ON THE CANONICAL MAPPING OF A TOWER OF LOCALLY HERMITIAN SYMMETRIC SPACES OF FINITE VOLUME

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ABSTRACT. *Let $\{M_i\}$ be a tower of coverings of non-compact locally Hermitian symmetric spaces of finite volume and $\overline{M}_i$ be a smooth compactification of M_i. We show in this paper that there exists $i_o > 0$ such L^2 canonical sections on M_i gives an embedding of M_i when $i \geqslant i_o$, and that $2K_{\overline{M}_i}$ is very ample for those compactifications with boundary divisors being union of smooth complex tori in simple normal crossing. The proof also verifies the convergence of normalized dimension of the space of holomorphic k forms to its von-Neumann dimension on the universal covering, as well as stability of Bergman kernel in taking the limit on the level of the tower.*

§1. Introduction

1.1 Let $\widetilde{M} = G/K$ be a Hermitian symmetric space of non-compact type, where G is a semi-simple Lie group and K is a maximal compact of G. Let Γ be a lattice. Then $M = \Gamma\backslash G/K$ has finite volume with respect to the Bergman metric on $\widetilde{M}$. M is a projective algebraic variety if Γ is cocompact, and a quasi-projective manifold if Γ is non-cocompact. M is smooth if we assume that Γ is neat. In this article, by a finite locally Hermitian symmetric space, we always mean $\Gamma\backslash\widetilde{M}$ with Γ cofinite and neat. It is known that given any locally Hermitian symmetric space of finite covolume, there is a finite covering corresponding to a subgroup Γ' of finite index of the associated lattice Γ, such that Γ' is neat and the corresponding locally Hermitian symmetric space is finite in our sense.

There are two main goals in this article. The first is to install a method to study stability properties of Bergman kernel and L^2-embedding properties on a tower of non-compact Kähler manifolds, implemented in this case to a tower of finite locally Hermitian symmetric space. The second is to introduce a method to study effective Kodaira embedding type statements for appropriate compactification $\overline{M}$ of a non-compact Kähler manifold M by relating geometry of $\overline{M}$ to L^2 embedding properties of M, implemented to appropriate finite Hermitian symmetric spaces.

We recall some notations. By a tower of coverings $\{M_i\}$ of M, we mean a sequence of finite coverings $M_{i+1} \to M_i$ with $M_1 = M$, such that $\pi_1(M_{i+1}) < \pi_1(M_i)$ is a normal subgroup of $\pi_1(M_1)$ with finite index and $\cap_{i=1}^{\infty}\pi_1(M_i) = \{1\}$. For a locally symmetric space $\Gamma\backslash G/K$, a tower of coverings corresponds to a sequence of nested normal subgroups $\{\Gamma_i\}$ of the lattice Γ. Our main interest is to investigate whether interesting geometric properties of M_i, such as birational properties, can be reflected from the universal covering $\widetilde{M}$ of M_i.

Key words: very ampleness, non-compact locally Hermitian symmetric space, compactificaitions, tower of coverings

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The questions are interesting from the view of both geometry and automorphic forms, since the pull-back of pluricanonical sections to $\widetilde{M}$ gives rise to Γ sections of canonical sections on $\widetilde{M}$ and can be interpreted as automorphic forms of certain weights.

We say that a Hermitian holomorphic line bundle (L, h) on a non-compact manifold complex M equipped with a Kähler metric g is L^2 very ample if its L^2-sections with respect to the Hermitian metric h and the volume form induced by g give an embedding of M. Similarly, given any $N > 0$, we say that L is L^2 *N-jet generating* if the L^2-sections of L with respect to the Hermitian metric h and the volume form induced by g generates N-th jet at every point on M.

In the case that L is the canonical line bundle K_M, we equip the line bundle with the metric induced by the Bergman metric. The first result of the paper is as follows.

Theorem 1. *Let M be a finite locally Hermitian symmetric manifold of non-compact type as explained. Let $\{M_i\}$ be a tower of coverings of M. Then there exists $i_o > 0$ such that K_{M_i} is L^2 very ample with respect to the Bergman metric on M_i for $i \geqslant i_o$. Furthermore, given any $N > 0$. There exists i_N such that K_{M_i} is L^2 N-jet generating if $i \geqslant i_N$.*

The main difficulty with the finite case comparing to the compact case is that the injectivity radius of the canonical metric at a point approaches 0 as the point approaches to infinity, no matter how high a level of covering is taken. We overcome the difficulty by utilizing the fact that the manifolds involved can be compactified. For this we need to devise a method to handle the dificulty that in terms of the local coordinates chart centered at a point at a compactifying divisor on a smooth compactification of M, the canonical metric blows up as one approaches the compactifying divisor.

1.2 Theorem 1 in turn reflects geometric properties of its appropriate smooth compactification $\overline{M}$ of M. A natural question is about the birational properties of some compactification of finite Hermitian symmetric spaces. Earlier results in this direction of relating properties of M to $\overline{M}$ include [Mu], [T], and more recently [WY1], [WY3] and references therein. Here is our contribution in this direction.

Theorem 2. *Let M be a finite locally Hermitian symmetric manifold. Let $\{M_i\}$ be a tower of coverings of M. Denote by $\overline{M}_i$ a smooth compactification of M_i. Assume that compactifying divisor $D_i = \overline{M}_i - M_i$ is a divisors in simple normal crossing with each of the irreducible component being a smooth Abelian variety. Then there exists $i_o > 0$ such that $2K_{\overline{M}_i}$ is very ample on $\overline{M}_i$ for $i \geqslant i_o$.*

We remark that the conditions are satisfied by all smooth complex ball quotients after passing to some appropriate finite unramified coverings, cf. [Mo], [AMTR].

To put the result in perspective, note that from the work of Mumford [M] and Tai [T], there exists $i_o > 0$ such that $\overline{M}_i$ is of general type for $i \geqslant i_o$. In the special case that M a complex ball quotient,for $n \geqslant 3$, it is proved in [BT] that $K_{\overline{M}_1}$ is ample. From [N1], [WY2], it is also known that $\overline{M}_i$ is Kobayashi hyperbolic for $i \geqslant i_o$. Theorem 2 gives very ampleness of $2K_{\overline{M}_i}$ for i sufficiently large. Note also that without going to a finite unramified covering, it may not even be true that $K_{\overline{M}}$ is ample in the case of $n = 2$. Examples include those of Hirzebruch in [Hi]. The results in this paper can also be considered as a type of effective Kodaira Embedding Theorem in terms of unramified coverings instead of power of canonical line bundle as given in Fujita conjecture, cf. [F], [AS], [He].

The proof of Theorem 2 relies on Theorem 1 together with a careful study of the geometry near the compactifying divisor, which is an abelian variety. It is at this place that we need to have extra leeway given by $2K_{\overline{M}_i}$ to work with instead of $K_{\overline{M}_i}$.

We have the following immediate corollary to Theorem 2, which for complex ball quotients follow from [BT].

Corollary 1. *Let M be a finite locally Hermitian symmetric spaces in the setting of Theorem 2 above. There exists $i_o > 0$ such that $K_{\overline{M}_i}$ is very ample if $i > i_o$.*

1.3 Along the way of proving Theorem 1, we have also proved stability of the Bergman kernel, namely the convergence of the normalized dimension of the space of L^2-holomorphic forms on M_i to its von-Neumann dimension on the universal covering $\widetilde{M}$, again in the cofinite setting. There are a lot of research results known for cocompact lattices, more recent ones include [LZ], [W], [YY], [YuZ] and references therein. The results for non-compact ones are much more limited. In such situation, the corresponding results for cusps forms, namely forms vanishing at the infinity of M, have been proved by Savin in [S]. Together with the techniques developed in [Y2, Y5], a stability result in the sense of pointwise convergence of the k-th Bergman kernel on M_i to the corresponding one on $\widetilde{M}$ is also obtained.

Theorem 3. *Let M be a (non-compact) locally Hermitian symmetric manifold of finite volume. Let $\{M_i\}$ be a tower of coverings of M.*
(a). For $k = 0, \dots, n$,

$$\frac{h^k_{(2)}(M_i, K_{M_i})}{[\Gamma, \Gamma_i]} \to h^k_{v,(2)}(\widetilde{M}, K^2_{\widetilde{M}})$$

as $i \to \infty$.
(b). Let x_o be a point in a fundamental domain Σ of $M_1 = M$ on $\widetilde{M}$. Then

$$B^0_{(2)}(M_i, K_{M_i})(x) \to B^0_{(2)}(\widetilde{M}, K_{\widetilde{M}})(x)$$

uniformly in a neighborhood U of x_o as $i \to \infty$. Furthermore, given any differential operator $D_l = \frac{\partial^l}{\partial z_{i_1} \cdots \partial z_{i_l}}$ of degree l on Σ,

$$D_l B^0_{(2)}(M_i, K_{M_i})(x) \to D_l B^0_{(2)}(\widetilde{M}, K_{\widetilde{M}})(x)$$

uniformly in a neighborhood U of x_o as $i \to \infty$.

1.4 Here we describe the structure of the paper. Some preliminary discussions about tools needed are explained in §2. The main new technical input of this article is given in the proof of Proposition 1 in §3, which allows us to control the geometry near infinity. In §4, we combines the ingredient in §3 with results in [Y2, Y5] to give a proof of Theorem 3 about stability of the Bergman kernels. In §5, we give a proof of Theorem 1, following a new formulation which improves and simplifies the formulations taken in [Y2, Y5]. In §6, we analyze the geometry around a compactifying divisor and use L^2-estimates to construct enough sections to give a proof of Theorem 2.

§2. Preliminaries and formulation

2.1 Let us recall some standard terminologies involving L^2-cohomology. Let M be a complete Kähler manifold. Let (L,h) be a Hermitian line bundle on M. Denote by $H^i_{(2)}(M,L)$ the space of L^2 $\square_{\overline{\partial}}$-harmonic L-valued $(0,i)$-forms on M with respect to the Hermitian metric h of the line bundle L and the volume form on M. This corresponds to the reduced L^2 cohomology on $\widetilde{M}$. The L^2-norm of φ is defined by

$$\begin{aligned}\|\varphi\|^2 &= \int_M \varphi\wedge *\varphi. \\ &= \int_M |\varphi|_h^2\omega^n,\end{aligned}$$

where ω is the Kähler form on M and $*\varphi$ is the Hodge dual of φ with respect to h and ω. In the setting of Hermitian symmetric space of non-compact type M and L being a multiple of K_M, the metric h is induced from the Bergman metric, and ω^n is the volume form of the Bergman metric with Kähler form ω on M. We would also omit h in the subscript when there is no danger of confusion.

Let $\{f_k\}$ be an orthonormal basis of $H^i_{(2)}(M,L)$. The Bergman kernel is defined to be

$$B^{0,i}_{M,L}(x,y) := \sum_k f_k(x)\wedge *f_k(y).$$

As such we are regarding $B^{p,0}_M$ as a section of $p_1^*(\Omega^{0,i}_M\otimes L)\otimes p_2^*(\Omega^{n,n-i}_M\otimes L^*)$, where p_a is the projection of $M\times M$ into the a-th factor, $a=1,2$.

We are mainly interested in the trace of the kernel, $B^{0,i}_{M,L}(x,x)$. We define the von-Neumann dimension of L-valued i-form to be

$$h^{0,i}_{v,(2)} = \int_\Sigma B^{0,i}_{M,L}(x,x),$$

where Σ is a fundamental domain of M.

As the Bergman kernel is independent of the choice of a basis, for each fixed point $x\in M$, the trace of the Bergman kernel

$$B^{0,i}_{M,L}(x,x) = \sup_{f\in H^{0,i}_{(2)}(M,L),\|f\|=1} |f(x)|_h^2 = \sup_{f\in H^{0,i}_{(2)}(M,L),\|f\|=1} |f_U(x)|^2 h\omega^n,$$

where $\|\cdot\|$ stands for the L^2-norm, and we have written $f = f_U(dz_{j_1}\wedge\cdots\wedge dz_{j_i})\otimes e$ in terms of local coordinates $(z_1,\ldots,z_n)$ and local basis e of L.

Let Σ_i a the fundamental domain of M_i in $\widetilde{M}$. We may assume that $\Sigma_i\subset\Sigma_{i+1}$. $\Sigma=\Sigma_1$ is a fundamental domain of M on $\widetilde{M}$. We fix such a Σ in the following discussions.

The von-Neumann arithmetic genus on the universal covering $\widetilde{M}$ of M is defined by

$$\chi_{v,(2)}(\widetilde{M}) = \sum_{i=0}^n (-1)^i h^{0,i}_{v,(2)}(\widetilde{M}) = \sum_{i=0}^n (-1)^i h^{i,0}_{v,(2)}(\widetilde{M}),$$

from the Hodge identities on a complete Kähler manifold, where $h^{0,i}_{v,(2)}(\widetilde{M}) = \dim H^{0,i}_{v,(2)}(\widetilde{M})$. In general, for L a holomorphic line bundle on $\widetilde{M}$ invariant under deck-transformation, define

$$\chi_{v,(2)}(\widetilde{M},L) \;=\; \sum_{i=0}^{n}(-1)^i h^{0,i}_{v,(2)}(\widetilde{M},L).$$

2.2 In this paper we are mainly interested in holomorphic forms. These are also the space of harmonic forms of type $(i,0)$ on a Kähler manifold. Let L be a Hermitian holomorphic line bundle. We will use $\mathcal{H}^{i,0}_{(2)}(M,L)$ to denote the space of holomorphic $(i,0)$ L-valued forms which are L^2 with respect to the Bergman metric on M and the Hermitian metric on L. The corresponding Bergman kernel is

$$B^{i,0}_M(x,y) := \sum_k f_k(x)\wedge\overline{f_k}(y).$$

The space of all such L^2 holomorphic sections of the canonical line bundle is also denoted by $\Gamma_{(2)}(M,K_M)\cong\mathcal{H}^{n,0}(M,\mathcal{O})$.

From Hodge theory, these spaces reflect the reduced L^2-cohomology on a complete non-compact manifold, cf. [Da].

2.3 We recall some standard facts about some compactification of M, which is cofinite in our sense.

Lemma 1. *([BB], [AMRT], [SY], [MZ], [Mo]) (a). M admits a smooth compactification $\overline{M}$, so that $D=\overline{M}-M$ is a normal crossing divisor.*
(b). D can be chosen to be a union of algebraic tori.
(c). In the case that M is a complex ball quotient, $\overline{M}$ can be found such that D is a disjoint union of Abelian varieties.

For arithmetic lattice Γ, this follows from Bailey-Borel [BB] compactification and resolution of singularity of Hironka. A smooth toroidal compactification is given by Ash-Mumford-Rapoport-Tai in [AMRT] for arithmetic Γ. For such compactification, D is a union of algebraic tori in the study of toric varieties.

In the case that Γ is non-arithmetic, this only happens if $\widetilde{M}\cong B^n_{\mathbb{C}}$. In such case, there is the work of Siu-Yau [SY] on the compactification of M by a finite number of cusps, and Mok in [Mo] showed that a resolution in terms of toroidal compactification still works in this case. In particular, for $\widetilde{M}=B^n_{\mathbb{C}}$, it is known that D is a disjoint union of tori, cf. [AMRT], [Mo].

In general, compactification satisfying (a) follows from the work of [MZ], [Y1] once we are given a non-compact Kähler manifold equipped with a Kähler metric of bounded sectional curvature, Ricci curvature bounded from above by a negative constant, and finite volume.

2.4 For a general Hermitian symmetric space $\widetilde{M}$ and a finite $M=\widetilde{M}/\Gamma$, the compactifying divisor D has irreducible components which may have non-trivial intersections among them. In such case, for any $x\in D\subset\overline{X}$, there is a neighborhood $\overline{U}\subset\overline{X}$ of x in $\overline{X}$ with local coordinates $(z_1,\dots,z_n)$ such that $\overline{U}\cap D=\{z_1\cdots z_k=0\}$ $(1\le k\le n)$ and the complement $U:=\overline{U}-\overline{U}\cap D\cong(\Delta^*_{1/2})^k\times\Delta^{n-k}_{1/2}\subset(\Delta^*)^k\times\Delta^{n-k}$, where the Δ_r,Δ^*_r refer to disk or puncture disks of radius r in $\mathbb{C}$. By a Poincaré metric on U, by mean the restriction of the product of the Poincaré metrics on the disks and puncture disks Δ and Δ^*. Let us first give some estimates on the Bergman metric near the boundary divisor.

Lemma 2. *The Bergman metric near a point D in a toroidal compactification $\overline{M}$ of M is quasi-isometric to a Poincaré metric with Kähler form given by*

$$\omega_P \;:=\; \frac{\sqrt{-1}dz_1\wedge d\overline{z_1}}{|z_1|^2(\log|z_1|^2)^2}+\cdots+\frac{\sqrt{-1}dz_k\wedge d\overline{z_k}}{|z_k|^2(\log|z_k|^2)^2}+\sum_{i=k+1}\sqrt{-1}g_{i\bar{i}}dz_i\wedge d\overline{z}_i \qquad (1)$$

in some local coordinates near a point on D, where

$$\frac{c_1}{(\log|z|)^\alpha}\leqslant g_{i\bar{i}}\leqslant c_2 \qquad (2)$$

with constants $a, c_1, c_2 > 0$.

Proof Let $\omega_{U,P}$ be the Poincaré metric on U. Since the Bergman metric on M has holomorphic sectional curvature bounded from below by a negative constant and $\omega_{U,P}$ has holomorphic sectional curvature bounded from above by a negative constant, the usual Schwarz Lemma of Ahlfors, cf. [Ro], implies that there is a constant $c_3 > 0$ such that

$$g_B \leqslant c_3 g_{U,P}.$$

On the other hand, the normal component of g_B is well-known to be of the form of the standard Poincaré metric on the punctured disk. For the component of the metric parallel to the direction of the divisor, from the fact that the metric is good in the sense of Mumford in [Mu], we know that there is a lower bound of $\frac{c_1}{(\log|z|)^\alpha}$ for some positive constant $\alpha > 0$. □

§3. Convergence in asymptotic dimension

3.1 Recall that we denote by $h^i_{v,(2)}(\widetilde{M}, K_{\widetilde{M}})$ the von Neumann dimension of the space of L^2-holomorphic j forms on $\widetilde{M}$ with respect to M. We need the following result in later sections.

Proposition 1. *For $k = 0, \ldots, n$,*

$$\frac{h^k_{(2)}(M_i, K_{M_i})}{[\Gamma,\Gamma_i]} \to h^k_{v,(2)}(\widetilde{M}, K^2_{\widetilde{M}})$$

as $i \to \infty$.

For the space of cusp forms, which correspond to L^2-holomorphic forms vanishing at the cusps, a similar result was obtained Savin in [S]. In our situation, there may be L^2 holomorphic forms which do not vanish at the infinity. The proof here is also very different. The more geometric argument is applicable to general non-compact Kähler manifolds with appropriate conditions imposed as well.

As mentioned earlier, this type of statement was classical for compact M_i and follows for example from [K] and [Y2]. Let us explain the non-compact case here.

We know from standard facts that $h^i_{(2)}(\widetilde{M}, K^2_{\widetilde{M}}) = 0$ for $0 \leqslant i < n$ and > 0 for $i = n$. We try to relate the geometric properties of M_i to M. Though the basic strategy is the same as for the compact analogues used in [Y2] and [Y4], there is the complication given by non-compactness of M_i, which renders Kazhdan's approach in [K] not applicable without preventing escape of some sort of mass to the infinity. Eventually we show that such a loss of mass to infinity cannot happen and prove that $\frac{h^k_{(2)}(M_i,K_{M_i})}{[\Gamma,\Gamma_i]} \leqslant h^k_{(2)}(\widetilde{M},K^2_{\widetilde{M}})$ for each k, from which index theorem type of argument is applied to prove that equality of the two quantities for $k=n$.

3.2

Lemma 3. *Let $f \in \Gamma_{(2)}(M,K_M)$ with $\|f\|=1$. There exists an open neighborhood U of D in $\overline{M}$ and such that*

$$\sup_{x\in U}|f|^2_o(x) \leqslant \sup_{y\in\partial U}|f|^2_o(y) \quad \text{and} \quad \sup_{x\in U}|f|^2_g(x) \leqslant \sup_{y\in\partial U}|f|^2_g(y)$$

where the first one is measured with respect to the Euclidean metric g_o on U, and the second with respect to the Kähler metric $g=g_B$, for U sufficiently small .

Proof Let us for simplicity assume first the case that $\widetilde{M}=B^n_{\mathbb{C}}$. In such case, as mentioned in **2.2**. there exists $\overline{M}=M\cup D$ and $D=\cup_{i=1}^n T_i$ is a disjoint union of a finite number of tori of complex dimension $n-1$. The canonical line bundle $K_{\overline{M}}$ restricts to K_M on M. A holomorphic section $f\in\Gamma_{(2)}(M,K_M)$ extends as a meromorphic section of $K_{\overline{M}}$. Since L^2 norm of K_M is conformal invariant, we know that f is L^2 respect to a smooth Hermitian metric on $\overline{M}$ as well. One consider a trivialization of $K_{\overline{M}}$ on a neighborhood V of a point $z\in D$. From Fubini Theorem, we may assume that the restriction of f to s generic disk Δ transversal to D is actually L^2 on D. Consider the Laurent expansion of f on D. It follows that from L^2 finiteness that $f|_D$ cannot have any pole along D. This also follows from Riemann Extension Theorem. Hence actually f extends to a holomorphic section of $K_{\overline{M}}$.

The first estimates in the lemma now follows from maximal principle.

For the second estimates, let us first consider the case of complex ball quotients. In this case, the volume form of g_B is of form

$$\frac{|dz^1\wedge\cdots\wedge d\overline{z}^n|^2}{|z_1|^2|\log|z_1||^{n+1}}$$

in suitable local coordinates around D, with D given locally by $z_1=0$, cf. [Mo]. Hence the induced metric on the canonical line bundle is $|z_1|^2|\log|z_1||^{n+1}$ and is decreasing in z_1.

For a general Hermitian symmetric space $\widetilde{M}$ and a finite $M=\widetilde{M}/\Gamma$, the only difference is that the compactifying divisor D may have irreducible components which have non-trivial intersections among them. In such case, for any $x\in D\subset\overline{X}$, there is a neighborhood $\overline{U}\subset\overline{X}$ of x in $\overline{X}$ with local coordinates $(z_1,\ldots,z_n)$ as given in **2.4**. The Bergman metric has estimates given by (1) and (2). In particular, the volume form is of form

$$\frac{|dz^1\wedge\cdots\wedge d\overline{z}^n|^2}{\prod_{j=1}^k(|z_j|^2\cdot|\log|z_j||^{\gamma_j})}$$

on U of form $\Delta^*_{\frac{1}{2}}\times\Delta_{\frac{1}{2}}$ in local coordinates for some fixed numbers γ_j, as explained in §2. Again the metric on the canonical line bundle is $\prod_{j=1}^k\left(|z_j|^2\cdot(\log|z_j|^2)^{\gamma_j}\right)$ and is decreasing as z gets closer to the compactifying divisor. The earlier argument applies again.

□

3.3

Lemma 4. *Let $1 \leqslant l \leqslant n$. Let $f \in \Gamma_{(2)}(M, \Omega^l_{M_i})$. There exists an open neighborhood W of D in $\overline{M}$ and a constant $C > 0$ independent of f and x such that*

$$\|f\|_{M_i - \pi_i^{-1}(W)} := \left(\int_{M_i - \pi_i^{-1}(W)} |f|^2 \right)^{1/2} \geqslant C \|f\|_{M_i}.$$

Proof Consider first the case of $l = n$. Let us illustrate the proof for a standard neighborhood of a point on D as discussed in **2.4** to be of form $U \cong (\Delta^*_{1/2})^k \times \Delta^{n-k}_{1/2}$ with $k = 1$. The other cases are exactly the same except that the notation is more complicated. Again, as explained in **2.3**, up to quasi-isometry we may assume that

$$\omega_B \sim \frac{\sqrt{-1} dz_1 \wedge d\overline{z_1}}{|z_1|^2 (\log |z_1|^2)^2} + \sum_{i=2}^{n} \frac{\sqrt{-1}}{(\log |z_1|)^{\alpha_i}} dz_i \wedge d\overline{z}_i \tag{3}$$

for some $\alpha_i > 0$. Here we say that two Kähler forms ω_1 and ω_2 satisfies $\omega_1 \sim \omega_2$ if there exists a constant $c > 0$ such that $\frac{1}{c}\omega_1 \leqslant \omega_2 \leqslant c\omega_1$.

Note that $\partial U \cap \Sigma$ is a relatively compact set on Σ, the fundamental domain of M in $\widetilde{M}$. Hence on identifying Σ with M, the injectivity radius of a point on ∂U in M is bounded from below by a positive constant. Hence we may assume that there exists $r_1 > 0$ so that the Euclidean ball $B_{r_1}(x)$ is embedded in U for each point $x \in \partial U$. Replacing r_1 by $\min(\frac{1}{4}, r_1)$ if necessary, we may assume that $r_1 < \frac{1}{4}$. It follows from Cauchy estimates that there exists a constant $c > 0$ such that

$$\int_{B_{r_1}(y)} |f|^2 \geqslant c |f(y)| \tag{4}$$

for each $y \in \partial U$. Let $V = \cup_{x \in \partial U} B_{r_1}(x)$. Let $W = U - V$. It follows that for each $x \in U_1$,

$$|f(x)|^2 \leqslant \frac{1}{c} \int_{B_{r_1}(y)} |f|^2 \leqslant \frac{1}{c} \int_V |f|^2. \tag{5}$$

Integrating over $x \in U_1$, we get

$$\int_W |f(x)|^2 \leqslant c_1 \int_V |f|^2 \tag{6}$$

for $c_1 = \frac{\mathrm{Vol}(U_{\frac{1}{2} - r_1}(0), g_o)}{c}$. Since $\pi^{-1}(U)$ can be considered as a disjoint union $\cup_j \gamma_j U$ for γ_i in the deck transformation group of π_i and γ_j acts by isomorphism, we conclude that the same estimates holds for each $\gamma_j U$ and hence

$$\|f\|^2_{\pi_i^{-1}(W)} \leqslant c_1 \|f\|^2_{\pi_i^{-1}(V)} \leqslant c_1 \|f\|^2_{M_i - \pi_i^{-1}(W)}. \tag{7}$$

Since the left hand side of the expression above is just $\|f\|^2_{M_i} - \|f\|^2_{M_i - \pi_i^{-1}(W)}$, we conclude that

$$\|f\|^2_{M_i - \pi_i^{-1}(W)} \geqslant \frac{c_1}{1 + c_1} \|f\|^2_{M_i}, \tag{8}$$

which concludes the proof for $k = n$.

Consider now $l < n$. In such case, the L^2-norm is not conformal and depends on the metric and we need to pay more attention. Again let us illustrate the proof for $U \cong (\Delta^*_{1/2})^k \times \Delta^{n-k}_{1/2}$ with $k = 1$, since the other cases are exactly the same except that the notation is more complicated. .

An holomorphic k-form on M is of form

$$f = \sum_J f_{j_1,\dots,j_l} dz^{j_1} \wedge \cdots \wedge dz^{j_l} = \sum_J f_J dz^J,$$

where J denotes the tuple $(j_1, \dots, j_l)$ and each f_J is a holomorphic function on U.

I

Consider a fixed subscript I. In such case, up to quasi-isometry, we may assume that the metric takes the form (3) and

$$\int_U |f_I|^2_g dV_g = \int_U |f_I|^2_g \det g dV_{g_o} = \int_U |f_I|^2_{g_o} h dV_{g_o}, \tag{9}$$

$$h = \begin{cases} \frac{1}{|z_1|^2(\log|z_1|)^2} \prod_{s\notin I} \frac{1}{(\log|z_1|)^{\alpha_{j_s}}}, & \text{if } 1 \notin I, \\ \prod_{s\notin I} \frac{1}{(\log|z_1|)^{\alpha_{j_s}}}, & \text{if } 1 \in I. \end{cases} \tag{10}$$

in our local coordinates. The estimates in (4), (5) and (6) are replaced by

$$\int_{B_{r_1}(y)} |f|^2 h \geqslant c^I |f(y)| \tag{11}$$

$$|f(x)|^2 \leqslant \frac{1}{c^I} \int_V |f|^2 h \tag{12}$$

$$\int_W |f(x)|^2 h \leqslant c_1^I \int_V |f|^2 h, \tag{13}$$

$$\tag{14}$$

where $c^I > 0$ is a constant and $c_1^I = \frac{\mathrm{Vol}(U_{\frac{1}{2}-r_1}(0),h)}{c^I}$. Note that the reason that the argument works is that $\mathrm{Vol}(U_{\frac{1}{2}-r_1}(0), h)$ is finite with respect to our h defined in (10).

Hence as in (8), we get

$$\|f_I\|^2_{M_i - \pi_i^{-1}(W)} \geqslant c_2^I \|f_I\|^2_{M_i} \tag{15}$$

for $c_2^I = \frac{c_1}{1+c_1}$.

Summing over all I, we conclude that there exists a constant $c > 0$, taking to be the minimum of all the c_2^I, such that

$$\|f\|_{M_i - \pi_i^{-1}(W)} \geqslant c\|f\|_{M_i}. \tag{16}$$

□

3.4

Lemma 5. *In terms of the earlier notation with $0 \leqslant k \leqslant n$, we have*

$$\int_{M_i - \pi_i^{-1}(W)} B^{k,0}_{M_i} \geqslant c \int_{M_i} B^{k,0}_{M_i}.$$

Proof This follows from the definition of Bergman kernel as mentioned in **2.1**. □

3.5 Proof of Proposition 1 Consider first $0 \leqslant k < n$. It is a classical fact that there is no L^2-holomorphic k-forms on the bounded symmetric domain $\widetilde{M}$ for $k < n$ with respect to the Bergman metric, cf. [Y6]. Hence it follows that $h^k_{v,(2)}(\widetilde{M}) = 0$.

We claim that $\lim_{k\to\infty} \frac{h^k_{(2)}(M_i, K_{M_i})}{[\Gamma,\Gamma_i]} = 0$ for $0 \leqslant k < n$. The case of $k = 0$ is trivial. Assume now that $1 \leqslant k \leqslant n-1$. Let $\epsilon > 0$. Suppose that $\frac{h^k_{(2)}(M_i, \mathcal{O}_{M_i})}{[\Gamma,\Gamma_i]} \geqslant \epsilon$ for a (sub)sequence of i. From Kähler identity, $h^k_{(2)}(M_i, \mathcal{O}_{M_i}) = h^{k,0}(M)$, which implies $\frac{1}{[\Gamma,\Gamma_i]}\int_{M_i} B^{k,0}_{M_i} \geqslant \epsilon$ Hence we conclude that $\frac{1}{[\Gamma,\Gamma_i]}\int_{M_i - \pi^{-1}W} B^{k,0}_{M_i} \geqslant c\epsilon$ Since the Bergman kernel is invariant under automorphism, this implies that

$$\int_{\Sigma - p^{-1}(U)} B^{k,0}_{M_i} \geqslant c\epsilon,$$

where $p : \widetilde{M} \to M$ is the universal covering map. Hence in terms of extremal sections as described in **2.1**, there exists a holomorphic k-form f_i on M_i with $\|f_i\|_{M_i} = 1$ and $|f_i(x_o)|_g \geqslant \epsilon$ at some point $x_o \in \Sigma - p^{-1}(U)$. Since $\Sigma - p^{-1}(U)$ is relatively compact, a normal family argument as given by Kazhdan [K], see also [Y2], leads to a section $f \in \mathcal{H}^{k,0}_{(2)}(\widetilde{N})$ with f non-trivial at a point $x_o \in \Sigma - p^{-1}(U)$. This contradicts the earlier statement that $\mathcal{H}^{k,0}_{(2)}(\widetilde{N}) = 0$ for $0 \leqslant k < n$ and $\widetilde{M}$ a bounded symmetric domain. Hence the claim is proved. Hence the Proposition is proved for $k < n$.

Now consider $k = N$. The conclusion of Lemma 3 in fact implies that the supremum of an L^2-section is bounded away from D. Hence same argument as above implies that

$$\frac{h^k_{(2)}(M_i, K_{M_i})}{[\Gamma, \Gamma_i]} \leqslant h^k_{v,(2)}(\widetilde{M}, K^2_{\widetilde{M}}). \tag{17}$$

The equality will follows from covering index type of statement. For complex rank one case this also follows from the result of Barbarsh-Moscovici [BM]. In all the other cases, the equality follows from a result of Rohlfs-Speh [RS] as used in §5 of [S].

□

§4. Pointwise convergence and stability

4.1 We begin with an observation. One problem that we need to deal with for a cofinite locally Hermitian symmetric space M is that the injectivity radius of M is 0, since the injectivity radius of a point on M tends to 0 as the point approaches the infinity divisor. Nevertheless, we have the following if a point is fixed on the universal covering. Recall a fundamental domain of M_i is given by D_i and $D_i \subset D_{i+1}$ with $D = D_1$.

Lemma 6. *Let $x \in D \subset \widetilde{M}$ be a fixed point. Let $\tau_i(x)$ be the injectivity radius of $\pi_i(x) \subset M_i$. Then $\lim_{i\to\infty} \tau_i(x) = \infty$.*

Proof Assume that the contrary is true and there exists $\gamma_i \in \Gamma_i$ with $d(\gamma_i x, x) \leqslant R$, a constant. Then $\gamma_i(x) \to y \in B_R(x)$, after passing to a subsequence of $\{\gamma_i\}$ if needed. Hence the distance with respect to the Bergman metric, $d(\gamma_i^{-1}\gamma_{i+1}(x), x) = d(\gamma_{i+1}(x), \gamma_i(x)) \to 0$ as $i \to \infty$, since Γ acts by isometry. This contradicts proper action of Γ on $\widetilde{M}$. $\square$

4.2 The following is the main result of this section. Sometimes it is also called stability result for the kernel involved. For the Bergman kernel $B^0_{(2)}(M_i, K_{M_i})$, we take the notation that $B^0_{(2)}(M_i, K_{M_i})(x) := B^0_{(2)}(M_i, K_{M_i})(x, x)$ as trace of the kernel function.

Proposition 2. *Let $x_o \in \Sigma \subset \widetilde{M}$. Then*

$$B^0_{(2)}(M_i, K_{M_i})(x) \to B^0_{(2)}(\widetilde{M}, K_{\widetilde{M}})(x)$$

uniformly in a neighborhood U of x_o as $i \to \infty$. Furthermore, given any differential operator $D_l = \frac{\partial^l}{\partial z_{i_1} \cdots \partial z_{i_l}}$ of degree l on Σ,

$$D_l B^0_{(2)}(M_i, K_{M_i})(x) \to D_l B^0_{(2)}(\widetilde{M}, K_{\widetilde{M}})(x)$$

uniformly in a neighborhood U of x_o as $i \to \infty$.

To prepare for the proof of Proposition 2, we make the following observation.

Lemma 7. $B^0_{(2)}(M_i, K_{M_i})(x) \leqslant C$ *for all $x \in D$ and $i \in \mathbb{N}$.*

Proof Note that $B^0_{(2)}(M_i, K_{M_i})$ is biholomorphic invariant. Hence it suffices to consider $x \in D$. Again we consider nested fundamental domains so that $D = D_1$ and $D_i \subset D_{i+1}$. Let $V \subset U$ be neighborhoods of D on M. Consider now separately V and $M - V$.

For $x \in M - V$, the injectivity radius of M is uniquely bounded from below by a constant τ_o. Hence the injectivity radius of M_i at x is uniquely bounded from below by τ_o as well. Hence for $f \in H^0_{(2)}(M_i, K_{M_i})$ with $\|f\|_{M_i,g} = 1$, it follows from Cauchy's estimates that

$$|f|_g(x) \leqslant C\|f\|_{M_i,g} = C$$

for some constant $C > 0$, cf. [Y2].

Consider now $x \in U$. It follows from Lemma 3 that $\sup_{x\in U} |f|^2_g(x) \leqslant C \sup_{y\in\partial U} |f|^2_g(y)$. As $\partial U \subset M - V$, we conclude from the last paragraph that $\sup_{y\in\partial U} |f|^2_g(y) \leqslant C$. We conclude that

$$\sup_{x\in U} |f|^2_g(x) \leqslant CC_1$$

for another positive constant C_1.

The above two paragraphs conclude the proof of the lemma. $\square$

4.3 Proof of Proposition 2 Proposition 1 implies that

$$\lim_{i\to\infty} \int_D B^0_{(2)}(M_i, K_{M_i}) = \int_D B^0_{(2)}(\widetilde{M}, K_{\widetilde{M}}). \tag{18}$$

We also know that for any point $x \in D$,

$$\lim_{i\to\infty} B^0_{(2)}(M_i, K_{M_i})(x) \leqslant B^0_{(2)}(\widetilde{M}, K_{\widetilde{M}})(x). \tag{19}$$

Our goal is to show that

$$\lim_{i\to\infty} B^0_{(2)}(M_i, K_{M_i})(x) = B^0_{(2)}(\widetilde{M}, K_{\widetilde{M}})(x) \tag{20}$$

uniformly for all $x \in U$, a neighborhood. Using Lemma 6.

The rest of the argument is similar to the ones in [Y3], [Y4] and [Y6]. We follow the steps outlined in §4 of [Y6], which we outline here briefly for completeness of presentation. The main estimate is to evaluate each of the terms on the right hand side of the following.

$$\begin{aligned} |B_{M_i,K}(x,x) - B_{\widetilde{M},K}(x,x)| &\leqslant |k_{\widetilde{M},K}(t,x,x) - B_{\widetilde{M},K}(x,x)| \\ &+|k_{M_i,K}(t,x,x) - k_{\widetilde{M},K}(t,x,x)| + |k_{M_i,K}(t,x,x) - B_{M_i,K}(x,x)|. \end{aligned} \tag{21}$$

At this point we remark that $B_{M_i,K} = B_{M_i,\Omega^n}$ is the Bergman kernel of holomorphic n-forms. Similarly, $B_{\widetilde{M},K} = B_{\widetilde{M},\Omega^n}$, $k_{M_i,K} = k_{M_i,\Omega^n}$ and $k_{\widetilde{M},K} = k_{\widetilde{M},\Omega^n}$. As argued in §4 of [C], In the finite volume case,the heat kernel for Ω^n on M_i is still related to the heat kernel for L^2 smooth n-forms on $\widetilde{M}$ by

$$k_{M_i,\Omega^n}(t,x,y) = \sum_{\gamma\in\Gamma_i} k_{\widetilde{M},\Omega^n}(t,\hat{x},\gamma\hat{y})$$

where $x = \pi_i(\hat{x},\hat{y})$. This is verified for example by the argument given in §4 of [C] and is used in our argument similar to the compact case treated in [Y3], [Y4]. Note that as $\widetilde{M}$ is a Hermitian symmetric space of non-compact type, there is a rich history of study on the spectral behavior. In particular, the heat kernel $k_{M_i,\Omega^n}(t,x,y)$ for Ω^n on $\widetilde{M}$ has similar estimates as heat kernel for functions on $\widetilde{M}$ when the distance $d(x,y)$ is large, cf [LM], Theorem 1. Together with Lemma 3, the discussions in [Y3], [Y4], as summarized in [Y6] are applicable.

Hence we need to estimate each of the three expressions in the right hand side of (21). The estimate of the first term is given by

$$|k_{\widetilde{M},K}(t,x,x) - B_{\widetilde{M},K}(x,x)| \quad \leqslant \quad \frac{\epsilon}{3}$$

uniformly according to Lemma 2 of [Y4]

By Lemma 1 of [Y3], the second term is estimated by

$$\begin{aligned} |k_{M_i,K}(t,x,x) - k_{\widetilde{M},K}(t,x,x)| &= \sum_{\gamma\in\Gamma-\{1\}} k_{\widetilde{M},K}(t,x,\gamma x) \\ &\leqslant ce^{-\frac{d^2(x,\gamma x)}{4t}} \\ &\leqslant (\frac{\epsilon}{3})^2 \end{aligned}$$

We will choose $t = d(x,\gamma x)$, which is at least τ_i.

The third term of inequality (21) is estimated by

$$\begin{aligned} & |k_{M_i,K}(t,x,x) - B_{M_i,K}(x,x)| \\ \leqslant\ & |k_{\widetilde{M},K}(t,x,x) - B_{\widetilde{M},K}(x,x)| + |k_{M_i,K}(t,x,x) - k_{\widetilde{M},K}(t,x,x)| \\ & +|k_{M_i,K}(t,x,x) - B_{M_i,K}(x,x)| \\ \leqslant\ & \frac{\epsilon}{3} \end{aligned}$$

where the last inequality follows from convergence of the Bergman kernel as in Lemma 5 of [Y3], by applying the arguments in Lemma 1, Lemma 3 of [Y4].

We conclude that given any $\epsilon > 0$, there exists $i_o > 0$ such that for all $i \geqslant i_o$ and $x, y \in D_i$,

$$|B_{M_i,K}(x,x) - B_{\widetilde{M},K}(x,x)| \leqslant \epsilon. \tag{22}$$

Letting $\epsilon \to 0$, we get the uniform convergence of $B_{M_i,K}(x,x)$ to $B_{\widetilde{M},K}(x,x)$ as $i \to \infty$.

The same argument, replacing $B_{M_i,K}$ everywhere by $D_l B_{M_i,K}$ and $B_{\widetilde{M},K}$ by $D_l B_{\widetilde{M},K}$, we will get the uniform convergence of

$$D_l B^0_{(2)}(M_i, K_{M_i})(x,x) \to D_l B^0_{(2)}(\widetilde{M}, K_{\widetilde{M}})(x,x)$$

for x in a small neighborhood of $x_o \in \widetilde{M}$ as was used in [Y3]-[Y6]. This also follows from the exposition in Lemma 4 and Proposition 2 of [Y4], since the kernels $B_{M_i,K}(x,y)$ is holomorphic in x and conjugate holomorphic in y, and we may apply Cauchy type of estimates to $M \times \overline{M}$.

This concludes the proof of Proposition 2. □

4.4 Proof of Theorem 3

Theorem 3 is now a consequence of Proposition 1 and Proposition 2. □

§5. L^2-very ampleness and L^2 jet generating.

5.1 We are going to finish the proof of Theorem 1 in this section. The idea is similar to those in [Y3] and [Y6] for compact cases but with some modifications needed to take care of the geometry at the infinity, making use of the results in the last section. As usual, we break the proof into proof of base point freeness, immersion and separation of points.

5.2 *Base Point Freeness*

From Proposition 1, we know that after passing to a certain M_i for i sufficiently large, we may assume that $\Gamma_{(2)}(M_i, K_{M_i}) \neq \emptyset$. Hence for simplicity of notation, we may replace M by M_i if necessary and assume that $\Gamma_{(2)}(M, K_M) = H^0_{(2)}(M, K_M) \neq \emptyset$.

From the first paragraph in the proof of Lemma 1, we know that a section $s \in \Gamma_{(2)}(M, K_M)$ extends holomorphically to a holomorphic section of $\Gamma(\overline{M}, K_{\overline{M}})$, which we use the same notation to represent. Denote the base locus of $\Gamma_{(2)}(M, K_M)$ on M by $\mathrm{Bs}_{(2)}(K_M)$. Then as $\overline{M}$ is projective algebraic, there are finite number of subvarieties $V_j \subset M, j = 1, \dots q$ so that $\mathrm{Bs}_{(2)}(K_M) = \cup_{j=1}^q (V_J)$. We are going to use induction to show that there exists a covering M_i of M, so that there exists section $s \in \Gamma_{(2)}(M_i, K_{M_i})$ with pull back s not vanishing on $p_i^* V_j$ for each j, where $p_i : M_i \to M$ is the covering map.

Choose a point $x_1 \in V_1$. There exists a neighborhood U_1 of D so that $x_1 \in \overline{M} - U_1$. Hence ∂U_1 is relatively compact in M. It follows from Proposition 2 that there exists $s \in \Gamma_{(2)}(M_i, K_{M_i})$ with $s(p_i^*(x_i)) \neq 0$, for $i \geqslant i_o$ sufficiently large. Hence $V_{11} := \mathrm{Bs}_{(s)}(K_M) \cap V_1$ is a proper subvariety of V_1 and hence of lower dimension. We may now apply induction. After a finite number of steps, and passing to a corresponding cover M_{V_1}, the base locus of the canonical sections on M_{V_1} does not intersect the pull-back of V_1. Note by pulling back canonical sections of V_1 by $p_{V_1} : M_{V_1} \to M$, we see that $\mathrm{Bs}_{(2)}(K_{M_{V_1}})$ is contained in $p_{V_1}^{-1}(\cup_{j=2}^q V_j)$.

Repeat the above procedure for V_2 in place of V_1, we conclude that after passing to an appropriate coverings M_{V_2}, the base locus of the canonical sections of M_{V_2} does not intersect V_2. In other words, in terms of the projection map

$$p_{V_2} : M_{V_2} \to M_{V_1} \to M,$$

we have $\mathrm{Bs}_{(2)}(K_{M_{V_2}}) \subset p_{V_2}^{-1}(\cup_{j=3}^q V_j)$.

Repeat the procedure using induction, we conclude that after passing to a covering M_{V_q}, $\mathrm{Bs}_{(2)}(K_{M_{V_q}}) = \emptyset$. Hence there exists a covering $M_{i_o} := M_{V_a}$ in the tower such that $K_{M_{i_o}}$ is base point free. This also implies that K_{M_i} is base point free for $i \geqslant i_o$ by pulling back the canonical sections.

5.3 *Immersion*

Let $s_1, \ldots, s_l$ be a unitary basis of $\Gamma_{(2)}(M, K_M)$. Let $t_{i,j} := d(s_i/s_j)$ be the meromorphic function on M coming from the quotients of the two sections. Then the locus of non-immersion is the set

$$A_M := \{x \in M : \mathrm{rank}([t_{i,k}]_{1 \leqslant i \leqslant l}) < n \text{ for all } k \leqslant l\},$$

cf. [Y2] or [Y5]. Hence if we set B_k to be the matrix $[t_{i,j}]_{1 \leqslant i \leqslant l}$,

$$A_M := \{x \in M : \text{ all } (n-1) \times (n-1) \text{ minors of } B_k \text{ determinant} = 0, \forall 1 \leqslant k \leqslant l\}.$$

The corresponding set for L^2 sections of $K_{\widetilde{M}}$ is

$$A_{\widetilde{M}} := \{x \in \widetilde{M} : \mathrm{rank}(d(\tilde{s}_i/\tilde{a})_{1 \leqslant i \leqslant n}) < n \ \forall \tilde{s}_i, \tilde{a} \in \Gamma_{(2)}(\widetilde{M}, K_{\widetilde{M}})\},$$

Note that $A_{\widetilde{M}} =$. In fact as $A_{\widetilde{M}}$ is invariant under biholomorphism on $\widetilde{M}$ and $\widetilde{M}$ is homogeneous, it suffices for us to check that $\mathrm{rank}(d(\tilde{s}_i/\tilde{a})_{1 \leqslant i \leqslant n})(0) = n$ at the point $0 \in \widetilde{M}$ in representing $\widetilde{M}$ as a bounded symmetric domain in $\mathbb{C}^n$. For this purpose, it suffices for us to take $\tilde{a} = 1 dz^1 \wedge \cdots \wedge dz^n$ and $\tilde{s}_i = z_i dz^1 \wedge \cdots \wedge dz^n$.

The argument as given in the part for base-point freeness leads to a proof.

5.4 *Separation of points*

Consider $M_i \times M_i$. The non-separate locus is $S_i \subset M_i \times M_i$. Similar to earlier discussions, we let

$$S_i = \{(x, y) \in M_i \times M_i : \frac{s}{t}(x) = \frac{s}{t}(y) \ \ \forall s, t \in \Gamma(M_i, K_{M_i})\}.$$

Similar to the argument in the earlier subsections, $S_{i+1} \subset S_i$ for all i by considering pull-back canonical sections. On the universal cover $\widetilde{M}$, clearly a corresponding set

$$\widetilde{S} = \{(x, y) \in \widetilde{M} \times \widetilde{M} : \frac{s}{t}(x) = \frac{s}{t}(y) \ \ \forall s, t \in \Gamma_{(2)}(\widetilde{M}, K)\}$$

is empty, by considering L^2 holomorphic n-forms obtained from multiplying bounded holomorphic functions to $dz^1 \wedge \cdots \wedge dz^n$.

The argument of the subsection for base-point freeness leads to result that $S_i = \emptyset$ for $i \geqslant i_o$ for some i_o sufficiently large.

5.5 *Generation of jets*

Since this involves just more number of derivatives comparing to the prove of immersion, exactly similar type of arguments can be applied with the availability of Proposition 2, see also page 221 of [Y3]. We skip the details here.

5.6 Proof of Theorem 1

Theorem 1 now is a direct consequence of discussions in **5.2-5.4**. □

§6. On certain smooth compactification

6.1 Throughout this section, we consider those M with compactification as stated in Theorem 2. To have a clearer picture of the argument, the reader may just consider M to be a finite complex ball quotient with smooth toroidal compactification as given in [AMRT] and [Mo], which are also the main examples we have in mind for this paper.

To relate the geometry of M with $\overline{M}$, let us first make the following observation.

Lemma 8. $H^0_{(2)}(M, K_M) \cong H^0(\overline{M}, K_{\overline{M}})$.

Proof Let $\varphi \in H^0_{(2)}(M, K_M)$. As L^2-norm on K_M is independent of metric chosen, we may assume that φ is L^2 respect to the Euclidean metric on $\overline{M}$. On a small coordinate neighborhood U of a point $p \in D \in \overline{M}$, we may write $\varphi = \psi dz^1 \wedge dz^2 \wedge \cdots \wedge dz^n$ in terms of local holomorphic coordinates (z_1, z_2). It follows from the L^2 assumption and the Riemann Extension Theorem that ψ can be extended holomorphically across D. Hence φ can be extend as a holomorphic section of $H^0(\overline{M}, K_{\overline{M}})$. □

6.2 The following lemma in the case of complex ball quotients of complex dimension $n \geqslant 2$ is already proved in [BT], which is stronger for these special cases in the sense that $i_o = 1$ and we do not need to go to higher levels in the tower. For any Hermitian locally symmetric space of complex dimension n, we assume as in Theorem 2 that each irreducible component D_j of the compactifying divisor $D = \sum_{j=1}^p D_j$ is an Abelian variety of complex dimension $n-1$.

Lemma 9. *There exists $i_o > 0$ such that $K_{\overline{M}_i}$ is ample on $\overline{M}_i$ for $i \geq i_o$.*

Proof Since an irreducible component $D_{i,j}$ is a torus, the canonical line bundle $K_{D_{i,j}}$ is trivial. It follows from the Adjunction formula that $K_{\overline{M}_i}|_{D_{i,j}} = -D_{i,j}|_{D_{i,j}}$. The latter is a positive line bundle from [AMRT] for arithmetic Γ. In the case that M is a complex ball quotient and Γ is non-arithmetic, we only need to consider complex ball quotients and in the case, this follows from [Mo].

To check the ampleness of $K_{\overline{M}_i}$, it suffices for us to check that $K^k_{\overline{M}_i} \cdot V > 0$ for every complex subvariety $V \subset \overline{M}_i$ of complex dimension $0 \leqslant k \leqslant n$ from Nakai-Moishezon criterion. cf. [Ha].

In the case of $k = n$ so that $V = \overline{M}$, this follows from the Lemma 8 and Theorem 1.

Suppose V is (irreducible) of complex dimension $d < N$. It suffices for us to show that $K_{\overline{M}_i}|_V$ is big. Consider first the case that $V \cap M_i = V \cap (\overline{M}_i - D_i) \neq \emptyset$. In such case, $V \cap M_i$ is an analytic subvariety of dimension d in M. I follows again from Theorem 1 that sections of $\Gamma_{(2)}(M_i, K_{M_i})$ restricted to V give immersion and separate points on $V \cap M$. Again from

Lemma 8, such sections are restriction of sections of $\Gamma(\overline{M}_j, K_{\overline{M}_j})$. In particular, $K_{M_j}|_V$ is big. On the other hand, if $V \subset D_{i,j}$ for some j. It follows Adjunction Formula that

$$K_{\overline{M}_i}|_{D_{i,j}} = -D_{i,j}|_{D_{i,j}} \tag{23}$$

is an ample line bundle on $D_{i,j}$. Hence $K_{\overline{M}_i}|_{D_{i,j}}$ is big again. The lemma nows follows from Nakai-Moishizon criterion. □

6.3 We observe that in general, $\pi_i : M_i \to M_{i-1}$ extends to a holomorphic mapping $\pi_i : M'_i \to \overline{M}_{i-1}$ for some compactification M'_i of M_i, by Nagata's Compactification Theorem, as explained in [WY2], §4.1-4.2. In the case of toroidal compactification for a locally Hermitian symmetric space, it is well-known that π_i extends to $\pi_i : \overline{M}_i \to \overline{M}_{i-1}$ as utilized in [Mu], [T]. Notice in the case of non-arithmetic quotients in complex rank one case, the behavior near a compactifying divisor is again the same as arithmetic case as explained in [Mo]. Hence a tower of coverings $\{M_i\}_{i=1}^{\infty}$ can be regarded as the restriction of coverings a tower $\{\overline{M}_i\}_{i=1}^{\infty}$ to the quasi-projective parts, for which $\pi_i : \overline{M}_i \to \overline{M}_{i-1}$ is ramified instead of unramified covering.

In fact, we may assume that π_i is ramified along D_i, by considering a sub tower of sufficiently large gap if necessary, according the earlier work of [M], [T] or [WY2]. In particular, this is proved in Lemma 4.3 of [WY2]. We refer the reader to §4 of [WY2] for more related discussions, but just recall the following statement that we need.

Lemma 10. *([WY2], Lemma 4.3). Let $r > 0$ be a positive number. There exists $i_o > 0$ such that for all $i \geqslant i_o$, the ramification order of $p_i : \overline{M}_i \to \overline{M}_1$ along D_i is at lease r.*

6.4

Lemma 11. *Let $r > 0$ be a positive number. There exists $i_o > 0$ such that there exist sections $s_1, \dots, s_{N_i} \in \Gamma(\overline{M}_i, K_{\overline{M}_i})$ vanishing along D_i to an order greater than r for $i \geqslant i_o$ and they generate K_{M_i} on M_i.*

Proof Write $D_i = D_{ij}, j = 1, \dots, n_i$ be the irreducible components of D_i. Fix i_1 such that $\Gamma_{(2)}(M_{i_1}, K_{M_{i_1}})$ is very ample on M_{i_1} from Theorem 1. Let $i > i_1$ and $q_i := p_{i_1+1} \circ p_{i_1+1} \circ \cdots \circ p_i : \overline{M}_i \to \overline{M}_{i_1}$.

From Lemma 7, π_i is ramified. Let $r > 0$ be a fixed number. Let n_{ij} be the ramification index along D_{ij}. Let $\varphi \in \Gamma(\overline{M}_{i_1}, K_{\overline{M}_{i_1}})$. It follows that $q_i^* \varphi$ is a global section in $K_{\overline{M}_i}$ with vanishing order along D_{ij} given by $n_{ij} \geqslant r$. This is just a reflection of the usual Hurwitz formula that $K_{\overline{M}_i} = \pi_i^* K_{\overline{M}_{i-1}} + \sum_j (n_{ij} - 1)(D_{ij})$. □

6.5

Lemma 12. *There exists $i_o > 0$ such that $2K_{\overline{M}_i}|_{D_i}$ is very ample for $i \geq i_o$.*

Proof From (23), we know that $(D_{i,j}, K_{M_i}|_{D_{i,j}})$ is ample for each irreducible component $D_{i,j}$ of $D_i = \overline{M}_i - M_i$. From Lefschetz's Theorem, cf. [GH], we know that $3K_{M_i}|_{D_{i,j}}$ is very ample as $D_{i,j}$ is an abelian variety. The theorem of Lefschetz was generalized by Ohbuchi [O] to the statement that for an ample line bundle L on an abelian variety A, $2L$ is very ample except in the special situation that the pair (A, L) is isomorphic to $(A_1 \times A_2, \mathcal{O}(D_1 \times A_2 + A_1 \times D_2)$, where $A_i, i = 1, 2$, is an abelian variety and D_i is an ample line bundle on A_i with $\dim(A_1, \mathcal{O}(D_1)) = 1$. In our situation, it is clear that $K|_{D_{i,j}} = -D_{i,j}|_{D_{i,j}}$ is not of the above form. Hence $2K_{M_i}|_{D_{i,j}}$ is very ample from the result of Ohbuchi. □

6.6 We can now complete the proof of Theorem 2.

Proof of Theorem 2

Consider the exact sequence

$$\begin{array}{llllll} 0 & \longrightarrow & H^0(\overline{M}_i, 2K_{\overline{M}_i} - D_i) & \longrightarrow & H^0(\overline{M}_i, 2K_{\overline{M}_i}) & \stackrel{\beta}{\longrightarrow} & H^0(D_i, 2K_{\overline{M}_i}|_{D_i}) \\ & \longrightarrow & H^1(\overline{M}_i, 2K_{\overline{M}_i} - D_i) & \longrightarrow & \cdots & & \end{array} \tag{24}$$

First we claim that $H^1(\overline{M}_i, 2K_{\overline{M}_i} - D_i) = \{0\}$. Recall we have sections $s_i, i = 1, \ldots, N_i$ obtained in Lemma 11 vanishing to order r along D_i. When $r > 1$, we may regard s_i as a section $t_i \in \Gamma(\overline{M}_i, K_{\overline{M}_i} - D_i)$. It follows that $h := (\sum_{i=1}^{N_i} |t_i|^2)^{-1}$, in local coordinate charts, will give a singular Hermitian metric for $K_{\overline{M}_i} - D_i$ which is smooth of K_{M_i} as $s_i, i = 1, \ldots, N_i$ has no common zero on M_i from Lemma 11, since they generate K_{M_i}. The metric is singular only along D_i. It follows that h is a singular Hermitian metric with positive curvature as a current, which is smooth everywhere on M_i and has Lelong number at least r along D_i. It follows from Nadel's vanishing theorem [N2], or L^2-estimates, that

$$H^1(\overline{M}_i, 2K_{\overline{M}_i} - D_i) = H^1(\overline{M}_i, K_{\overline{M}_i} + (K_{\overline{M}_i} - D_i)) = \{0\}$$

and the claim is proved.

From the claim, the homomorphism β in the exact sequence (24) is surjective. Now from Lemma 12, the global sections in $H^0(D_i, 2K_{\overline{M}_i}|_{D_i})$ separate points on D_i. including infinitesimal ones if $i \geqslant i_o$, which is to be assumed below. Since these sections are the restriction of sections in $H^0(\overline{M}_i, 2K_{\overline{M}_i})$ from the exact sequence and claim above, we conclude that sections of $\Gamma(\overline{M}_i, 2K_{\overline{M}_i})$ separate points on D_i, including infinitesimal ones.

From Lemma 8 and Theorem 1, we also know that $\Gamma(\overline{M}_i, K_{\overline{M}_i})$ separates points, including infinitesimal ones, on M_i. Squares of such sections gives sections in $\Gamma(\overline{M}_i, 2K_{\overline{M}_i})$ separating points, including infinitesimal ones on M_i. Hence to complete the proof of Theorem 2, it remains for us to show that $\Gamma(\overline{M}_i, 2K_{\overline{M}_i})$

(i) gives an immersion at points of D_i, and
(ii) separates points x, y with $x \in M_i$ and $y \in D_i$.

In both cases, the argument of the claim above using L^2-estimates or Nadel's vanishing theorem [N2] shows that they are valid, by choosing r to be sufficiently large. Let us simply illustrate this for (i).

Let $x \in D_i \subset \overline{M}_i$. Let U be a small coordinate neighborhood of x in $\overline{M}_i$, with x given by 0 in coordinates $(z_1, \ldots, z_n)$. We can take this as the pull back of some neighborhood of $\pi_i(x)$ on $\overline{M}$ as well. Assume that i is sufficiently large so that the vanishing order r_i of π_i satisfies $r_i \geqslant n+2$ after applying Lemma 10. Let $s_1, \ldots, s_{N_i}$ be the sections of $\Gamma(\overline{M}_i, K_{\overline{M}_i})$ obtained in Lemma 11 vanishing to order r_i at x and generates K_{M_i} on M_i. Let $h = (\sum_{i=1}^{N_i} |s_i|^2)^{-1}$, which gives a singular Hermitian metric for $K_{\overline{M}_i}$. We also equip $\overline{M}_i$ with a smooth Kähler metric g_1 with Kähler form ω_1. Then it follows that $h_2 := h \cdot \frac{1}{\det g_1}$ is a singular Hermitian metric on $2K_{\overline{M}_i}$, with curvature satisfying

$$\sqrt{-1}\partial\overline{\partial}\log h_2 + \mathrm{Rc}(g_1) = \sqrt{-1}\partial\overline{\partial}\log h_1 \geqslant c\omega_1 \tag{25}$$

as a current, for some constant $c > 0$.

Let ρ be a smooth function supported in a smaller neighborhood U_ϵ of U. Let e_K^2 be a local basis of $K_{\overline{M}_i}$ on U. The expression $\overline{\partial}(\rho z_i e_K^2)$ has compact support in U. Extend it by 0 outside of U to get a smooth $K^2_{\overline{M}_i}$-valued $(0,1)$-form on $\overline{M}_i$. Standard L^2-estimates, cf. [Ho], allows us to find a solution of

$$\overline{\partial}u = \overline{\partial}(\rho z_i e_K^2) \tag{26}$$

with estimates

$$\int_M |u|^2_{h_2} \leqslant \int_M \frac{1}{c}|\overline{\partial}(\rho z_i e_K^2)|_{h_2} \tag{27}$$

The right hand side of the above inequality is finite since $\overline{\partial}(\rho z_i e_K^2)$ vanishes in a neighborhood of x. Hence the left hand side of (27) is finite, which implies that u vanishes to order at least 2 at x since h has a pole of large order at x. We conclude that $f_i := \rho z_i e_K^2 - u \in \Gamma_{(2)}(\overline{M}_i, K_{\overline{M}_i})$ when evaluated at 0 satisfies

$$\frac{\partial}{\partial z_i} f_i(0) = (\frac{\partial}{\partial z_i} z_i(0)) e_K^2(0) = e_K^2(0),$$

which is non-zero. Since i is arbitrary, we conclude that sections of $\Gamma(\overline{M}_i, 2K_{\overline{M}_i})$ gives an immersion at x. Since this applies to all $x \in D$, (i) is valid.

(ii) is proved similarly. Note that from Theorem 1 and Lemma 11, we can find sections in $\Gamma(\overline{M}_i, K_{\overline{M}_i})$ vanishing to arbitrary predetermined order at $y \in M_i$ if i is sufficiently large. Hence the argument earlier as in (1) allows us to construct sections in $\Gamma(\overline{M}_i, 2K_{\overline{M}_i})$ separating a point $x \in D$ and a point $y \in M_i$. Hence (ii) is also valid.

Theorem 2 follows.

□

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