A SURFACE OF MAXIMAL CANONICAL DEGREE+CORRIGENDUM

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Abstract. It is known since the 70's from a paper of Beauville that the degree of the rational canonical map of a smooth projective algebraic surface of general type is at most 36. Though it has been conjectured that a surface with optimal canonical degree 36 exists, the highest canonical degree known earlier for a minimal surface of general type was 16 by Persson. The purpose of this paper is to give an affirmative answer to the conjecture by providing an explicit surface.

1. Introduction

1.1. Let M be a minimal surface of general type. Assume that the space of canonical sections $H^0(M, K_M)$ of M is non-trivial. Let $N = \dim_{\mathbb{C}} H^0(M, K_M) = p_g$, where p_g is the geometric genus. The canonical map is defined to be in general a rational map $\Phi_{K_M} : M \dashrightarrow P^{N-1}$. For a surface, this is the most natural rational mapping to study if it is non-trivial. Assume that Φ_{K_M} is generically finite with degree d. It is well-known from the work of Beauville [B], that $d := \deg \Phi_{K_M} \leqslant 36$. We call such degree the canonical degree of the surface, and regard it 0 if the canonical mapping does not exist or is not generically finite. The following open problem is an immediate consequence of the work of [B] and is implicitly hinted there.

Problem *What is the optimal canonical degree of a minimal surface of general type? Is there a minimal surface of general type with canonical degree* 36*?*

Though the problem is natural and well-known, the answer remains elusive since the 70's. The problem would be solved if a surface of canonical degree 36 could be constructed. Prior to this work, the highest canonical degree known for a surface of general type is 16 as constructed by Persson [Pe]. We refer the readers to [DG], [Pa], [T] and [X] for earlier discussions on construction of surfaces with relatively large canonical degrees. The difficulty for the open problem lies in the lack of possible candidates for such a surface.

Note that from the work of [B], a smooth surface of canonical degree 36 is a complex two ball quotient $B^2_{\mathbb{C}}/\Sigma$, where Σ is a lattice of $PU(2, 1)$. Hence it is infinitesimal rigid and can neither be obtained from deformation nor written as a complete intersection of hypersurfaces in projective spaces.

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The purpose of this paper is to give an answer to the problem above by presenting explicitly a surface with canonical degree 36. Comparing to earlier methods, we look for such a surface from a new direction, namely, arithmetic lattices coming from recent classification of fake projective planes given in [PY] and [CS]. In fact, the surface here is constructed from a fake projective plane originally studied in Prasad-Yeung [PY].

Theorem 1. *There exists a smooth minimal surface of general type M with a generically finite canonical map $\Phi_{K_M} : M \to P^2_{\mathbb{C}}$ of degree 36, constructed from an appropriate unramified covering of a well-chosen fake projective plane of index 4.*

The example obtained above corresponds to an arithmetic lattice Σ associated to a non-trivial division algebra over appropriate number fields as discussed in [PY]. Arithmetic lattices coming from non-trivial division algebra are sometimes called arithmetic lattices of the second type, cf. [Ye2]. In contrast, geometric complex two ball quotients studied extensively in the literature correspond to a class of examples commensurable with Deligne-Mostow surfaces in [DM], or those constructed by Hirzebruch [H], cf. [DM]. Further examples in this latter direction can be found in the recent paper of Deraux-Parer-Paupert [DPP]. The lattices involved are sometimes called arithmetic or integral lattices of first type, which are defined over number fields instead of non-trivial division algebras. Up to this point, the effort to construct an example of optimal canonical degree in the form of a lattice of first type has not been successful.

1.2. The idea of proof Theorem 1 is as follows. The key observation is to relate a well-chosen fake projective plane to possible existence of a surface of optimal canonical degree. An appropriate normal cover of a fake projective plane of degree four gives the Euler number expected for a candidate surface. We need to guarantee the vanishing of the first Betti number to achieve the correct dimension of the space of the canonical sections. After this, the main part of argument is to ensure that the canonical map is generically finite and base point free, which turns out to be subtle. In this paper, we choose an appropriate covering corresponding to a congruence subgroup of the lattice associated to an appropriate fake projective plane, which ensures that the first Betti number is trivial and the Picard number is one. The latter condition makes the surface geometrically simple for our arguments. We divide the proof into three steps, proving that the rational canonical map is generically finite, that the map has no codimension one base locus, and that the map has no codimension two base locus. We make extensive use of the finite group actions given by the covering group. In this process we have to utilize the geometric properties of the fake projective plane and relate to finite group actions on a projective plane and on a rational line. We also need to utilize vanishing properties in [LY] of sections of certain line bundles which are numerically small rational multiples of the canonical line bundle, related to a conjecture on existence of exceptional objects in [GKMS].

We would like to explain that the software package Magma was used in this paper, but only very elementary commands are used. Starting with the presentation of our fake projective plane given in [CS], only one-phrase commands as used in calculators are needed (see the details in §3).

More examples and classification of surfaces of optimal canonical degree arising from fake projective planes would be discussed in a forthcoming work with Ching-Jui Lai.

1.3. The author is indebted to Donald Cartwright for his numerous helps related to the use of Magma and lattices associated to fake projective planes. He would also like to thank Ching-Jui Lai for many discussions, comments and help on the paper, to Carlos Rito for spotting some errors in earlier drafts and making helpful comments, and to Rong Du for bringing the problem to his attention. Part of the work was done while the author is visiting the Institute of Mathematics at the University of Hong Kong, and the author would like to express his gratitude for the hospitality of the institute.

2. Preliminaries

2.1. For completeness of presentation, let us explain why the maximal degree is bounded from above by 36 as is observed in [B]. Let S be the rational image of Φ_{K_M}. Denote by F and P the fixed and movable parts of the canonical divisor K_M respectively and $\pi : \widehat{M} \to M$ the resolution of P. Let $\pi^*P = F_{\widehat{M}} + P_{\widehat{M}}$ be the similar decomposition on $\widehat{M}$. Let $h^1(M) = \dim_{\mathbb{C}} H^1(M, \mathcal{O}_M), p_g = \dim_{\mathbb{C}} H^2(M, \mathcal{O}_M)$ and $\chi(\mathcal{O}_M)$ be the the arithmetic genus of M respectively. Then

$$\begin{aligned}(\deg S)d = P^2_{\widehat{M}} &\leqslant P^2 \\ &\leqslant K^2_M \\ &\leqslant 9\chi(\mathcal{O}_M) \\ &= 9(p_g - h^1 + 1) \\ &\leqslant 9(p_g + 1).\end{aligned}$$

where the first two inequalities were explained in [B]. Hence

$$d \leqslant 9(\frac{p_g + 1}{\deg S}).$$

However, from Lemma 1.4 of [B], we know that $\deg S \geqslant p_g - 2$. We conclude that

$$d \leqslant 9(\frac{p_g + 1}{p_g - 2}) \leqslant 36,$$

since $p_g \geqslant 3$ from the fact that the canonical mapping is generically finite.

Tracing back the above argument, it follows that the equality holds only if the fixed part of M is trivial, $p_g = 3$ and $h^1 = 0$, and that the Miyaoka-Yau inequality becomes an equality. From the work of Aubin and Yau, see [Ya], the latter condition implies that it is the quotient of a complex two ball by a lattice in $PU(2, 1)$. Moreover, we see that the canonical mapping is base point free by tracing through the argument above.

Note that a complex two ball quotient is infinitesimally rigid from the result of Calabli-Vesentini [CV]. Hence such a surface cannot be constructed from complete intersections.

3. Description of the surface

3.1. Recall that a fake projective plane is a smooth compact complex surface with the same Betti numbers as $P^2_{\mathbb{C}}$. This is a notion introduced by Mumford [M] who constructed the first example. All fake projective planes have recently been classified into twenty-eight non-empty classes by the work of Prasad-Yeung in [PY]. Together with the work of Cartwright-Steger [CS], we know that there are precisely 100 fake projective planes among those 28 classes. It is known that a fake projective plane is a smooth complex two ball quotient $B^2_{\mathbb{C}}/\Pi$ for some lattice $\Pi \subset PU(2,1)$, and has the smallest Euler number among smooth surfaces of general type. We refer the readers to [Ré] and [Ye2] for surveys about fake projective planes.

The fake projective planes $X = B^2_{\mathbb{C}}/\Pi$ are classified in the sense of lattices. In the notations explained in [PY], lattices Π are constructed as a subgroup of a lattice $\overline{\Gamma}$ which determined a class of fake projective planes classified.

3.2. In this paper, we are going to consider the following specific fake projective plane. In the formulation of [PY], the surface has the same defining number fields as Mumford's fake projective plane as constructed in **5.7, 5.11** of [PY], corresponding to $a = 7, p = 2$ in the notation there. In particular, two different lattices Π representing fake projective planes with automorphism group of order 21 are constructed. Each such lattice Π is a congruence subgroup as explained in **5.11** of [PY] and is different from the one of Mumford. According to [PY], the associated maximal arithmetic lattice $\overline{\Gamma}$ is an arithmetic lattice of second type in the sense that it is an arithmetic lattice defined from a non-trivial division algebra $\mathcal{D}$ with an involution of second type ι.

3.3. The maximal arithmetic group $\overline{\Gamma}$ to be used in this article corresponding to the class chosen above in [PY] (cf. Theorem 4.2, **5.9**, **5.11**). A presentation of the lattice is found with a procedure explained by Cartwright and Steger in [CS] and details given by the file `a7p2N/gp7 2generators reducesyntaxtxt` in the weblink of [CS], with generators and relations given by

$$\begin{aligned}\overline{\Gamma} := \langle z, b \quad | \quad & z^7, (b^2z^{-1})^3, (bz^{-1}b^3z^2)^3, (b^3z^{-2}bz^{-2})^3, b^3z^{-2}b^{-1}z^2b^{-2}z, \\ & b^3z^3bz^2b^{-1}z^{-1}b^3z, zb^2z^{-2}b^{-1}z^{-1}b^{-3}zb^{-1}z^{-1}b^3z, \\ & bzb^5z^{-2}b^2z^2b^2z^{-2}b^2z^3\rangle.\end{aligned}$$

The lattice associated to the fake projective plane is denoted by Π and is generated by the subgroup of index 21 in $\overline{\Gamma}$ with generators given by

$$b^3, z^2bz^{-1}b^{-1}, (zbz^{-1})^3, zbz^{-1}b^{-1}z, zb^{-1}z^{-2}b, (bz^{-1})^3,$$

which is one of the candidates found by command `LowIndexSubgroups` in Magma and is the one we used, denoted by $(a = 7, p = 2, \emptyset, D_32_7)$ in the notation of Cartwright-Steger (see file `registerofgps.txt` in the weblink of [CS]), the first two entries correspond to $a = 7, p = 2$ in the number fields studied in [PY]. Denote by X the resulting fake projective plane. It follows that $H_1(X, \mathbb{Z}) = \mathbb{Z}_2^4$, which follows after applying the Magma command `AbelianQuotient` to the presentation above.

Denote by $g_1, \ldots, g_6$ the elements listed above. Magma command `LowIndexSubgroups` allow us to find a normal subgroup Σ of index 4 in Π with generators given by

$$g_4, g_5 g_1^{-1}, g_6 g_2^{-1}, g_1^{-2}, g_2^{-2}, g_3^{-2}, g_5^{-1} g_1^{-1}, g_6^{-1} g_2^{-1}, g_1 g_2 g_3^{-1}, g_1 g_3 g_2^{-1}.$$

The corresponding ball quotient is denoted by $M = B_{\mathbb{C}}^2/\Sigma$. In the next few sections, we would show that M is a surface with maximal canonical degree.

4. Some geometric properties of the surface

4.1. We collect some general information about the surface M.

Lemma 1. *The ball quotient M is a smooth unramified covering of degree 4 of the fake projective plane X satisfying the following properties.*
(a). $b_1(M) = 0$ and $H_1(M, \mathbb{Z}) \cong \mathbb{Z}_2^5 \times \mathbb{Z}_4$.
(b). Picard number $\rho(M) = 1$.
(c). The lattice Σ is a congruence subgroup of Π.
(d). The automorphism group of M has order $Aut(M) = A_4$, the alternating group of 4 elements.
(e). $\Sigma \lhd \Pi$, $\Pi \lhd N_\Sigma$, $\Pi \lhd \overline{\Gamma}$ with $|\Pi/\Sigma| = 4$, $|N_\Sigma/\Sigma| = 12$, and $|N_\Sigma/\Pi| = 3$, where N_Σ is the normalizer of Σ in $\overline{\Gamma}$.
(f). The action of $\mathbb{Z}_3$ on M descends to an action of $\mathbb{Z}_3$ on X.
(g). The sequence of normal coverings $B_{\mathbb{C}}^2/\Sigma \xrightarrow{p} B_{\mathbb{C}}^2/\Pi \xrightarrow{q} B_{\mathbb{C}}^2/\overline{\Gamma}$ corresponds to normal subgroups $\Sigma \lhd \Pi \lhd \overline{\Gamma}$, with covering groups $\Pi/\Sigma = \mathbb{Z}_2 \times \mathbb{Z}_2$ and $\overline{\Gamma}/\Pi = \mathbb{Z}_7 : \mathbb{Z}_3$, the unique non-abelian group of order 21.

Proof From the presentation of Σ and Magma command `AbelianQuotient`, we conclude that $H_1(M) \cong \mathbb{Z}_2^5 \times \mathbb{Z}_4$. Hence (a) follows.

To prove (c), we consider the division algebra $\mathcal{D}$ associated to our fake projective plane mentioned above. Let

$$V = \{\xi \in \mathcal{D} : \iota(\xi) = \xi, \mathrm{Tr}(\xi) = 0\}.$$

V forms a vector space of dimension 8 over $\mathbb{Q}$. $\overline{\Gamma}$ has a representation on V, acting by conjugations. Hence there is a natural homomorphism $f : \overline{\Gamma} \to SL(8, \mathbb{Z})$. Considering reduction modulo 2, there exists a homomorphism $f_2 : \overline{\Gamma} \to SL(8, \mathbb{Z}_2)$ for which $|f_2(\overline{\Gamma})| = 64 \times 21$. From Magma, we can check that the image $f_2(\Pi)$ of Π has order 64, and so has index 21 in the image of $\overline{\Gamma}$. Recall that Π has index 21 in $\overline{\Gamma}$. Hence Π contains the kernel of f_2 and is a congruence subgroup of $\overline{\Gamma}$. The author is indebted to Donald Cartwright for explaining the above procedure checking congruence property.

Consider a normal subgroup Σ of Π with index 4 given by choice in the last section below. From Magma again, the order of $f_2(\Sigma)$ is 16 and hence is of index 4 in $f_2(\Pi)$. Again, as $\ker(f_2) \subset \Sigma$, we conclude that N is a congruence subgroup. Hence (c) is true.

Once we know that (c) is true, the facts about Picard number in (b) and $b_1(M) = 0$ in (a) also follow from the work of Rogawski [Ro] and Blasius-Rogawski [BR], see also [Re].

For (d) and (e), we check by magma that the normalizer N_Σ of Σ in $\bar{\Gamma}$ is a subgroup of index 7. Hence we know that the automorphism group of M given by N_Σ/Σ is a group of order 12. In fact, this corresponds to the group $(a = 7, p = 2, \emptyset, 2_7)$ in the notation of Cartwright-Steger in file `registerofgps.txt` in the weblink of [CS], since that is the only group of right order in $\bar{\Gamma}$ supporting a unramified covering of index 12. From Magma, we check that the quotient group $H := N_\Sigma/\Sigma$ is a non-abelian with $[H, H] = \mathbb{Z}_3$ and actually $H = A_4$ after comparing with the library of small groups in Magma. Magma also allows us to show that $\Pi \lhd N_\Sigma$.

(f) follows from the fact that $C = \mathbb{Z}_2 \times \mathbb{Z}_2$ is a normal subgroup of A_4. Recall that Σ is a normal subgroup of Π with quotient C so that we may write $\Pi = C\Sigma$. Let $x \in B^2_{\mathbb{C}}$. By definition, for $\gamma \in \mathbb{Z}_3 < A_4$, the action of γ at the Σ cosets satisfies

$$\gamma(\Sigma x) = \gamma\Sigma\gamma^{-1} \cdot \gamma x = \Sigma(\gamma x).$$

We need to show the same is true for a Π coset. This follows from

$$\gamma(\Pi x) = \gamma(C\Sigma x) = \gamma C\gamma^{-1} \cdot \gamma\Sigma x = C\Sigma(\gamma x) = \Pi x$$

where we used the fact that C is a normal subgroup of A_4.

(g) follows from the above description as well.

□

Remark As a consequence of the Universal Coefficient Theorem, the torsion part of the Néron-Severi group corresponds to the part in $H_1(M, \mathbb{Z})$, namely, $\mathbb{Z}_2^5 \times \mathbb{Z}_4$.

4.2. We also recall the following result which is related to a conjecture of Galkin-Katzarkov-Mellit-Shinder in [GKMS].

Lemma 2. *Let H be the ample line bundle on X on the fake projective plane X as studied above, so that $K_X = 3H$ as defined in [PY],* **10.2, 10.3**. *Then*
(a). $H^0(X, 2H) = 0$.
(b). There is no $Aut(X)$ invariant sections in $H^0(X, 2H + e)$, where e is any torsion line bundle on X.

Proof Part (a) follows from Theorem 1.3 or Lemma 4.2 of Galkin-Katzarkov-Mellit-Shinder [GKMS], Theorem 1 of Lai-Yeung [LY], or Theorem 0.1 of Keum [K]. Part (b) follows directly from the proof of Theorem 1 of [LY].

□

4.3. Recall that from construction in §3, M is an unramified covering of a specific fake projective plane X of index 4. Since X is a fake projective plane, the Betti numbers and Hodge numbers of X are the same as the corresponding ones on $P^2_{\mathbb{C}}$. It follows that $\chi(\mathcal{O}_X) = 1$. Hence $\chi(\mathcal{O}_M) = 4\chi(\mathcal{O}_X) = 4$. Since $h^1(M) = 0$, it follows that $p_g = 4 - h^1(M) - h^0(M) = 3$. We conclude that $h^0(M, K_M) = 3$. Let $\{s_1, s_2, s_3\}$ be a basis of $H^0(M, K_M)$. The linear system associated to the basis gives rise to a rational mapping

$$\Phi : x \dashrightarrow [s_1(x), s_2(x), s_3(x)] \in P^2_{\mathbb{C}}.$$

Let S be the rational image of Φ. We know that $\dim_{\mathbb{C}} S = 1$ or 2.

Lemma 3. *The action of the covering group $\mathcal{G} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ on M induces an action of $\mathcal{G}$ on $\Phi(M) \subset P^2_{\mathbb{C}}$ through the canonical rational map $\Phi : M \dashrightarrow \Phi(M) \subset P^2_{\mathbb{C}}$.*

Proof From factorization of rational maps in complex dimension two, there exists a morphism $\pi : \widehat{M} \to M$, where $\widehat{M}$ is a sequence of blow-ups of M, and holomorphic map $f : \widehat{M} \to P^2_{\mathbb{C}}$ such that $f = \Phi \circ \pi$.

We observe that the action of $\mathcal{G}$ on M lifts to an action on $\widehat{M}$. To see this, observe that the base locus of the canonical map is invariant under $\mathrm{Aut}(M)$. $\widehat{M}$ is obtained from M from a series of blow-ups and we know that the induced action is biholomorphic outside of the blown-up locus. Since the transformation $\gamma \in \mathcal{G}$ comes from a fractional linear transformation of an element in $PU(2,1)$, the transformation is locally linear around any fixed point and in particular lifts to the blown up divisors. Hence $\mathcal{G}$ acts holomorphically on $\widehat{M}$.

We define an action of $\mathcal{G}$ on $P^2_{\mathbb{C}}$ as follows. For $\gamma \in \mathcal{G}$ and $z \in P^2_{\mathbb{C}}$ satisfying $z = f(x)$, define

$$\gamma z = f(\gamma(x)).$$

To see that it is well-defined, suppose $f(x) = [g_1(x), g_2(x), g_3(x)]$ for some $g_i(x)$, $i = 1, 2, 3$, corresponding to a basis of the linear system associated to K_M. Assume that $f(x) = f(y)$. Then since g_1, g_2, g_3 form a basis for the space of sections in K_M, we conclude that there exists a constant k such that $g_i(x) = kg_i(y)$ for all $1 \leqslant j \leqslant 3$. Hence $g(x) = kg(y)$ for all $g \in \Gamma(M, K)$, from which we deduce that $\gamma^* g_i(x) = \gamma^* g_i(y)$. We conclude that

$$\begin{aligned} f(\gamma x) &= [\gamma^* g_1(x), \gamma^* g_2(x), \gamma^* g_3(x)] \\ &= [\gamma^* g_1(y), \gamma^* g_2(y), \gamma^* g_3(y)] \\ &= f(\gamma y), \end{aligned}$$

from which we conclude that $\mathcal{G}$ acts on $\Phi(M) \subset P^2_{\mathbb{C}}$.

□

5. Generically finiteness

5.1. The goal of this section is to show that the rational mapping Φ is dominant. First we make the following observation. Recall as in [PY] that $K_X = 3H_X$ for some line bundle on X which corresponds to a $SU(2,1)$-equivariant hyperplane line bundle H on $\widetilde{M} \cong B^2_{\mathbb{C}}$. In the following, the descends of H to M and X would be denoted by H_M and H_X respectively, or simply H when there is no danger of confusion. In particular, $H_M = p^* H_X$. Hence $K_M = 3p^* H_X$. As $K_M^2 = 36$ and the Picard number of M is 1, there are the following two different cases to consider,

Case (A), $H_M = 2L$, where L is a generator of the Neron-Severi group of M modulo torsion, or
Case (B), H_M is a generator of the Neron-Severi group.

5.2.

Lemma 4. *The canonical map Φ of M is generically finite.*

Proof Assume that $\dim_{\mathbb{C}} S = 1$. We claim that the rational image $C = \Phi(M)$ has genus 0. Assume on the contrary that C has genus at least 1. As mentioned earlier, there exists a morphism $\pi : \widehat{M} \to M$, and a holomorphic map $f : \widehat{M} \to P^2_{\mathbb{C}}$ such that $f = \Phi \circ \pi$. By Hurwitz Formula, the blown up divisors are mapped to a point on C and hence actually Φ extends across any possible base point set of Φ to give a holomorphic $\Phi : M \to P^2_{\mathbb{C}}$. As M has Picard number 1 from Lemma 1, this leads to a contradiction since the fibers are contracted. Hence the Claim is valid.

In general, we may write $K_M = F + P$, where F is the fixed part and P is the mobile part. In our case here, from the claim, it follows that $C = \Phi(M)$ is a rational curve. Since $\dim(\Phi(M)) = 1$, as mentioned in [B] **1.1**, page 123, we may write

$$K_M \equiv F + 2Q, \qquad (1)$$

where F is the fixed part of K_M, $2Q$ is the mobile part of K_M and Q is an irreducible curve. Here we denote the numerical equivalence of two divisors A and B by $A \equiv B$.

Our next step to prove the claim that F is trivial. Assume on the contrary that F ins non-trivial.

Consider first *Case (A)*. If F is non-trivial, it follows that $F \equiv bL$, where b is even and hence $b \geqslant 2$, which in turn implies that $Q \equiv cL$, where $c = 3 - \frac{b}{2} \leqslant 2$ from the decomposition of K_M above.

Consider first the case that $b = 2$ so that $F \equiv 2L$. It follows that $H_M \equiv 2L$ since $\rho(M) = 1$. Hence H is the same as $2L$ up to a torsion line bundle in $\mathbb{Z}_2^5 + \mathbb{Z}_4$, from Lemma 1 and the Universal Coefficient Theorem.

Hence $F \equiv H_M$ on M. As F on M by definition is invariant under $\mathrm{Aut}(M)$, it descends to X to give an effective divisor G on X. It follows that $G \equiv H_X$ on X. As $H_1(X, \mathbb{Z}) = \mathbb{Z}_2^4$, $G = H_X + e_2$ for some two torsion line bundle on X from the Universal Coefficient Theorem. This implies that $2H_X = 2G$ is effective on X, contradicting Lemma 2a.

The only other possibility is that $b = 4, c = 1$. In such case, we would have $Q \equiv L$. Hence we may choose F to be a generator of the Néron-Severi group modulo torsion on M. In such case, we may write $H = 2Q + e$, where e is a torsion line bundle corresponding to an element in $H_1(M, \mathbb{Z}) = \mathbb{Z}_2^4 \times \mathbb{Z}_4$ from Universal Coefficient Theorem. It follows that $K_M = 3H_M = 6Q + 3e$. Since $K_M = F + 2Q$, we conclude that $F = 4Q + 3e$. Hence $F = 2H_M + e$.

As the canonical line bundle K_M is invariant under the automorphism group of M, we know that the dimension one component F of the canonical line bundle is invariant under $\mathrm{Aut}(M)$. It follows that F descends as an effective divisor G on the fake projective plane X. The line bundle H is clearly invariant as a holomorphic line bundle under $\mathrm{Aut}(M)$ from construction. It follows from $e = F - 2H$ that e is invariant as a holomorphic line bundle under $\mathrm{Aut}(M)$. We conclude that $H^0(X, 2H_X + e) \neq 0$ on X, since it contains the effective divisor G, where $p : M \to X$ is the covering map. Recall that from our setting, the coverings $B^2_{\mathbb{C}}/\Sigma \to B^2_{\mathbb{C}}/\Pi \to B^2_{\mathbb{C}}/\overline{\Gamma}$ corresponds to normal subgroups $\Sigma \lhd \Pi \lhd \overline{\Gamma}$. Hence from construction G is invariant under $\mathrm{Aut}(X) = \overline{\Gamma}/\Pi$. This contradicts Lemma 2b. Hence F is trivial for *Case (A)*.

Consider now *Case (B)*. In such case, as $K_M = 3H_M$, equation (1) implies that $F \equiv H_M$. Again, as argued earlier in *Case (A)*, F on M descends to X to give an effective divisor $G \equiv H_X$ on X. Furthermore, $G = H_X + e_2$ for some two torsion line bundle on M so that $2H_X = 2G$ is effective on X, contradicting Lemma 2a.

Hence the claim about triviality of F is proved. We conclude that $K_M = P$. In general, P may have still have codimension two base point set, which is a finite number of points in this case. From equation (1), we may write $K_M = 2Q$ for an effective divisor Q on M, where Q is the pull-back of $\mathcal{O}(1)$ on $\Phi(M) \subset P^2_{\mathbb{C}}$, here we recall that $\Phi(M)$ is a rational curve as discussed earlier. Now applying Lemma 3, we see that $\mathcal{G}$ induces an action on the rational image $\Phi(M) \subset P^2_{\mathbb{C}}$. As $\Phi(M)$ is a rational curve, from Lefschetz Fixed Point Theorem, $\mathcal{G}$ has two fixed points on $\Phi(M)$. Let a be such a fixed point on $\Phi(M)$. The fiber $\pi(f^{-1}(a))$ above the fixed point a corresponds to an effective divisor Q_1 in the class of Q on M as mentioned above.

Note that $K_M = 2Q$ also implies that only Case (I) may occur, that is, $K_M \equiv 6L$, where L is a generator of the Néron-Severi group on M and hence that $Q_1 \equiv 3L$. On the other hand, Q_1 as constructed is fixed by $\mathcal{G}$ as a set. Hence Q_1 as a variety is invariant under the action of the Galois group $\mathcal{G}$ and descends to X to give rise to an effective divisor R_1 on X. Note that Q_1 contains all base points of K_M and hence the orbits of any base point, which is assumed to be non-trivial. Hence $Q_1 = p^*R_1$ and is connected. On the other hand, $R_1 \equiv cH_X$ on X, where $1 \leqslant c \leqslant 3$ is a positive integer. Hence

$$Q_1 = p^*R_1 \equiv cH_M \equiv 2cL,$$

which contradicts the earlier conclusion that $Q_1 \equiv 3L$. Hence P has no base point set.

If follows that Φ is a morphism and fibers over a rational curve. However, this contradicts the fact that M has Picard number 1.

In conclusion, $\dim_{\mathbb{C}} S \neq 1$ and hence has to be 2.

□

6. Codimension one component of base locus

6.1. The goal of this section is to show that there is no fixed component in the linear system associated to K_M.

Lemma 5. *The base locus of Φ_{K_M} does not contain dimension one component.*

Proof Let L be the generator of the Néron-Severi group modulo torsion. Since the Picard number is 1, we know that $L \cdot L = 1$ from Poincaré Duality. Replacing L by $-L$ if necessary, we may assume that L is ample. Now we may write

$$K_M = F + P,$$

where F is the fixed part and P is the moving part.

We claim that F is trivial. Assume on the contrary that F is non-trivial.

From construction, the covering $p : M \to X$ is a normal covering of order 4 and we may write $X = M/\mathcal{G}$, where $\mathcal{G} = \mathbb{Z}_2 \times \mathbb{Z}_2$ is a order 4 group corresponding to deck transformation of the covering. Hence $\mathcal{G}$ is a subgroup of the automorphism group of M. From definition, F is invariant under the automorphism group of M and hence is invariant under $\mathcal{G}$. It follows that F descends to an effective divisor G on X. As X has Picard number 1, we know that $K_X \equiv \beta G$ for some positive rational number β, observing that $p^*K_X = K_M$ is numerically an integral multiple of F.

From the remark in Section **4** and the descriptions in Section **3**, we know that the set of torsion line bundles on X is given by $\mathbb{Z}_2^4$. Hence from $K_X = 3H_X$, either

(I) $\beta G = 3H_X$ and $K_X = \beta G$, or
(II) $\beta G = 3H_X + e_X$ and $K_X = \beta G + e_X$, where e_X is a 2-torsion line bundle on X.

Case (I) cannot occur, since in such case there is a non-trivial section for $\Gamma(X, K_X)$, contradicting that X is a fake projective plane.

Hence we only need to consider Case (II). In such case, there are the following three subcases.

(IIa) $G \equiv H_X$, or
(IIb) $G \equiv 2H_X$, or
(IIc) $G \equiv 3H_X$.

In Case (IIa), $G = H_X + e_X$. Hence $2H = 2G$ is effective. This is impossible from Lemma 2.

In Case (IIb), again from Lemma 2, we can rule out $G = 2H$ and conclude that $G = 2H_X + e_X$ for some two torsion line bundle e_X. The argument of the last two paragraphs of §5 leads to a contradiction.

For Case (IIc), we have $G = 3H + e_X$, $K_X = G + e_X$. There are a few subclasses.

Case (IIci), $p^*G = F$ is irreducible. In such case, $K_M = F + e_{M2}$, where e_{M2} is a two torsion line bundle on M. However, as $K_M = F + P$, it follows that the movable part of K_M is numerically trivial. This is a contradiction.

Case (IIcii), $p^*G = F_1 + F_2$ consists of two irreducible components. In such case, $F_2 = \sigma F_1$ for some $\sigma \in \mathcal{G} = \mathbb{Z}_2 \times \mathbb{Z}_2$. By taking dot product with a generator of the Néron-Severi group modulo torsion, we conclude that $F_2 \equiv F_1$ and hence $F_2 = F_1 + e'_{M2}$ for some two torsion line bundle e'_{M2}. In such case $K_M = F_1 + F_2 + p^*(e_X)$. From construction, we know that $F_1 + F_2 \equiv 3H_M \equiv K_M$ on M. Again, this leads to a contradiction since P would then be numerically trivial.

Case (IIciii), $p^*G = F_1 + F_2 + F_3 + F_4$ consists of four irreducible components. In such case, we can reach similar contradiction by similar argument as above. Alternatively, we see from similar argument as in the last paragraph that

$$K_M \equiv F_1 + F_2 + F_3 + F_4 \equiv 4F_1.$$

This leads to a contradiction since we either have *Case (A)*, $K_M \equiv 6L$, where L is a generator of Néron-Severi group modulo torsion, or *Case (B)*, $K_M \equiv 3H_M$ with H_M being the generator of the Neron-Severi group of M modulo torsion.

We conclude that the base locus of Φ_{K_M} has no codimension one components.

□

7. Zero dimensional components of the base locus

7.1. From Lemma 3, the Galois group $\mathcal{G}$ of the covering $p : M \to X$ induces an action on $P^2_{\mathbb{C}}$. Let S and T be the order two automorphisms generated by the first and the second factor of $\mathbb{Z}_2$ on $\mathcal{G}$ respectively. From the results of [HL] (see also [S], [W]), as homology class of $P^2_{\mathbb{C}}$ corresponds to the canonical class on M is invariant under $\mathrm{Aut}(M)$, we know that the fixed point set of each of $\{S, T, ST\}$ consists of a line and an isolated point, so that the three points form vertices of a triangle and the three line segments form the sides of the triangle. Denote the triangle by $\Delta_{P_1P_2P_3}$. Hence we may assume that S fixes the point P_1 and the line ℓ_1 is the line through P_2 and P_3. Similarly for S and T. The vertices are the fixed points of $\mathcal{G}$.

Since a line on $P^2_{\mathbb{C}}$ is defined by a linear equation $a_1x_1 + a_2x_2 + a_3x_3 = 0$ on homogeneous coordinate $[x_1, x_2, x_3] \in P^2_{\mathbb{C}}$, it corresponds to the zero set of a holomorphic section $s \in \Gamma(M, K_M)$. Hence the pull back of ℓ_i on M, defined by $\pi(f^{-1}(\ell_i))$ is given by the zero set of $s_i \in \Gamma(M, K_M)$.

Lemma 6. *There is no zero dimensional component in $\cap_{i=1}^3 Z_{s_i}$ for s_i as defined above.*

The rest of the section is devoted to the proof of the lemma, which we resort to counting of intersection numbers and group actions. For this purpose, we first make some observations.

From Stein Factorization, we may decompose $f = g \circ h$ into holomorphic maps, where $h : \widehat{M} \to S$ has connected fiber, $f : S \to P^2_{\mathbb{C}}$ is finite and S is a normal surface. The degree of f is the same as the degree of g. Since f is generically finite, we know that there can at most be a finite number of dimension one fibers for h and hence for f. Suppose C is a dimension one fiber of f. Let $\widehat{s}$ be a section in $\Gamma(\widehat{M}, P_{\widehat{M}})$. We make the following claim.

Claim: $\widehat{s} \cdot C = 0$ and $\widehat{s}$ does not intersect C if $\widehat{s}$ does not share a component with C.

To prove the claim, we let D be a hyperplane section on $P^2_{\mathbb{C}}$ which avoids the set of points which are the image of all such contracted components C. From projection formula, $f^*D \cdot C = 0$. On the other hand, $f^*D \in \Gamma(\widehat{M}, P_{\widehat{M}})$. Hence $\widehat{s} \cdot C = f^*D \cdot C = 0$. This implies that $\widehat{s}$ does not intersect C if $\widehat{s}$ does not share a component with C.

In the following we are going to apply the *claim* several times. In our situation, since the Picard number of M and X are both 1, C would descend to a divisor C_1 of $\Gamma(X, H + \epsilon)$ or $(X, 2H + \epsilon)$ for some $Aut(X)$-invariant torsion line bundle ϵ in the fake projective space X, which does not exists from the vanishing results in [LY].

7.2. We may assume that $\pi : \widehat{M} \to M$ is a resolution of M invariant under $\mathrm{Aut}(M)$, so that $f : \widehat{M} \to P^2_{\mathbb{C}}$ is a morphism. From construction $\pi(f^{-1}(\ell_i))$, $i = 1, 2, 3$, is invariant under $\mathbb{Z}_2 \times \mathbb{Z}_2$ and hence is a $\mathbb{Z}_2 \times \mathbb{Z}_2$-invariant section s_i of $\Gamma(M, K_M)$. Note that they are linear independent by construction and hence span $\Gamma(M, K_M)$, which has dimension 3. Since each of them is invariant under the Galois transformation group $\mathbb{Z}_2 \times \mathbb{Z}_2$ of $p : M \to X$, each descends to a global section t_i of $K_X + \tau$, where

τ is a torsion line bundle. Since $p^*\tau = 0$, we know that τ is a $\mathbb{Z}_2 \times \mathbb{Z}_2$-torsion line bundle. If $\tau = 0$, we reach a contradiction since $H^0(X, K_X) = 0$. Hence we conclude that $s_i \in \Gamma(X, K_X + \tau_i)$, where τ_i are non-trivial 2-torsion line bundles. We note that they span all the possible sections of bundles of form $K_X + \tau$ in the orbit of $\mathbb{Z}_7$ of sections of $\Gamma(X, K_X + \tau_i)$, where τ is a 2-torsion line bundle, since $\dim(\Gamma(M, K_M)) = 3$. Here as mentioned in **3.2**, we know from the computation of Cartwright and Steger that $H_1(X, \mathbb{Z}) = \mathbb{Z}_2^4$, hence the bundle $K + \tau_i$ is invariant under $\mathbb{Z}_7$ as a line bundle. Now for sections of $\Gamma(X, K_X + \tau_i)$, if a section is not invariant, the space would have dimension greater than 1, which when lifted to X and taken together with s_1, s_2, s_3, would lead to $\dim(\Gamma(M, K_M)) > 3$.

Let $B = p(A)$. Since $\{s_1, s_2, s_3\}$ is $\mathbb{Z}_2 \times \mathbb{Z}_2$ invariant, the zeros divisors t_i all pass through each point of B on X. As $K_X \cdot K_X = 9$, it follows that B has at most 9 points.

In the following we would denote by $\widehat{s}_i$ the proper transform of s_i in the $\mathrm{Aut}(M)$-invariant minimal resolution $\widehat{M}$ of M associated to the birational map Φ_{K_M}. The covering map p induces an isomorphism of a small neighborhood of base point of $s_i, i = 1, \ldots, 3$ to a small neighborhood of $t_i, i = 1, \ldots, 3$. For convenience, we would denote by $\widehat{t}_i$ the proper transform of t_i on $\widehat{X}$, the induced modification of X corresponding to $\widehat{M} \to M$.

7.3 In terms of the notation of **2.1**, we note that $P^2 = p^*P \cdot P_{\widehat{M}} = F_{\widehat{M}} \cdot P_{\widehat{M}} + P_{\widehat{M}}^2$. The sequence of estimates of degrees can be written as

$$\deg \Phi = \deg(f) = P_{\widehat{M}}^2 = P^2 - F_{\widehat{M}} \cdot P_{\widehat{M}} \leqslant P^2 \leqslant K_M^2 = 36. \tag{2}$$

7.4 Now we recall that $\mathrm{Aut}(X)$ is the abelian group $G = 7 : 3 = \mathbb{Z}_7 \rtimes \mathbb{Z}_3$ of order 21, where $\mathbb{Z}_3$ acts on $\mathbb{Z}_7$ by a homomorphism $\mathbb{Z}_3 \to \mathrm{Aut}(\mathbb{Z}_7)$. G has a normal Sylow subgroup of order 7, denoted by $\mathbb{Z}_7$. There are also seven Sylow subgroups of order 3. $\mathbb{Z}_7$ has three fixed points on X, and each Sylow 3-subgroup has 3 fixed points on X, according to a result of Keum and Cartwright-Steger. Let $1 \neq \gamma \in G$. Note that $\gamma^* t_i$ would be another section of some $K_X + \tau$, where τ is a 2-torsion. As mentioned in the last paragraph, from dimension considering, it follows that τ has to be one of the $\tau_i, i = 1, 2, 3$ mentioned earlier, and $\gamma^* t_i$ has to pass through each point of T as well. As $|G| = 21$ and the set B, which has cardinality at most 9, is invariant under an automorphism of M, we conclude that each point Q in B is actually fixed by some element $\gamma \in G$. In our case, there is a unique subgroup $\mathbb{Z}_7$ of order 7 and seven subgroups $\mathbb{Z}_3$ of order 3 acting on X. We consider the subgroup $\mathbb{Z}_3$ of $\mathrm{Aut}(X)$ descended from $\mathrm{Aut}(M)$ as mentioned in Lemma 1(f). The group of order 3 and the group of order 7 generates $\mathrm{Aut}(M)$. There are two cases to consider,

Case I: Q is a fixed point of a subgroup H of $\mathrm{Aut}(M)$ isomorphic to $\mathbb{Z}_3$, and
Case II: Q is a fixed point of the subgroup of $\mathrm{Aut}(M)$ isomorphic to $\mathbb{Z}_7$.

7.5 Consider first Case I. For simplicity, we just call the group involved $\mathbb{Z}_3$. Assume now that a point Q_1 in B is a fixed point of $\mathbb{Z}_3$. Then Q_1 lies on t_1 and is not fixed by $\mathbb{Z}_7$, as it is well-known that no point on M is fixed by the whole group $G = 7 : 3$. Hence the orbit of Q_1 has seven points $Q_i, i = 1, \ldots, 7$ and all lies on t_1.

As $t_1 \cdot t_2 = 9$, we conclude that apart from the seven points Q_1 which are base locus, t_1 intersects t_2 either twice at a point or once at two points, which we denote by W. Since each point in $p^{-1}(W)$ is mapped to the point $\ell_1 \cap \ell_2$ on $P^2_{\mathbb{C}}$ and ℓ_1 intersects ℓ_2 in simple normal crossing, we conclude that the degree $\deg(f) \geqslant 8$. Here we recall from the discussion following the claim in **7.1** that $p^{-1}(W)$ does not contain any one dimensional component. Recall that each t_i is fixed as a set by $\mathbb{Z}_7$. Hence A has 28 points $R_i, i = 1, \ldots, 28$ on M. After resolving A, the base point set of K_M, each R_i gives rise to an exceptional curve F_i, which intersects each proper transform $\widehat{s}_i$ of s_i. $\widehat{s}_1 \cdot F_i > 0$ for each $i = 1, \ldots, 28$. Hence $F \cdot P_{\widehat{M}} \geqslant 28$ in (2). From (2) it follows that $\deg(f) \leqslant 36 - 28 = 8$. Hence we conclude that $\deg(f) = 8$ and each proper transform of $\widehat{s}_i$ intersects F_i only once for each i. Moreover, the exceptional divisor over each R_i is a single rational curve F_i for each $i = 1, \ldots, 28$.

Now for each fixed F_i, $f(F_i)$ intersects $\ell + 1$. Let $x \in f(F_i) \cap \ell_1$. We observe that $f^{-1}(x)$ contains $\gamma(F_i \cap f^{-1}(x))$ for all $\gamma \in \mathbb{Z}_2 \times \mathbb{Z}_2$. As the degree $\deg(f) = 8$, the degree of the curve $\gamma(F_i)$ in $P^2_{\mathbb{C}}$ is either 1 or 2.

Consider the action of a subgroup $\mathbb{Z}_3$ of G on X. Either

(a): it leaves t_i invariant as a set for all i, or
(b): permutes among t_i, $i = 1, 2, 3$.

Consider first Case (a). From [Su], [HL] or [W], as the homology class of $P^2_{\mathbb{C}}$ corresponding to the canonical class on M is invariant under $\mathrm{Aut}(M)$, we know that the fixed point set of $H \cong \mathbb{Z}_3$ on $P^2_{\mathbb{C}}$ consists either of a line and a point, or three points. We claim that the first case cannot happen. Otherwise the line has to be one of $\ell_i, i = 1, \ldots, 3$ as the fixed point set contains $f(\widehat{s}_i \cap \widehat{s}_j)$ for $i \neq j$. However, if say ℓ_1 is fixed by $\mathbb{Z}_3$, it implies that $\widehat{s}_1$ is fixed pointwise under $\mathbb{Z}_3$, since we know that the degree of f is 8, which is not divisible by 3. Now we used the fact that M as an unramified covering of X is an arithmetic ball quotient division algebra and hence supports no totally geodesic curves, which in turn implies that a non-trivial finite group action on M has only a finite number of fixed points, cf. [Ye1], p.19-21. In particular, there is no fixed point of $\mathbb{Z}_3$ on $\Phi^{-1}(y)$ for a generic $y \in \ell_i \subset P^2_{\mathbb{C}}$. We conclude that $\widehat{t}_1$ and hence t_1 is fixed pointwise under $\mathbb{Z}_3$, which is a contradiction since $\mathbb{Z}_3$ has isolated fixed points on M. Hence the claim is proved.

Hence the induced action of $\mathbb{Z}_3$ on $P^2_{\mathbb{C}}$ has three fixed points. We aso know that on each rational line ℓ_i, the induced action of $\mathbb{Z}_3$ has two fixed points. Since there are three lines, it follows that the fixed points of $\mathbb{Z}_3$ has to be the three points P_1, P_2, P_3 corresponding to $\ell_i \cap \ell_{i+1}$.

In terms of our earlier notation, we note that each $Q_i, i = 1, \ldots, 7$, which is a $\mathbb{Z}_7$-orbit of Q_1, lies in B as well, since the divisors $t_j, j = 1, 2, 3$ are invariant under $\mathbb{Z}_7$ as we note earlier. Note that the pull-back of each t_j is just a single irreducible component s_j, which follows from Lefschetz Hyperplane Theorem. In fact, any extra component would lead to $\dim(\Gamma(M, K_M)) > 3$ and a contradiction as well. Hence each $R_j \in A$ for $j = 1, \ldots, 28$.

Observe from Lemma 1 that A_4 is the automorphism group of M and hence contains four $\mathbb{Z}_3$-subgroups $H_i, i = 1, \ldots, 4$, permuted under conjugation by $\mathbb{Z}_2 \times \mathbb{Z}_2$. Moreover, the action of H_i descends to X. Let H_1 be the $\mathbb{Z}_3$-group studied in the last paragraph. On X, the points gQ_i, where $g \in \mathbb{Z}_7$, are fixed by the $\mathbb{Z}_3$-subgroup gH_1g^{-1} of $\mathrm{Aut}(X)$. Since $\mathrm{Aut}(X) = 7 : 3$ contains precisely seven such subgroups under conjugacy of elements of $\mathbb{Z}_7$, we know that four of the seven groups are given by $H_i, i = 1, \ldots, 4$. Consider now the four points $R_{1i} \in p^{-1}(Q_1)$, where $i = 1, \ldots 4$. The set is invariant under H_1. Hence we may assume that R_{11} is fixed under H_1. We claim that each of $R_{1i}, i = 2, 3, 4$, is invariant under some H_j for $j = 2, 3, 4$. This follows from the fact that the deck transformation of the covering $p : M \to X$ is precisely $\mathbb{Z}_2 \times \mathbb{Z}_2$, and that hR_{11} is fixed by hH_1h^{-1} for $h \in \mathbb{Z}_2 \times \mathbb{Z}_2$. This argument actually folds for $R_{ji}, i = 1, \ldots, 4$ for each $j = 1, \ldots, 4$. Hence there are sixteen such points R_{ji}. Rename them as $R_i, i = 1, \ldots, 16$. It follows that $f(F_i \cap \widehat{s}_j)$ lies on ℓ_j for $i = 1, \ldots, 16$ and $j = 1, 2$.

Recall from earlier discussions that the action of $H_k \cong \mathbb{Z}_3, k = 1, \ldots, 4$ on $P^3_{\mathbb{C}}$ has three fixed points $P_k, k = 1, 2, 3$. From earlier discussions, we also know that $F_i \cap \widehat{s}_j$ only at one point. Since both F_i and $\widehat{s}_j$ are invariant under $\mathbb{Z}_3$, it follows that $F_i \cap \widehat{s}_j$ for each i and j is invariant under $\mathbb{Z}_3$. Hence the same is true for $f(F_i \cap \widehat{s}_j)$. It follows that for $j = 1, 2$ and $i = 1, \ldots, 16$, $f(F_i \cap \widehat{s}_j)$ is one of the three fixed points mentioned earlier. Since they also lie on ℓ_1 and ℓ_2 by definition, it follows that $f(F_i \cap \widehat{s}_j) = \ell_1 \cap \ell_2 = P_3$. Since there are at least 16 points in the preimage of P_3 as constructed, this contradicts $\deg(f) = 8$ derived earlier. Here we have used the *Claim* in §7.1.

Consider now Case (b). In terms of earlier notation $H_i \cong \mathbb{Z}_3$ induced an action on $P^3_{\mathbb{C}}$, the image of Φ. From construction, we know that H_i leaves the three lines $\cup_{j=1}^3 \ell_j$ invariant as a set, and permutes the three lines. From the results of [Su], [HL] or [W], H_i acts as elements in $U(3)$ and the fixed point set consists either of (i) three fixed points, or (ii) a point and a line L. First we observe that (ii) cannot happen, for otherwise L intersects ℓ_1 and there is a fixed point of H_i on ℓ_1. This implies that the fixed point has to be either $\ell_1 \cap \ell_2$ or $\ell_1 \cap \ell_3$. In the first case, H_i has to permute between ℓ_1 and ℓ_2, which is not possible as H_i has order 3 and does not leave ℓ_1 invariant. Similar contradiction arises in the second case. Hence only (i) occurs. Choose homogeneous coordinates on $P^2_{\mathbb{C}}$ so that ℓ_1 be defined by $Z_1 = 0$ and $Z_2 = \gamma Z_1, Z_3 = \gamma^2 Z_1$, where γ is a generator of H_1. It follows that we may represent γ in terms of our basis

$$\gamma = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

For $H_i, i = 2, 3$, a generator γ_i has to be of form

$$\gamma_i = \begin{pmatrix} 0 & 0 & \theta_{i1} \\ \theta_{i2} & 0 & 0 \\ 0 & \theta_{i3} & 0 \end{pmatrix}$$

where $\theta_{ij}, j = 1, 2, 3$ are third roots of unity. It follows from direct computation that $\theta_{i1} \cdot \theta_{i2} \cdot \theta_{i3} = 1$ so that if we write $\theta_{ik} = \theta_{i1} \cdot \omega'_{ik}$ for $k = 2, 3$, we have $\omega'_{i3} = (\omega'_{i2})^2$. Hence in terms of the chosen homogeneous coordinates on $P^2_{\mathbb{C}}$, the fixed points for each H_i is given by $U_1 = [1,1,1], U_2 = [1,\eta,\eta^2], U_3 = [1,\eta^2,\eta]$, where η is a third root of unity.

Recall also that H_i has a fixed point at Q_i in our earlier notation for $i = 1,\ldots,4$. Consider now $p^{-1}(Q_1) = R_{1j}, j = 1,\ldots 4$. We have assumed that R_{11} is fixed by H_1. Now since the exceptional divisor F_1 consists of a rational curve and we have an action of $H_1 \cong \mathbb{Z}_3$ acting on F_1, there are at least two points on F_1 fixed by H_1. Similarly, as the three points $R_{1j}, j = 2,3,4$ are obtained from action of $\mathbb{Z}_2 \times \mathbb{Z}_2$ on R_{11}, we see that each R_{1j} is fixed by some conjugate of H_1 and hence by one of H_2, H_3, H_4. In other words, each of the four points $R_{1j}, j = 1,\ldots,4$ is fixed by precisely one H_k for some $k = 1,\ldots,4$. This holds for all the 16 points $p^{-1}(Q_j) = \{R_{j1}, R_{j2}, R_{j3}, R_{j4}\}$, $j = 1,\ldots,4$. Each of them gives rise to two fixed points of some H_k on the exceptional divisor F_i. Hence there are altogether 32 such points. Now the action of each of the four groups H_i, $i = 1,\ldots,4$ on $P^2_{\mathbb{C}}$ has fixed point set given by $\{U_1, U_2, U_3\}$. It follows that the degree of the mapping Φ is at least $32/3$, noting that there may be other points in the preimage. Since $32/3 > 8$, this contradicts our earlier conclusion that $\deg(\Phi) = 8$.

7.6 Consider now Case II and again denote by $\mathbb{Z}_7$ the unique $\mathbb{Z}_7$ subgroup of $\mathrm{Aut}(M)$. Assume now that a point Q_1 in B is a fixed point of $\mathbb{Z}_7$. Under the action of $1 \neq \gamma \in H_i, i = 1,\ldots 4$, a $\mathbb{Z}_3$ subgroup descends to M mentioned earlier, we know that that $\gamma Q_1 \neq Q_1$ and hence has to be fixed by a conjugate of $\mathbb{Z}_7$ subgroup of G. As such a Sylow 7-subgroup is unique, the group is just the $\mathbb{Z}_7$ group studied. Hence γQ_1 is fixed by $\mathbb{Z}_7$ as well. Moreover, the same argument implies that γQ_1 lies in the base locus of $t_j, j = 1,2,3$ and hence $\gamma Q_1 \in B$. It follows that all the three fixed points of $\mathbb{Z}_7$ on X lie in B. As discussed earlier, each t_i is fixed as a set by $\mathbb{Z}_7$. Then Q_1 lies on t_i and is not fixed by $\mathbb{Z}_3$. Hence its orbit by $\mathbb{Z}_3$ consists Q_1, Q_2 and Q_3 lying on t_i for $i = 1,2,3$. This leads to 12 base points R_j, $j = 1,\ldots,12$, on M after pulling back by π. Resolving each base point in a $\mathrm{Aut}(M)$-invariant manner, it follows as before that $F \cdot \widehat{s}_i = F \cdot P_{\widehat{M}}$ is a positive multiple of 12. Note that for each of $i = 1,2,3$, the behavior of $F \cdot s_i$ at all the points R_j are all the same for all $1 \leqslant j \leqslant 12$, since s_i is invariant under each $H_j, j = 1,\ldots,4$. Since $F_i \cdot P_{\widehat{M}} > 0$ for each irreducible component F_i of F, we conclude from (2) that $F \cdot P_{\widehat{M}}|_{R_j} = 1$ or 2, and $F|_{R_j}$ can have either

Case (a), one component, or
Case (b), two components.

Moreover,

$$\deg\Phi = \deg(f) = 36 - F \cdot P_{\widehat{M}} \leqslant 24. \tag{3}$$

Now we observe that t_1 and t_2 cannot intersect at any other points apart from base locus. Otherwise there would be at least 7 such points in the orbit of $\mathbb{Z}_7$ on t_1. This leads to 28 points of the intersection of s_1 and s_2 on M. Unless $\widehat{s}_1$ and $\widehat{s}_2$ share a component C which is f exceptional and mapped to the point $\ell_1 \cap \ell_2$, the claim in §7.1 implies that $\deg(f) > 28$, contradicting $\deg(f) \leqslant 24$. However if such a component C exists, as C does not have a component in the exception divisor of π as studied in §7.1, we conclude that s_1 and s_2 share some C_1 from M. This implies that t_1 and t_2 share some component C_2 on X. In such a case, C_2 is a section of $(X, H+\epsilon)$ or $(X, 2H+\epsilon)$ for some Aut(X)-invariant torsion line bundle ϵ in the fake projective space X, which does not exist from the vanishing results in [LY].

Note that the three fixed points of $\mathbb{Z}_7$ on X are permuted by any subgroup isomorphic to $\mathbb{Z}_3$ in Aut(X), and so does the base points of K_M under the action of Aut(M). Hence the behavior of the base locus at the 12 points of base locus on X are the same. Consider one such base point R_a. Suppose $F_{ai}, i = 1, \dots, N$ are the irreducible components of the resolution F_a of the point R_a so that the proper transform of $\pi^*\Phi_{K_M}$ is base point free. We note that a resolution in a small neighborhood of a point $R_a \in A$ can be considered as the resolution of the corresponding point $Q_a \in B$, since the mapping $p : M \to X$ is etale. Hence by doing surgery, we may assume that there is a resolution $\pi : \hat{X} \to X$ for which an exceptional fiber G_a at Q_a is isomorphic to an exceptional fiber F_a at R_a. Similarly, we let $\widehat{t_j}$ be the proper transforms of t_i. Now since Q_a is a fixed point of $\mathbb{Z}_7$ on X, $\mathbb{Z}_7$ acts on the exceptional fiber G_a at Q_a. The induced action of $\mathbb{Z}_7$ should leave each G_{ai} which intersect with some $\widehat{t_j}$ invariant. Otherwise, there would be at least seven such components, giving rise to $G_a \cdot \widehat{t_i} \geqslant 7$. This is translated to the conclusion that $F_a \cdot \widehat{s}_i \geqslant 7$ on $\widehat{M}$. Since there are 12 such base points, it would lead to $F \cdot K_M \geqslant 12 \cdot 7$ which violates (3).

Consider now *Case a*. There is only one irreducible component in G_a at Q_a. Since G_a is a rational curve, $\mathbb{Z}_7$ has two fixed points only. Since $G_a \cap \widehat{t_i}$ is a fixed point, we may assume that the two fixed point are $\widehat{Q}_1 = G_a \cap \widehat{t_1}$, and $\widehat{Q}_2 = G_a \cap \widehat{t_2} = G_a \cap \widehat{t_3}$. This is reflected correspondingly for $\widehat{s}_i$ on R_j. Since $F_a \cdot \widehat{s}_i = F_a \cdot P_{\widehat{M}}$ for each i, this number can either be 1 or 2 from (3). $G_a \cdot \widehat{t_i}$ cannot be 2, for otherwise the intersection of $\widehat{s}_2$ and $\widehat{s}_3$ at F_a satisfies $\widehat{s}_2 \cdot \widehat{s}_3|_{F_a} = \widehat{t_2} \cdot \widehat{t_3}|_{G_a} \geqslant 2$, where the notation refers to intersection along F_a or G_a. Since there are twelve such points R_a, by looking at the preimage of $\ell_2 \cap \ell_3$, this implies that $\deg \Phi \geqslant 24$, contradicting (3) since $F \cdot P_{\widehat{M}} = F \cdot \widehat{s}_i = 24$ in such case. Hence we conclude that $F_a \cdot \widehat{s}_i = 1$ for each $a = 1, \dots, 12$. In particular, we conclude from this and $F \cdot \widehat{s}_i = 12$ that each $\widehat{s}_i$ intersects F_a normally for each $i = 1, \dots, 3$ and $\widehat{s}_2$ intersects $\widehat{s}_3$ normally. This implies that on M, s_1 intersects s_2 and s_3 transversally respectively, and s_2 intersects s_3 with multiplicity two at R_a. Hence t_1 intersects t_2 and t_3 transversally respectively, and t_1 intersects t_2 with multiplicity two at Q_a. This means that $(t_1 \cdot t_2 + t_2 \cdot t_3 + t_3 \cdot t_1)|_{Q_a} = 4$, where $t_k \cdot t_l|_{Q_a}$ refers to multiplicity of intersection of t_k and t_l at Q_a. Recall now that on X, the zero divisors $t_i \cdot t_j = K_M^2 = 9$ for $i \neq j$. Hence

$$27 = t_1 \cdot t_2 + t_2 \cdot t_3 + t_3 \cdot t_1 = \sum_{i=1}^{3} (t_1 \cdot t_2 + t_2 \cdot t_3 + t_3 \cdot t_1)|_{Q_i} = 12,$$

which is a contradiction.

Consider now *Case b*. In this case, an exceptional fiber G_a at Q_a consists of two irreducible components G_{a1} and G_{a2} meeting at a point W_{a0} on $\widehat{M}$. W_{a0} is fixed by $\mathbb{Z}_7$. Denote by W_{ai} the other fixed point of $\mathbb{Z}_7$ on G_{ai}, $i = 1, 2$. From (3) as before, we know that $\deg(\Phi) = 12$ and $F \cdot P_{\widehat{M}} = 24$. As there are twelve points $R_i, i = 1, \ldots, 12$ under consideration, we conclude that $F_a \cdot \widehat{s}_i = 2$ for all $a = 1, \ldots, 12$ and $i = 1, 2, 3$. As in *Case a*, G_a meets $\widehat{t}_i$ only at one of the three fixed points of $\mathbb{Z}_7$. If W_{a1} does not lie in at least one of $\widehat{t}_j, j = 1, 2, 3$, as $P_{\widehat{M}} \cdot G_{a1} > 0$, it follows that all $\widehat{t}_i, i = 1, 2, 3$ intersects G_{a1} at the point W_{a0}, which however contradicts that $P_{\widehat{M}}$ is base point free. Similarly, if W_{a2} does not lie in one of $\widehat{t}_j, j = 1, 2, 3$, it leads to the same contradiction. If on the other hand W_{a0} does not lie in at least one of $\widehat{t}_j, j = 1, 2, 3$, all the $\widehat{t}_i, i = 1, 2, 3$ meet F_{a1} at the two points W_{a1}. Again it follows that W_{a1} is a base point of $P_{\widehat{M}}$ and leads to a contradiction.

Hence after renaming index if necessary, we may assume that $W_{a1} \in G_a \cap \widehat{t}_1, W_{a2} \in G_a \cap \widehat{t}_2$ and $W_{a0} \in G_a \cap \widehat{t}_3$. However, this implies correspondingly that $F \cdot \widehat{s}_3 \geqslant \sum_{a=1}^{12} F_a \cdot \widehat{s}_3 = 24$. From (3), it follows that $F \cdot \widehat{s}_3 = 24$. Hence we conclude that $F \cdot \widehat{s}_i = F \cdot P_{\widehat{M}} = 24$ for $i = 1, 2$. This implies that $\widehat{t}_i$ intersects G_{ai} with multiplicity 2 at W_{ai} and hence s_i intersects F_{ai} to multiplicity 2, where $i = 1, 2$. Now from the paragraph immediately after (3), we conclude that $\widehat{s}_i$ cannot intersect $\widehat{s}_j$ at any point except for the union of the fibers F, which implies that $\widehat{s}_i \cdot \widehat{s}_j = 0$ for $i \neq j$ from the discussion above. This however contradicts the fact that $\widehat{s}_i \cdot \widehat{s}_j = P_{\widehat{M}} \cdot P_{\widehat{M}} > 0$.

In conclusion, both *Case (a)* and *Case (b)* leads to a contradiction. Hence Case II does not occur. This concludes the proof of Lemma 2.

□

7.7 Proof of Lemma 6

From the discussions in **7.4**, every point in the base locus has to be in the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbit of one of the fixed point set of either a subgroup of order 7 or 3 of the automorphism group of $\mathrm{Aut}(X)$. The discussions in **7.5** and **7.6** implies that there is no base locus corresponding to the fixed point set of $\mathrm{Aut}(M)$. Lemma 6 follows.

□

8. Conclusion of proof

8.1. The discussions of the previous few sections can be summarized into the following proposition.

Proposition 1. *The linear system associated to* $\Gamma(M, K_M)$ *is base point free and the image of* Φ_{K_M} *is* $P^2_{\mathbb{C}}$.

Proof From Lemma 5, the base locus of K_M is of dimension 0. From Lemma 6, we know that it is base point free. From Lemma 4, we know that the image of Φ_{K_M} has complex dimension 2 and hence has to be $P^2_{\mathbb{C}}$.

□

8.2. We can now complete the proof of our main result.

Proof of Theorem 1

We use the fake projective plane $X = B^2_{\mathbb{C}}/\Pi$ with Π as given in Section **3**. Let $M = B^2_{\mathbb{C}}/\Sigma$ be a $\mathbb{Z}_2 \times \mathbb{Z}_2$ cover of X as above. From Lemma 1, we conclude that $h^{1,0}(M) = 0$, from which we conclude from the discussions in the proof of Lemma 4 that $h^0(M, K_M) = 3$. Hence the canonical map Φ is apriori a birational map from M

to $P^2_{\mathbb{C}}$. Lemma 1 also implies that the Picard number $\rho(M) = 1$. From Proposition 1, we conclude that the canonical map is base point free and hence is a well-defined holomorphic map. The degree of the canonical map is given by

$$\int_M \Phi^*\mathcal{O}(1)\cdot\Phi^*\mathcal{O}(1) = K_M \cdot K_M = 4K_X \cdot K_X = 36,$$

since X is a fake projective plane and hence $K_X \cdot K_X = 9$. The surface is minimal since it is a complex ball quotient and hence does not contain rational curves due to hyperbolicity of M. Theorem 1 follows.

□

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Corrigendum

1. In the paper [Y], the proof of Lemma 6 contains the following error pointed out by Keum and F. Catanese (cf. arXiv:1801.05291). In the middle of **7.2** for the proof of Lemma 6, there is the wrong claim that the bundle $K_X + \tau_i$ and hence t_i is invariant under $\mathbb{Z}_7$ on X. The goal here is to give a completely new proof of Lemma 6 and hence Theorem 1. All the unexplained notations are referred to [Y].

Recall that $X = B^2_{\mathbb{C}}/\Pi$ is a fake projective plane with Π one of the lattices found in **5.11, A3** of [16], and is labelled as $(a = 7, p = 2, \emptyset, D_3 2_7)$ in [4] as a surface in the class corresponding to $(a = 7, p = 2, \emptyset, D_3 2_7)$. The automorphism group of X is $G = \langle a, b \mid a^7 = b^3 = 1, bab^{-1} = a^2\rangle$. It contains just one subgroup $\langle a\rangle$ of order 7, and the subgroup $\langle b\rangle$, whose seven distinct conjugates are the Sylow 3 subgroups of G.

The surface M that is to be proved to have canonical degree 36 is constructed as follows. It is known from the file of registry of surface of [4] that $H_1(X/G, \mathbb{Z}) = \mathbb{Z}_2$, $H_1(X/\langle a\rangle, \mathbb{Z}) = \mathbb{Z}_2$, $H_1(X/\langle b\rangle, \mathbb{Z}) = \mathbb{Z}_2^2$ and $H_1(X, \mathbb{Z}) = \mathbb{Z}_2^4$. Let $\rho_1 : X \to X/\langle b\rangle$ and $\rho_2 : X \to X/\langle a\rangle$ and $\rho : X \to X/G$ be the projection maps. Denote by $A \cong \mathbb{Z}_2^2$ the first two factors of $\mathbb{Z}_2$ in $H_1(X, \mathbb{Z})$ so that $(\rho_1)_*(A) = H_1(X/\langle b\rangle, \mathbb{Z})$, and B the first factor of $\mathbb{Z}_2$ in A so that $\rho_*(A) = H_1(X/G, \mathbb{Z})$. In such case, $(\rho_2)_*(B) = H_1(X/\langle a\rangle, \mathbb{Z})$. Let $p : M \to X$ be the $\mathbb{Z}_2 \times \mathbb{Z}_2$ cover of X with fundamental group Σ obtained by kernel of the homomorphism of $\alpha : \Pi \to A$. In other words, $\Pi/\Sigma = A$. From construction, $p^*(K_X + \tau_i) = K_M$ for $i = 1, 2, 3$, since τ_i's corresponds to elements of A, and M is a covering of X given by $\ker \alpha$.

The argument before **7.2** of [Y] is valid. In particular, the following is proved,
(i) from **4.3** of [Y], $\Gamma(M, K_M)$ has dimension 3 and from the second paragraph of **7.1**, is spanned by $s_1, s_2, s_3 \in \Gamma(M, K_M)$, where the zero divisor $Z_{s_i} = \{s_i = 0\}$ of s_i is invariant as a set under A;
(ii) from the first paragraph of **7.2**, s_i descends to a section $t_i \in \Gamma(X, K_X + \tau_i)$ for some two torsion line bundle τ_i, which could be identified with a non-zero element in $A \cong \mathbb{Z}_2^2$ by the Universal Coefficient Theorem for $i = 1, 2, 3$, in the sense that the torsion line bundles can be identified with the torsion elements in $H_1(X, \mathbb{Z})$;
(iii) $h^0(X, K_X + \tau_i) = 1$ for all $i = 1, 2, 3$ since $h^0(M, K_M) = 3$ as mentioned above, and as $t_i \in H^0(X, K_X + \tau_i)$, we conclude that $t_i^2 \in \Gamma(X, 2K_X)$;
(iv) from Lemma 4 and 5 of [Y], there is no base locus of $\Gamma(M, K_M)$ except possibly at a finite number of points, that is, $\cap_{i=1}^3 Z_{s_i}$ contains at most a finite number of points, where Z_s denotes the zero divisor of s.

For Theorem 1, it suffices to prove Lemma 6 in §7 of [Y] in the sense that such isolated based points do not exist. In this Corrigendum, this is shown by relating sections of $\Gamma(M, K_M)$ to certain sections of $\Gamma(X, 2K_X)$. We remark that M as described above has a presentation given by **3.3** of [Y] after checking with Magma, though the explicit presentation is not needed for the proof of Theorem 1 here.

2. The automorphism group G has three 1-dimensional representations, χ_0, χ and $\bar{\chi}$, and two 3-dimensional irreducible representations π, $\bar{\pi}$. Here χ_0 is the trivial character, and $\chi(a) = 1$ and $\chi(b) = \zeta_3$, while $\pi(a)$ is the diagonal matrix with diagonal entries ζ_7, ζ_7^2 and ζ_7^4, $\pi(b)$ is the permutation matrix corresponding to the permutation $(1, 3, 2)$, and $\bar{\chi}$ and $\bar{\pi}$ are the complex conjugates of χ and π, and ζ_n is a fixed primitive n-th root of unity.

Consider the G-spaces $V = H^0(X, 2K_X)$ and $P^9_{\mathbb{C}} = P_{\mathbb{C}}(V)$. They contain one copy of the trivial representation, 2 copies of π and 1 copy of $\bar{\pi}$, so that $V = V_0 + 2V_1 + V_2$. This is given explicitly in [BK] (2.1):

$$\begin{aligned} & a(u_0 : u_1 : u_2 : u_3 : u_4 : u_5 : u_6 : u_7 : u_8 : u_9) \\ = \; & (u_0 : \zeta_7^6 u_1 : \zeta_7^5 u_2 : \zeta_7^3 u_3 : \zeta_7 u_4 : \zeta_7^2 u_5 : \zeta_7^4 u_6 : \zeta_7 u_7 : \zeta_7^2 u_8 : \zeta_7^4 u_9) \end{aligned} \tag{4}$$

$$\begin{aligned} & b(u_0 : u_1 : u_2 : u_3 : u_4 : u_5 : u_6 : u_7 : u_8 : u_9) \\ = \; & (u_0 : u_2 : u_3 : u_1 : u_5 : u_6 : u_4 : u_8 : u_9 : u_7) \end{aligned} \tag{5}$$

We know from Riemann-Roch and Kodaira Vanishing Theorem that $h^0(X, 2K_X) = 10$. Denote by $h^0(X, 2K_X)^H = \dim_{\mathbb{C}} \Gamma(X, 2K_X)^H$ the dimension of the subspace of sections invariant up to a scalar multiple under a group H. We find that $h^0(X, 2K_X)^{\langle b \rangle} = 4$. To see this, from [4] or [11], we know that the singular set of $X/\langle b \rangle$ consists of three $\frac{1}{3}(1, 2)$ points, the resolution of each is a chain of two (-2) curves. Hence if $\sigma : Y \to X/\langle b \rangle$ is the canonical resolution, $K_Y = \sigma^* K_{X/\langle b \rangle}$ so that $K_Y^2 = (\sigma^* K_{X/\langle b \rangle})^2 = 3$ and $c_2(Y) = 9$, which gives rise to $h^0(X, 2K_X)^{\langle b \rangle} = h^0(X/\langle b \rangle, 2K_{X/\langle b \rangle}) = 4$ from Riemann-Roch and Kawamata-Viehweg Vanishing Theorem. Alternatively, this also follows from the representation above. Furthermore, the representation above shows that $h^0(X, 2K_X)^G = h^0(X/G, 2K_{X/G}) = 1$, where $K_{X/G}$ is regarded as a $\mathbb{Q}$ line bundle.

We would use (i)-(iv) in **1**, the explicit computations of [BK], and study of $\langle b \rangle$ invariant sections to check that there is no isolated points in $\cap_{i=1}^3 Z_{s_i}$. We used Magma which is symbolic and exact.

3. Proof of Lemma 6 As summarized in (i)-(iv) above, $H^0(M, K_M)$ is generated by $s_i, i = 1, 2, 3$, which descend to effective sections t_i of $K_X + \tau_i$ on X. From our setting in **1**, each $\tau_i, i = 1, 2, 3$ is invariant under $\langle b \rangle$ corresponding to the non-trivial element in $H_1(X/\langle b \rangle, \mathbb{Z}) = \mathbb{Z}_2^2$ from the Universal Coefficient Theorem. One of those three, say denoted by τ_3, is invariant under G corresponding to $H_1(X/G, \mathbb{Z}) = \mathbb{Z}_2$. Hence under the action of $\langle a \rangle$, the torsion line bundle τ_3 is fixed, while τ_1, τ_2 are not invariant under $\langle a \rangle$ and hence a acts freely within each orbit $\langle a \rangle \tau_1$ and $\langle a \rangle \tau_2$. The two orbits are disjoint, for if $a^k \tau_1 = \tau_2$ for some $1 \leqslant k \leqslant 6$, then

$$\tau_2 = b\tau_2 = ba^k \tau_1 = ba^k b^{-1} \tau_1 = a^{2k} \tau_1 = a^k \tau_2,$$

contradicting the free action of $\langle a \rangle$ on the orbit of τ_2. It follows that the cardinality of the set $\cup_{i=1}^2 \langle a \rangle \tau_i$ is 14, which together with the trivial bundle 0 and τ_3 exhaust the torsion line bundles of X corresponding to $H_1(X, \mathbb{Z}) = \mathbb{Z}_2^4$ from the Universal Coefficient Theorem. In addition to τ_3, we denote the remaining fourteen non-trivial 2-torsion line bundles by $\sigma_j, j = 1, \ldots, 14$.

Since $h^0(X, K_X + \tau_i) = 1$ for $i = 1, 2, 3$, it follows under the action of $\langle a \rangle$ as explained above that $h^0(X, K_X + \tau) = 1$ for all 2-torsion line bundle $\tau \neq 0$ corresponding to $H_1(X, \mathbb{Z}) = \mathbb{Z}_2^4$. Hence there are 14 sections $w_j, 1 \leq j \leq 14$ in $\cup_{i=1}^{14}\Gamma(X, K_X + \sigma_i)$ corresponding to $H_1(X, \mathbb{Z}) = \mathbb{Z}_2^4$. The 14 divisors consist of two $\langle a \rangle$ orbits, $\langle a \rangle t_2$ and $\langle a \rangle t_3$. The square of each such section gives rise to a section of $V = \Gamma(X, 2K_X)$. From (2), the vector space $\Gamma(X, 2K_X)^{\langle b \rangle} \cong \Gamma(X/\langle b \rangle, 2K_{X/\langle b \rangle})$ has a basis given explicitly from (2) by $v_0 := u_0$, $v_1 := u_1 + u_2 + u_3$, $v_2 := u_4 + u_5 + u_6$, $v_3 := u_7 + u_8 + u_9$.

We already know that $t_1^2 = u_0$ from [BK], which also follows from the command `sDomain in Magma.` t suffices for us to show that $\cap_{j=1}^3 Z_{t_i} = \emptyset$, for which we would give two arguments.

The first proof is to use reduction at a finite field F_p, where p is chosen to be 23 for convenience, and utilizing comparsion theorem of Grothendieck as given in SGA, XII 7, [G]. By checking the three 7×7 minors of Jacobians of the defining functions of X given in [BK] using Magma, we verify that X is smooth for $p = 23$ and hence has good reduction at 23, which we denote by X^p. As $\pi_1(X)$ is residually finite, we can identify the topological fundamental group with its etale fundamental group. The first homology group $H_1(X, \mathbb{Z})$, as $\pi_1(X)$ modulo its commutator, is identified with abelianization of the maximal quotient of the etale fundamental group by a prime relatively prime to p. The same principle holds for the resolution of $X/\langle b \rangle$ at its three singular points and hence for $X/\langle b \rangle$. As $p \neq 2$ and $H_1(X, \mathbb{Z})^{\langle b \rangle} = \mathbb{Z}_2^2$, we conclude that $H_1(X^p, \mathbb{Z})^{\langle b \rangle} = \mathbb{Z}_2^2$. In particular, a non-trivial 2-torsion line bundle on M gives rise to a non-trivial 2-torsion line bundle on M_p. This implies that $t_i, i = 1, 2, 3$ would give 3 different image $t_{p,i}$ on reduction modulo p. Recall that sections of $H^0(X, 2K_X)^{\langle b \rangle}$ are spanned by square of sections of $H^0(X, K_X + \epsilon)^{\langle b \rangle}$ for some 2-torsion line bundle ϵ. From earlier discussion, $H^0(X^p, 2K_{X^p})^{\langle b \rangle}$ has dimension 4 and $t_i^2, i = 1, 2, 3$ are linear combinations of $v_j, j = 1, \ldots, 4$. It follows that $t_{p,i}^2$ has to be a linear combination of v_j and is reducible or non-reduced modulo p. We apply `IsDomain` in Magma to each section $\sum_{i=0}^3 c_i v_i$ for $c_i \in \{0, \ldots, 22\}$ and find that there are exactly three quadruples c_i for which the sections are reducible or non-reduced, given by $v_0, v_0 + 14v_1$ and $v_0 + 22v_1 + 11v_2 + 19v_3$. Since we know already that there are three such sections coming from $t_{p,i}^2, i = 1, 2, 3$, these have to be $t_{p,i}^2$. We check that $\cap_{i=1}^3 Z_{t_{p,i}^2} = \emptyset$ on X_p from the command `HilbertPolynomial` in Magma, which gives value 0. This implies that $\cap_{j=1}^3 Z_{t_i^2} = \emptyset$ on X, which leads to $\cap_{j=1}^3 Z_{t_i} = \emptyset$ on X and hence $\cap_{j=1}^3 Z_{s_i} = \emptyset$ on M.

The second proof is more explicit. Recall that X is a Shimura variety and is defined over a number field $\mathbb{Q}(\sqrt{-7})$. Recall that $t_1^2 = u_0$. Since t_2, t_3 cannot be deformed as curves from the fact that $h^0(X, K_X + \tau_i) = 1$ for all 2-torsion τ_i, we know that they are rigid and can be defined over $\overline{\mathbb{Q}}$. Let $j = 2, 3$. Since the ring of integers $\mathcal{O}_{\mathbb{Q}(\sqrt{-7})}$ has a basis given by $1, \eta = \frac{1}{2}(1 + \sqrt{-7})$, we try $t_j^2 = v_0 + \sum_{j=2}^3 (\alpha_j + \beta_j \eta) v_j$ for some $\alpha_j, \beta_j \in \mathbb{Q}$. By considering reduction modulo $p = 11, 23, 29$ and using `IsDomain` as in the first method, we conclude that a candidate for t_2 is $\widehat{t}_2$ with $\widehat{t}_2^2 = v_0 + \eta v_1$, corresponding to $v_0 + 14v_1$ for $p = 23$. Using `IsDomain` command over $\mathbb{Q}(\sqrt{-7})$, we conclude that $\widehat{t_2}$ is either non-reduced or reducible and hence has to be square of some section of a bundle numerically equivalent to K_X from proof of Lemma 2 of [Y]. Hence $t_2 = \widehat{t_2}$ up to a scaling constant. Similar procedure leads to $t_3^2 = v_0 + (-6+2\eta)v_1 + (8-8\eta)v_2 - 4v_3$ up to a scalar. Since the image of t_3 in X^p in reduction modulo 23 is $t_{p,3}$ studied earlier, and $t_{p,3}$ does not have non-trivial intersection with the intersection of $Z_{t_1^2} \cap Z_{t_2^2}$ modulo $p = 23$ from command `HilbertPolynomial` in Magma, we conclude that $\cap_{j=1}^3 Z_{t_i^2} = \emptyset$. Alternatively, we show that $Z_{t_1^2} \cap Z_{t_2^2}$ actually occurs only at explicit points given by the three fixed points of $\mathbb{Z}_3$ on X. Using `HilbertPolynomial` , one shows that Z_{v_3} does not intersect $Z_{t_1} \cap Z_{t_2} = Z_{v_0} \cap Z_{v_1}$ but Z_{v_2} does. Hence if $t_3^2 = \sum_{i=0}^3 c_i v_i$ and $\cap_{i=1}^3 Z_{t_i} \neq \emptyset$, the only possibility is that $c_3 = 0$ by evaluating t_3^2 at $\cap_{j=0}^2 Z_{v_j}$. This contradicts the earlier fact that $t_{p,3}^2 = v_0 + 22v_1 + 11v_2 + 19v_3$, corresponding to $c_3 = 19 \pmod{23}$.

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