $\bar{u} \to \mathbb{A}\bar{u}$ in $O(N^3)$ by using Lemma 4.

To simplify the presentation, we shall only consider the part of $\mathbb{A}\bar{u}$ related to the term

$$(d_+(x,y)\,{}_{-1}\partial_x^{\frac{\alpha}{2}} u_N,\, {}_{-1}\partial_x^{\frac{\alpha}{2}} v_N)$$

in (4.9), since other three terms can be computed in a similar fashion. Taking $u_N = \sum_{k,j=0}^{N-2} \tilde{u}_{kj}\phi_k(x)\phi_j(y)$ and $v_N = \phi_l(x)\phi_i(y)$ in the above, and using Lemma 4, we find:

$$\begin{aligned}
&(d_+(x,y)\,{}_{-1}\partial_x^{\frac{\alpha}{2}} u_N, {}_{-1}\partial_x^{\frac{\alpha}{2}}\phi_l(x)\phi_i(y)),\\
&\quad = \Big(d_+(x,y)\sum_{k,j=0}^{N-2}\tilde{u}_{kj}\,{}_{-1}\partial_x^{\frac{\alpha}{2}}\phi_k(x)\phi_j(y),\, {}_{-1}\partial_x^{\frac{\alpha}{2}}\phi_l(x)\phi_i(y)\Big),\\
&\quad = (d_+(x,y)\sum_{k,j=0}^{N-2}\tilde{u}_{kj}\hat{\phi}_k(x)\phi_j(y),\ \hat{\phi}_l(x)\phi_i(y))_{w^{0,2-\alpha}(x)}.
\end{aligned} \tag{4.27}$$

Now we can use the following algorithms to compute the above:

Pre-computation: Compute the Gauss–Jacobi–Lobatto points and weights $\{x_j, w_j^x\}_{j=0}^N$ with respect to weight $w^{0,2-\alpha}(x)$, and the Gauss–Legendre–Lobatto points and weights $\{y_j, w_j^y\}_{j=0}^N$. Write $\hat{\phi}_l(x)$ in the form:

$$\hat{\phi}_l(x) = \sum_{m=0}^{l}\phi_{lm}J_m^{0,2-\alpha}(x), \tag{4.28}$$

and compute $\hat{\phi}_l(x_j)$ for $0 \le l, j \le N$.

Step 1: Compute

$$\tilde{u}(x_m, y_n) = \sum_{k,j=0}^{N-2}\tilde{u}_{kj}\hat{\phi}_k(x_m)\phi_j(y_n),\ m,n = 0,1,\cdots,N.$$

Table 1
Number of iterations for solving (4.26).

N	# of iteration					
	$\alpha = 1.2$, $\beta = 1.2$		$\alpha = 1.2$, $\beta = 1.8$		$\alpha = 1.8$, $\beta = 1.8$	
	without Precon.	with Precon.	without Precon.	with Precon.	without Precon.	with Precon.
20	209	13	217	13	138	13
30	433	13	447	13	269	14
40	730	14	723	14	416	15
50	1036	14	1049	15	588	15
60	1459	14	1427	15	722	15
70	1948	14	1876	15	981	15

Step 2: (Discrete Jacobi–Legendre transform) Determine $\{\tilde{w}_{kj}\}_{k,j=0}^{N}$ from

$$I_N(d_+\tilde{u})(x_m, y_n) = \sum_{k,j=0}^{N} \tilde{w}_{kj} J_k^{0,2-\alpha}(x_m) L_j(y_n),\ m, n = 0, 1, \cdots, N.$$

This can be done by using the Gauss–Jacobi and Gauss–Legendre quadrature [23] in $O(N^3)$ flops.

Step 3: Finally, thank to (A.1), we have

$$\begin{aligned}
&\big(d_+ \sum_{k,j=0}^{N-2} \tilde{u}_k \hat{\phi}_k(x)\phi_j(y), \hat{\phi}_l(x)\phi_i(y)\big)_{w^{0,2-\alpha}(x)} \\
&= \big(\sum_{k,j=0}^{N} \tilde{w}_{kj} J_k^{0,2-\alpha} L_j(y), \sum_{m=0}^{l} (\phi_{lm} J_m^{0,2-\alpha})\phi_i(y)\big)_{w^{0,2-\alpha}(x)} \\
&= \sum_{j=0}^{N}\sum_{k=0}^{l} (\tilde{w}_{kj}\gamma_k^{0,2-\alpha}\phi_{lk})(L_j(y), \phi_i(y)), \quad i, l = 0, 1, \cdots, N-2.
\end{aligned}$$

Note that $(L_j(y), \phi_i(y)) \neq 0$ only if $j = i, i+2$ so all steps above can be compute in $O(N^3)$ flops with matrix matrix product.

Let $\bar{\rho}$, $\bar{d}_\pm$ and $\bar{e}_\pm$ be some average constants of $\rho(x)$, $d_\pm(x)$ and $e_\pm(x)$, respectively. We define a bilinear form $\mathbf{X}_N \to \mathbb{R}$ by

$$\begin{aligned}
\tilde{A}(u_N, v_N) := \bar{\rho}(u_N, v_N) &+ \bar{d}_+({}_{-1}\partial_x^{\frac{\alpha}{2}} u_N, {}_{-1}\partial_x^{\frac{\alpha}{2}} v_N) + \bar{d}_-({}_x\partial_1^{\frac{\alpha}{2}} u_N, {}_x\partial_1^{\frac{\alpha}{2}} v_N) \\
&+ \bar{e}_+({}_{-1}\partial_y^{\frac{\beta}{2}} u_N, {}_{-1}\partial_y^{\frac{\beta}{2}} v_N) + \bar{e}_-({}_y\partial_1^{\frac{\beta}{2}} u_N, {}_y\partial_1^{\frac{\beta}{2}} v_N).
\end{aligned}$$

Then, it can be easily shown that there exist constants $c_1, c_2 > 0$ such that

$$c_1 \leq \frac{A(u_N, u_N)}{\tilde{A}(u_N, u_N)} \leq c_2, \tag{4.29}$$

which implies that the matrix associated to the bilinear form $\tilde{A}(\cdot,\cdot)$ is an optimal preconditioner for the matrix $\mathbb{A}$ associated to the bilinear form $A(\cdot,\cdot)$, i.e., the linear system (4.26) for a 2-D non-separable problem can also be solved, within any given accuracy threshold ϵ, in $O(N^3)$ flops.

We now present a numerical example to show the effectiveness of this iterative approach.

Example 5. We consider the problem (4.1)–(4.2) with a random function f and $\rho(x, y) = 1$, $d_+(x, y) = d_-(x, y) = 1 + \sin^2 x + \sin^2 y$, $e_+(x, y) = e_-(x, y) = 1 + \cos^2 x + \cos^2 y$.

We set the threshold $\epsilon = 10^{-8}$ and list in Table 1 the iteration numbers of the CG method for solving (4.26) directly or using the system with $\rho = d_\pm = e_\pm = 1$ as the preconditioner. We observe that the iteration numbers are bounded independent of N if the preconditioner is used.

5. Various extensions

We discuss in this section several direct extensions of the algorithms presented above.

5.1. Fractional PDEs in non-conserved form

We explain below how the algorithms presented in the previous section can be extended to deal with fractional PDEs in non-conservative form:

$$\begin{aligned}&\rho u(x,y) - d_+(x,y)\,{}_{-1}\partial_x^{\alpha} u(x,y) - d_-(x,y)\,{}_x\partial_1^{\alpha} u(x,y)\\ &\quad - e_+(x,y)\,{}_{-1}\partial_y^{\beta} u(x,y) - e_-(x,y)\,{}_y\partial_1^{\beta} u(x,y) = f(x,y), \quad (x,y)\in\Omega,\end{aligned} \tag{5.1}$$

with boundary condition (4.2).

We consider first the separable case, i.e., the coefficients satisfying (4.16). We define a non-symmetric bilinear form

$$\begin{aligned}\mathcal{A}(u,v) := \rho(u,v) &- ({}_{-1}\partial_x^{\frac{\alpha}{2}} u, {}_x\partial_1^{\frac{\alpha}{2}}(d_+(x)\,v)) - ({}_x\partial_1^{\frac{\alpha}{2}} u, {}_{-1}\partial_x^{\frac{\alpha}{2}}(d_-(x)\,v))\\ &- ({}_{-1}\partial_y^{\frac{\beta}{2}} u, {}_y\partial_1^{\frac{\beta}{2}}(e_+(y)\,v)) - ({}_y\partial_1^{\frac{\beta}{2}} u, {}_{-1}\partial_y^{\frac{\beta}{2}}(e_-(y)\,v)).\end{aligned} \tag{5.2}$$

Then, a spectral–Galerkin approximation to (5.1) with (4.2) is: Find $u_N\in\mathbf{X}_N$, such that

$$\mathcal{A}(u_N, v_N) = (\mathbb{I}_N f, v_N), \quad \forall v_N\in\mathbf{X}_N. \tag{5.3}$$

If we replace s^x_{kj} and s^y_{kj} in (4.17) by

$$s^x_{kj} = -\int_\Lambda {}_{-1}D_x^{\frac{\alpha}{2}}\phi_j(x)\,{}_xD_1^{\frac{\alpha}{2}}(d_+(x)\phi_k(x)) + {}_xD_1^{\frac{\alpha}{2}}\phi_j(x)\,{}_{-1}D_x^{\frac{\alpha}{2}}(d_-(x)\phi_k(x))dx, \quad S^x = \big(s^x_{kj}\big)_{k,j=0}^{N-2};$$

$$s^y_{kj} = -\int_\Lambda {}_{-1}D_y^{\frac{\beta}{2}}\phi_j(y)\,{}_yD_1^{\frac{\beta}{2}}(e_+(y)\phi_k(y)) + {}_yD_1^{\frac{\beta}{2}}\phi_j(y)\,{}_{-1}D_y^{\frac{\beta}{2}}(e_-(y)\phi_k(y))dy, \quad S^y = \big(s^y_{kj}\big)_{k,j=0}^{N-2}.$$

Then, taking $v_N = \phi_m(x)\phi_n(y)$ in (5.3) for $m,n = 0,1,\cdots,N-2$, we find that (5.3) reduces to the following linear system:

$$M^x U M^y + S^x U M^y + M^x U (S^y)^T = F. \tag{5.4}$$

Thanks to Lemma 4, the entries s^x_{kj} can be computed as follows:

$$\begin{aligned}s^x_{kj} &= -\int_\Lambda ({}_{-1}D_x^{\frac{\alpha}{2}}\phi_j(x)\,{}_xD_1^{\frac{\alpha}{2}}(d_+(x)\,\phi_k(x)) + {}_xD_1^{\frac{\alpha}{2}}\phi_j(x)\,{}_{-1}D_x^{\frac{\alpha}{2}}(d_-(x)\,\phi_k(x)))dx\\ &= -\int_{-1}^{1} ({}_{-1}D_x^{\alpha}\phi_j(x)(d_+(x)\,\phi_k(x)) + {}_xD_1^{\alpha}\phi_j(x)(d_-(x)\,\phi_k(x)))dx\\ &= -\int_{-1}^{1} ((1+x)^{1-\alpha}\breve{\phi}_j(x)(d_+\phi_k(x)) + (1-x)^{1-\alpha}\breve{\phi}_j(x)(d_-\phi_k(x)))dx,\end{aligned} \tag{5.5}$$

where

$$\breve{\phi}_j(x) = b_j J_j^{\alpha,1-\alpha}(x) - b_{j+1} J_{j+1}^{\alpha,1-\alpha}(x), \tag{5.6}$$

$$\breve{\phi}_j(x) = b_j J_j^{1-\alpha,\alpha}(x) + b_{j+1} J_{j+1}^{1-\alpha,\alpha}(x), \tag{5.7}$$

with $b_j = \frac{\Gamma(j+2)}{\Gamma(j-\alpha+2)}$. Hence, the above integration can be easily computed by using Gauss–Jacobi quadratures. Similarly, one can compute s^y_{kj}.

The matrix equation (5.4) is of the same form as (4.18), except that S^x and S^y are no longer symmetric. However, we can still apply the matrix diagonalization approach to solve (5.4) as long as the eigenvalues of S^x and S^y are all non-zero. This should be true under the condition that the original fractional PDE (5.1) with (4.2) admits a unique solution.

5.1.1. Numerical results

We consider (5.1)–(4.2) with the same coefficients and exact solution as in Example 3. We first compute all the eigenvalues of S^x and display them in Fig. 5. We observe that all eigenvalues have positive real part, indicating that the discrete problem (5.3) is well-posed.

In Figs. 6, we plot the errors in L^2-norm and $H^{\frac{\alpha}{2},\frac{\beta}{2}}$-norm in semi-log scale by using the matrix diagonalization method. We observe that, as in the conserved-form, the errors converge exponentially.

Next, we consider the problem (5.1)–(4.2) with general coefficients. As for the non-separable equation in conserved form, we shall use an iterative method. Since the problem is no longer symmetric, we shall use the preconditioned BICGSTAB

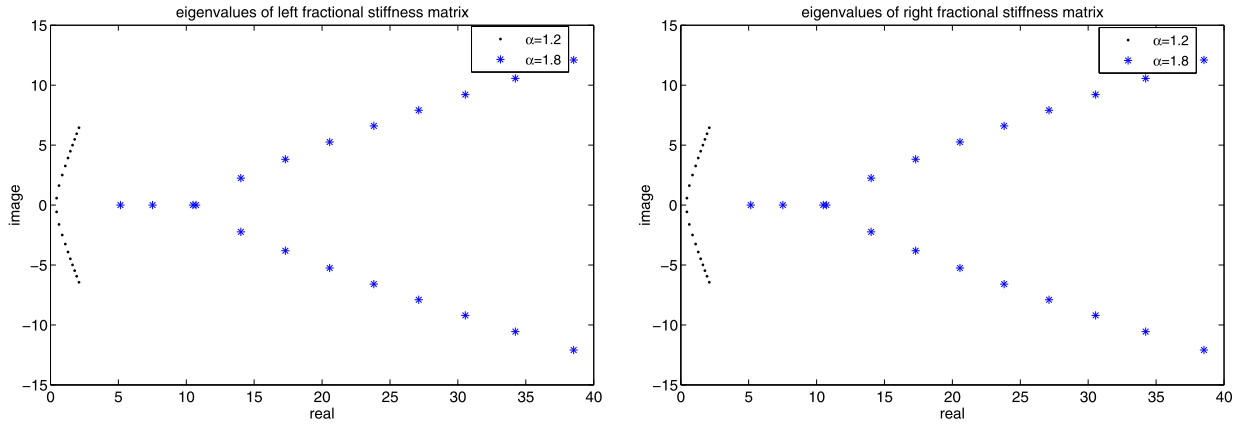

Fig. 5. Distribution of eigenvalues.

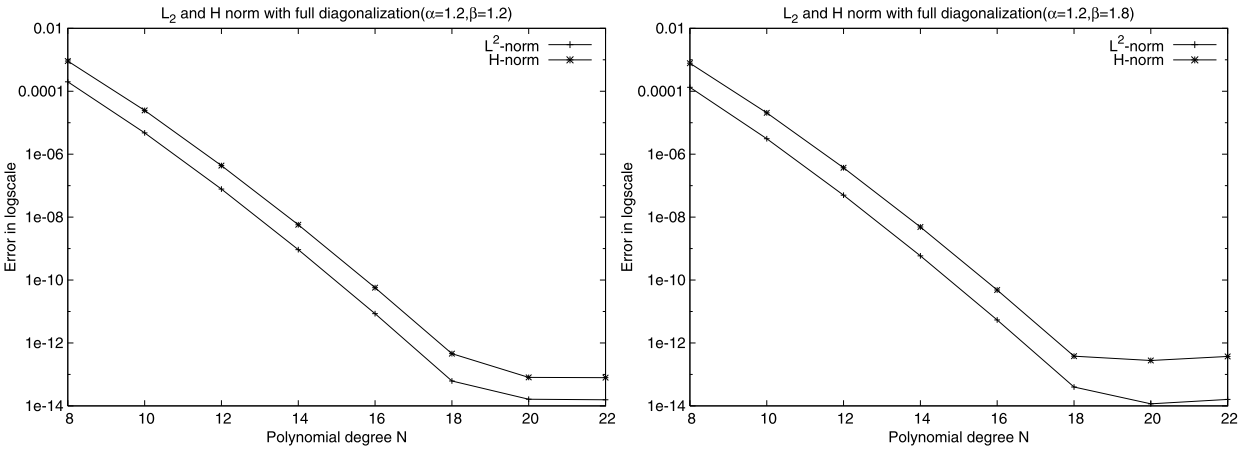

Fig. 6. $H^{\frac{\alpha}{2},\frac{\beta}{2}}$ and L^2 errors. Left: $\alpha = 1.2$, $\beta = 1.2$; Right: $\alpha = 1.2$, $\beta = 1.8$.

Table 2
Number of iterations for solving (5.1)–(4.2).

N	# of iteration					
	$\alpha = 1.2$, $\beta = 1.2$		$\alpha = 1.2$, $\beta = 1.8$		$\alpha = 1.8$, $\beta = 1.8$	
	without Precon.	with Precon.	without Precon.	with Precon.	without Precon.	with Precon.
20	124	14	170	14	100	14
30	148	14	342	14	138	16
40	166	14	486	14	198	16
50	186	14	588	14	226	16
60	192	14	802	14	218	16
70	192	14	778	14	258	16

method [6] with, once again, a problem with suitable constant coefficients as preconditioner. With reasonable conditions on the coefficients, one can expect that the convergence rate of the preconditioned BICGSTAB method will be independent of N.

We take (5.1)–(4.2) with the same coefficients and exact solution as in Example 4, and list in Table 2 the number of iterations needed with a threshold $\epsilon = 10^{-8}$. We also observe that the iteration numbers are bounded independent of N if a preconditioner is used.

5.2. Three or more dimensional problems

We only presented details of our algorithms and analysis for fractional elliptic PDEs in two-dimensional rectangular domains. However, it is clear that these approaches can be extended directly to fractional elliptic PDEs in three or more dimensional rectangular domains with a cost of $O(N^{d+1})$ (where d is the dimension). We leave the details to the interested readers.

5.3. Time–space fractional diffusion equation (TSFDE)

To illustrate the idea, we consider the following 1-D TSFDE

$$\begin{aligned}
&{}_0^C\partial_t^\alpha u(x,t) - d_+(x)\,{}_{-1}\partial_x^\beta u(x,t) - d_-(x)\,{}_x\partial_1^\beta u(x,t) = f(x,t), \quad (x,t)\in\Omega=\Lambda\times I,\\
&u(x,0)=u_0(x), \quad x\in\Lambda,\\
&u(-1,t)=u(1,t)=0, \quad \forall t\in I,
\end{aligned} \tag{5.8}$$

where $0<\alpha<1$, $1<\beta<2$, and $d_+(x), d_-(x)\geqslant c>0$ are given functions. Note that we consider here the time fractional derivative in Caputo form which allows us to deal with non-zero initial conditions. Thanks to Lemma 1 and following [13], a weak formulation of (5.8) is: for $f\in\mathcal{H}^{\frac{\alpha}{2},\frac{\beta}{2}}(\Omega)'$, find $u\in\mathcal{H}^{\frac{\alpha}{2},\frac{\beta}{2}}(\Omega)$, such that

$$({}_0\partial_t^{\frac{\alpha}{2}}u, {}_t\partial_T^{\frac{\alpha}{2}}v) - ({}_{-1}\partial_x^{\frac{\beta}{2}}u, {}_x\partial_1^{\frac{\beta}{2}}(d_+(x)\,v)) - ({}_x\partial_1^{\frac{\beta}{2}}u, {}_{-1}\partial_x^{\frac{\beta}{2}}(d_-(x)\,v)) = \mathcal{F}(v), \quad \forall v\in\mathcal{H}^{\frac{\alpha}{2},\frac{\beta}{2}}(\Omega), \tag{5.9}$$

where

$$\mathcal{H}^{s,\sigma}(\Omega)=H^s(I,L^2(\Lambda))\cap L^2(I,H_0^\sigma(\Lambda)), \text{ and } \mathcal{F}(v)=<f,v>+\Big(\frac{u_0(x)t^{-\alpha}}{\Gamma(1-\alpha)},v\Big).$$

Let us denote

$$\mathcal{X}_{M,N}:=\mathbb{P}_M^0\times\mathbb{P}_N=\text{span}\{\phi_k(x)L_j(y): 0\leq k\leq M-2,\ 0\leq j\leq N\}.$$

Then, the spectral–Galerkin approximation for (5.9) is: Find $u_{M,N}\in\mathcal{X}_{M,N}$ such that

$$\begin{aligned}
&({}_0\partial_t^{\frac{\alpha}{2}}u_{M,N}, {}_t\partial_T^{\frac{\alpha}{2}}v_{M,N}) - ({}_{-1}\partial_x^{\frac{\beta}{2}}u_{M,N}, {}_x\partial_1^{\frac{\beta}{2}}(d_+(x)\,v_{M,N})) - ({}_x\partial_1^{\frac{\beta}{2}}u, {}_{-1}\partial_x^{\frac{\beta}{2}}(d_-(x)\,v_{M,N}))\\
&\quad=\mathcal{F}(v_{M,N}), \quad \forall v_{M,N}\in\mathcal{X}_{M,N}.
\end{aligned} \tag{5.10}$$

Denote

$$\begin{aligned}
&u_{M,N}=\sum_{k=0}^{M-2}\sum_{j=0}^{N}\tilde{u}_{kj}\phi_k(x)L_j(t), \quad U=(\tilde{u}_{kj})_{k,j=0}^{M-2,N};\\
&f_{kj}=\mathcal{F}(\phi_k(x)L_j(t)), \quad F=(f_{kj})_{k,j=0}^{M-2,N};\\
&s_{kj}^t=\int_I {}_0D_t^{\frac{\alpha}{2}}L_j(t)\,{}_tD_T^{\frac{\alpha}{2}}L_k(t)dt, \quad S^t=\big(s_{kj}^t\big)_{k,j=0}^{N};\\
&s_{kj}^x=-\int_\Lambda {}_{-1}D_x^{\frac{\beta}{2}}\phi_j(x)\,{}_xD_1^{\frac{\beta}{2}}(d_+(x)\phi_k(x))+{}_xD_1^{\frac{\beta}{2}}\phi_j(x)\,{}_{-1}D_x^{\frac{\beta}{2}}(d_-(x)\phi_k(x))dx, \quad S^x=\big(s_{kj}^x\big)_{k,j=0}^{M-2};\\
&m_{kj}^t=\int_I L_k(t)\,L_j(t)dt, \quad M^t=(m_{kj}^t)_{k,j=0}^{N};\\
&m_{kj}^x=\int_\Lambda \phi_k(x)\phi_j(x)dx, \quad M^x=(m_{kj}^x)_{k,j=0}^{M-2}.
\end{aligned}$$

Taking $v=\phi_m(x)L_n(t)$ in (5.10), $m=0,1,\cdots,M-2$, $n=0,1,\cdots,N$, we find that (5.10) is equivalent to the following linear system:

$$M^xU(S^t)^T+S^xUM^t=F. \tag{5.11}$$

The above matrix system can once again be efficiently solved by the matrix diagonalization method using the following generalized eigenvalue problems:

$$M^x\bar{x}=\lambda^xS^x\bar{x}, \quad M^t\bar{t}=\lambda^tS^t\bar{t}. \tag{5.12}$$

Since S^x is the same as in Subsection 5.1, we only examine the eigenvalues $\{\lambda^t\}$ which are displayed in Fig. 7. We observe that since S^t is non-symmetric, the eigenvalues are not real but they all have positive real part. Hence, the system (5.11) can be solved by using the matrix diagonalization method in $O(N^3)$ operations.

Although we only considered 1-D TSFDEs above, it is clear that we can deal with multi-dimensional TSPDEs by combining the approaches here with those in previous sections.

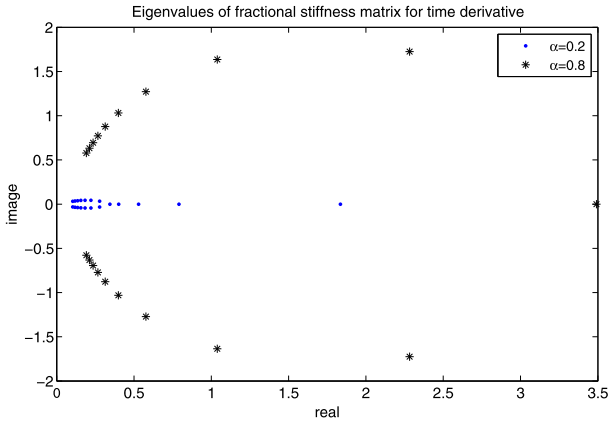

Fig. 7. Distribution of eigenvalues $\{\lambda^t\}$.

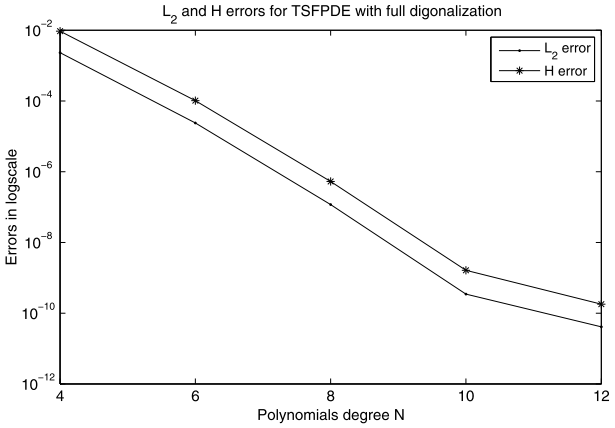

Fig. 8. $H^{\frac{\alpha}{2},\frac{\beta}{2}}$ and L^2 errors with spectral decomposition method for TDFDE, $\alpha = 0.8$, $\beta = 1.8$.

Example 6. Consider (5.8) with $T = 1$, $d_+(x) = 1 + x^2$, $d_-(x) = e^{-x}$, and the exact solution $u_{ex}(x,t) = e^{-t}\sin(\pi x)$.

The convergence w.r.t. to N with $M = 18$ is shown in Fig. 8. Since the solution is analytic, we have once again exponential convergence.

6. Concluding remarks

We developed in this paper efficient spectral–Galerkin algorithms and error analyses to solve multi-dimensional fractional elliptic equations with variable coefficients in conserved form as well as non-conserved form. The main idea was to extend the approach in [24] for usual PDEs to fractional PDEs. If the FPDEs are separable, we constructed a direct method by using a matrix diagonalization approach with a small multiple of N^{d+1} flops, where d is the dimension; while for non-separable FPDEs, we showed that the (matrix-free) matrix–vector product can be computed in $O(N^{d+1})$, and with a suitable separable FPDE with constant-coefficients as a (optimal) preconditioner, we can still solve the corresponding linear system with a preconditioned BICGSTAB method in $O(N^{d+1})$ flops. Hence, we have shown that multi-dimensional fractional PDEs can be solved with a spectral–Galerkin algorithm in essentially the same cost as related usual PDEs. We also discussed the extension of these approaches to deal with time–space FDEs.

We derived rigorous weighted error estimates which provide more precise and improved convergence rate than the usual non-weighted estimates for solutions with singularities at the boundary. We presented ample numerical results to validate our algorithms and error estimates.

We note that similar algorithms, based on Petrov–Galerkin methods, have been proposed in [32] for multi-dimensional FPDEs with one-sided fractional derivatives and constant-coefficients, as opposed to spectral–Galerkin methods for multi-dimensional FPDEs with double-sided fractional derivatives and variable-coefficients presented in this paper. The algorithms in [32] and our algorithms in this paper were both presented at the 2014 International Conference on Spectral and High-Order Methods which was held at Salt Lake City during June 23–27, 2014.

Appendix A. Jacobi polynomials and related properties

Due to the similarity between the Jacobi weights and the kernel function in the definitions of the fractional integrals/derivatives, Jacobi polynomials play essential roles in developing efficient spectral methods for FPDEs. We recall below some properties of Jacobi polynomials.

Let $J_n^{\alpha,\beta}(x)$ $(\alpha,\beta>-1)$ be the Jacobi polynomials which are orthogonal with respect to the weight function $w^{\alpha,\beta}(x)=(1-x)^\alpha(1+x)^\beta$ over $(-1,1)$, i.e.

$$\int_{-1}^{1} J_n^{\alpha,\beta}(x)J_m^{\alpha,\beta}(x)w^{\alpha,\beta}(x)=\gamma_n^{\alpha,\beta}\delta_{mn}, \tag{A.1}$$

where $\gamma_n^{\alpha,\beta}=\|J_n^{\alpha,\beta}(x)\|^2_{w^{\alpha,\beta}(x)}$, $\alpha,\beta>-1$. The Jacobi polynomials satisfy the three-term recurrence relation:

$$\begin{cases} J_0^{\alpha,\beta}(x)=1,\\ J_1^{\alpha,\beta}(x)=\frac{1}{2}(\alpha+\beta+2)x+\frac{1}{2}(\alpha-\beta),\\ J_{n+1}^{\alpha,\beta}(x)=(A_n^{\alpha,\beta}x-B_n^{\alpha,\beta})J_n^{\alpha,\beta}(x)-C_n^{\alpha,\beta}J_{n-1}^{\alpha,\beta}(x),\ n\geq 1, \end{cases} \tag{A.2}$$

where

$$A_n^{\alpha,\beta}=\frac{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)}{2(n+1)(n+\alpha+\beta+1)}, \tag{A.3}$$

$$B_n^{\alpha,\beta}=\frac{(\alpha^2-\beta^2)(2n+\alpha+\beta+1)}{2(n+1)(n+\alpha+\beta+1)(2n+\alpha+\beta)}, \tag{A.4}$$

$$C_n^{\alpha,\beta}=\frac{(n+\alpha)(n+\beta)(2n+\alpha+\beta+2)}{(n+1)(n+\alpha+\beta+1)(2n+\alpha+\beta)}. \tag{A.5}$$

In particular, we have the Legendre polynomials $L_n(x)=J_n^{0,0}(x)$. Below, we list some useful properties of the Jacobi polynomials that will be used later (cf. [23,25]):

$$J_n^{\alpha,\beta}(-x)=(-1)^nJ_n^{\beta,\alpha}(x). \tag{A.6}$$

$$J_n^{\alpha,\beta}(1)=\frac{\Gamma(n+\alpha+1)}{j!\Gamma(\alpha+1)},\ J_n^{\alpha,\beta}(-1)=(-1)^n\frac{\Gamma(n+\beta+1)}{j!\Gamma(\beta+1)}. \tag{A.7}$$

$$J_n^{\alpha+1,\beta}(x)=\frac{2}{2n+\alpha+\beta+2}\frac{(n+\alpha+1)J_n^{\alpha,\beta}(x)-(n+1)J_{n+1}^{\alpha,\beta}(x)}{1-x}. \tag{A.8}$$

$$J_n^{\alpha,\beta+1}(x)=\frac{2}{2n+\alpha+\beta+2}\frac{(n+\beta+1)J_n^{\alpha,\beta}(x)+(n+1)J_{n+1}^{\alpha,\beta}(x)}{1+x}. \tag{A.9}$$

Next, we recall some useful properties of the Jacobi polynomials in relation to fractional integrals/derivatives.

Lemma 5. *(See [2].) Let $\mu>0$ and $\forall x\in[-1,1]$, we have*

(i) For $\alpha>-1$, $\beta\in\mathbb{R}$

$$(1+x)^{\beta+\mu}\frac{J_n^{\alpha-\mu,\beta+\mu}(x)}{J_n^{\alpha-\mu,\beta+\mu}(-1)}=\frac{\Gamma(\beta+\mu+1)}{\Gamma(\beta+1)\Gamma(\mu)J_n^{\alpha,\beta}(-1)}\int_{-1}^{x}\frac{(1+s)^\beta J_n^{\alpha,\beta}(s)}{(x-s)^{1-\mu}}ds, \tag{A.10}$$

(ii) For $\alpha\in\mathbb{R}$, $\beta>-1$

$$(1-x)^{\alpha+\mu}\frac{J_n^{\alpha+\mu,\beta-\mu}(x)}{J_n^{\alpha+\mu,\beta-\mu}(1)}=\frac{\Gamma(\alpha+\mu+1)}{\Gamma(\alpha+1)\Gamma(\mu)J_n^{\alpha,\beta}(1)}\int_{x}^{1}\frac{(1-s)^\alpha J_n^{\alpha,\beta}(s)}{(s-x)^{1-\mu}}ds. \tag{A.11}$$

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