THE RIEMANN HYPOTHESIS FOR STIELTJES SPACES OF ENTIRE FUNCTIONS

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Abstract. The Riemann hypothesis originates as a conjecture concerning the zeros of a particular entire function which is motivated by properties of zeros of polynomials. The Riemann hypothesis is generalized as a conjecture about the zeros of entire functions which resemble the entire function chosen by Riemann. The Hermite class of entire functions identifies entire functions which resemble polynomials in being essentially determined by their zeros. Polynomial members of the Hermite class are applied by Stieltjes in an integral representation of nonnegative linear functionals on polynomials. Arbitrary members of the Hermite class are applied in an analogous construction [2] of Hilbert spaces whose elements are entire functions. The Riemann hypothesis for Hilbert spaces of entire functions [3] identifies a class of entire functions which resemble the original entire function of the Riemann hypothesis. The Riemann hypothesis is strengthened as the conjecture that the Riemann hypothesis for entire functions applies. The conjecture is shown [4] to be correct in a modified form appropriate to the singularity of the Euler zeta function. Since the proof is an elementary argument in Fourier analysis, delays in verification are due to unfamiliarity with concepts which are here reviewed.

The Riemann hypothesis is an assertion which applies to the zeros of an entire function which is not a polynomial. Entire functions which admit a relationship to zeros found in polynomials were discovered by Charles Hermite. The Hermite class of entire functions permits a reformulation of the Riemann hypothesis as the conjecture that some entire function belongs to the class.

A nontrivial entire function is said to be of Hermite class if it can be approximated by polynomials whose zeros are restricted to a given half–plane. For applications to the Riemann hypothesis the upper half–plane is chosen as the half–plane free of zeros. If an entire function $E(z)$ of z is of Hermite class, then the modulus of $E(x + iy)$ is a nondecreasing function of positive y which satisfies the inequality

$$|E(x - iy)| \leq |E(x + iy)|$$

for every real number x. These necessary conditions are also sufficient.

An entire function of Hermite class which has no zero is the exponential

$$\exp F(z)$$

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of an entire function $F(z)$ of z with derivative $F'(z)$ such that the real part of

$$iF'(z)$$

is nonnegative in the upper half–plane. The function

$$iF'(z) = a - ibz$$

is a polynomial of degree less than two by the Poisson representation of functions which are nonnegative and harmonic in the upper half–plane. The constant coefficient a has nonnegative real part and b is nonnegative.

If an entire function $E(z)$ of z is of Hermite class and has a zero w, then the entire function

$$E(z)/(z - w)$$

is of Hermite class. A sequence of polynomials $P_n(z)$ exists such that

$$E(z)/P_n(z)$$

is an entire function of Hermite class for every nonnegative integer n and such that

$$E(z) = \lim P_n(z)E_n(z)$$

uniformly on compact subsets of the upper half–plane for entire functions $E_n(z)$ of Hermite class which have no zeros.

An analytic weight function is defined as a function $W(z)$ of z which is analytic and without zeros in the upper half–plane. An entire function of Hermite class is an analytic weight function in the upper half–plane. Hilbert spaces of functions analytic in the upper half–plane were introduced in Fourier analysis by Godfrey Hardy.

The weighted Hardy space $\mathcal{F}(W)$ is defined as the Hilbert space of functions $F(z)$ of z, which are analytic in the upper half–plane, such that the least upper bound

$$\|F\|^2_{\mathcal{F}(W)} = \sup \int_{-\infty}^{+\infty} |F(x + iy)/W(x + iy)|^2 dx$$

taken over all positive y is finite. The classical Hardy space is obtained when $W(z)$ is identically one. Multiplication by $W(z)$ is an isometric transformation of the classical Hardy space onto the weighted Hardy space with analytic weight function $W(z)$.

An isometric transformation of the weighted Hardy space $\mathcal{F}(W)$ into itself is defined by taking a function $F(z)$ of z into the function

$$F(z)(z - w)/(z - w^-)$$

of z when w is in the upper half–plane. The range of the transformation is the set of elements of the space which vanish at w.

A continuous linear functional on the weighted Hardy space $\mathcal{F}(W)$ is defined by taking a function $F(z)$ of z into its value $F(w)$ at w whenever w is in the upper half–plane. The function

$$W(z)W(w)^-/[2\pi i(w^- - z)]$$

of z belongs to the space when w is in the upper half–plane and acts as reproducing kernel function for function values at w.

A Hilbert space of functions analytic in the upper half–plane which contains a nonzero element is isometrically equal to a weighted Hardy space if an isometric transformation of the space onto the subspace of functions which vanish at w is defined by taking $F(z)$ into

$$F(z)(z - w)/(z - w^-)$$

and if a continuous linear functional is defined on the space by taking $F(z)$ into $F(w)$ for every element w of the upper half–plane.

Examples of weighted Hardy spaces which apply in Fourier analysis are constructed from the gamma function discovered by Leonard Euler. The gamma function is a function $\Gamma(s)$ of s which is analytic in the complex plane with the exception of singularities at the nonpositive integers and which satisfies the recurrence relation

$$s\Gamma(s) = \Gamma(s + 1).$$

An analytic weight function

$$W(z) = \Gamma(s)$$

is defined by

$$s = \tfrac{1}{2} - iz.$$

A maximal dissipative transformation is defined in the weighted Hardy space $\mathcal{F}(W)$ by taking $F(z)$ into $F(z + i)$ whenever the functions of z belong to the space.

A relation T with domain and range in a Hilbert space is said to be maximal dissipative if a contractive transformation of the space into itself is defined for some, and hence every, λ in the right half–plane by taking

$$Tc + \lambda c$$

into

$$Tc - \lambda^- c$$

whenever c is in the domain of T. The transformation T is said to be dissipative if a contractive transformation with domain and range in the Hilbert space is defined for some, and hence every, element λ of the right half–plane.

The existence of a maximal dissipative transformation in a weighted Hardy space is a Riemann hypothesis for analytic weight functions.

Theorem 1. *A maximal dissipative transformation is defined in a weighted Hardy space $\mathcal{F}(W)$ by taking $F(z)$ into $F(z+i)$ whenever the functions of z belong to the space if, and only if, the function*

$$W(z + \tfrac{1}{2}i)/W(z - \tfrac{1}{2}i)$$

of z admits an extension which is analytic and has nonnegative real part in the upper half–plane.

Proof of Theorem 1. A Hilbert space $\mathcal{H}$ whose elements are functions analytic in the upper half–plane is constructed when a maximal dissipative transformation is defined in the weighted Hardy space $\mathcal{F}(W)$ by taking $F(z)$ into $F(z+i)$ whenever the functions of z belong to the space. The space $\mathcal{H}$ is constructed from the graph of the adjoint of the transformation which takes $F(z)$ into $F(z+i)$ whenever the functions of z belong to the space.

An element

$$F(z) = (F_+(z), F_-(z))$$

of the graph is a pair of analytic functions of z, which belong to the space $\mathcal{F}(W)$, such that the adjoint takes $F_+(z)$ into $F_-(z)$. The scalar product

$$\langle F(t), G(t)\rangle = \langle F_+(t), G_-(t)\rangle_{\mathcal{F}(W)} + \langle F_-(t), G_+(t)\rangle_{\mathcal{F}(W)}$$

of elements $F(z)$ and $G(z)$ of the graph is defined as the sum of scalar products in the space $\mathcal{F}(W)$. Scalar self–products are nonnegative in the graph since the adjoint of a maximal dissipative transformation is maximal dissipative.

An element $K(w,z)$ of the graph is defined by

$$K_+(w,z) = W(z)W(w - \tfrac{1}{2}i)^-/[2\pi i(w^- + \tfrac{1}{2}i - z)]$$

and

$$K_-(w,z) = W(z)W(w + \tfrac{1}{2}i)^-/[2\pi i(w^- - \tfrac{1}{2}i - z)]$$

when w is in the half–plane

$$1 < iw^- - iw.$$

The identity

$$F_+(w + \tfrac{1}{2}i) + F_-(w - \tfrac{1}{2}i) = \langle F(t), K(w,t)\rangle$$

holds for every element

$$F(z) = (F_+(z), F_-(z))$$

of the graph. An element of the graph which is orthogonal to itself is orthogonal to every element of the graph.

An isometric transformation of the graph onto a dense subspace of $\mathcal{H}$ is defined by taking

$$F(z) = (F_+(z), F_-(z))$$

into the function

$$F_+(z + \tfrac{1}{2}i) + F_-(z - \tfrac{1}{2}i)$$

of z in the half–plane

$$1 < iz^- - iz.$$

The reproducing kernel function for function values at w in the space $\mathcal{H}$ is the function

$$[W(z + \tfrac{1}{2}i)W(w - \tfrac{1}{2}i)^- + W(z - \tfrac{1}{2}i)W(w + \tfrac{1}{2}i)^-]/[2\pi i(w^- - z)]$$

of z in the half–plane when w is in the half–plane.

Division by $W(z + \frac{1}{2}i)$ is an isometric transformation of the space $\mathcal{H}$ onto a Hilbert space appearing in the Poisson representation of functions which are analytic and have nonnegative real part in the upper half–plane. The function

$$\phi(z) = W(z - \tfrac{1}{2}i)/W(z + \tfrac{1}{2}i)$$

of z admits an analytic extension to the upper half–plane. The function

$$[\phi(z) + \phi(w)^-]/[2\pi i(w^- - z)]$$

of z belongs to the space when w is in the upper half–plane and acts as reproducing kernel function for function values at w. Since multiplication by $W(z+\frac{1}{2}i)$ is an isometric transformation of the space into $\mathcal{H}$, the elements of $\mathcal{H}$ have analytic extensions to the upper half–plane. The function

$$[W(z + \tfrac{1}{2}i)W(w - \tfrac{1}{2}i)^- + W(z - \tfrac{1}{2}i)W(w + \tfrac{1}{2}i)^-]/[2\pi i(w^- - z)]$$

of z belongs to the space when w is in the upper half–plane and acts as reproducing kernel function for function values at w.

The argument is reversed to construct a maximal dissipative transformation in the weighted Hardy space $\mathcal{F}(W)$ when the function $\phi(z)$ of z admits an extension which is analytic and has nonnegative real part in the upper half–plane. The Poisson representation constructs a Hilbert space whose elements are functions analytic in the upper half–plane and which contains the function

$$[\phi(z) + \phi(w)^-]/[2\pi i(w^- - z)]$$

of z as reproducing kernel function for function values at w when w is in the upper half–plane. Multiplication by $W(z+\frac{1}{2}i)$ acts as an isometric transformation of the space onto a Hilbert space $\mathcal{H}$ whose elements are functions analytic in the upper half–plane and which contains the function

$$[W(z + \tfrac{1}{2}i)W(w - \tfrac{1}{2}i)^- + W(z - \tfrac{1}{2}i)W(w + \tfrac{1}{2}i)^-]/[2\pi i(w^- - z)]$$

of z as reproducing kernel function for function values at w when w is in the upper half–plane.

A transformation is defined in the space $\mathcal{F}(W)$ by taking $F(z)$ into $F(z+i)$ whenever the functions of z belong to the space. The graph of the adjoint is a space of pairs

$$F(z) = (F_+(z), F_-(z))$$

of elements of the space such that the adjoint takes the function $F_+(z)$ of z into the function $F_-(z)$ of z. The graph contains

$$K(w,z) = (K_+(w,z), K_-(w,z))$$

with

$$K_+(w,z) = W(z)W(w - \tfrac{1}{2}i)^- / [2\pi i(w^- + \tfrac{1}{2}i - z)]$$

and

$$K_-(w,z) = W(z)W(w + \tfrac{1}{2}i)^- / [2\pi i(w^- - \tfrac{1}{2}i - z)]$$

when w is in the half–plane

$$1 < iw^- - iw.$$

The elements $K(w,z)$ of the graph span the graph of a restriction of the adjoint. The transformation in the space $\mathcal{F}(W)$ is recovered as the adjoint of the restricted adjoint.

A scalar product is defined on the graph of the restricted adjoint so that an isometric transformation of the graph of the restricted adjoint into the space $\mathcal{H}$ is defined by taking

$$F(z) = (F_+(z), F_-(z))$$

into

$$F_+(z + \tfrac{1}{2}i) + F_-(z - \tfrac{1}{2}i).$$

The identity

$$\langle F(t), G(t)\rangle = \langle F_+(t), G_-(t)\rangle_{\mathcal{F}(W)} + \langle F_-(t), G_+(t)\rangle_{\mathcal{F}(W)}$$

holds for all elements

$$F(z) = (F_+(z), F_-(z))$$

and

$$G(z) = (G_+(z), G_-(z))$$

of the graph of the restricted adjoint. The restricted adjoint is dissipative since scalar self–products are nonnegative in its graph. The adjoint is dissipative since the transformation in the space $\mathcal{F}(W)$ is the adjoint of its restricted adjoint.

The dissipative property of the adjoint is expressed in the inequality

$$\|F_+(t) - \lambda^- F_-(t)\|_{\mathcal{F}(W)} \le \|F_+(t) + \lambda F_-(t)\|_{\mathcal{F}(W)}$$

for elements

$$F(z) = (F_+(z), F_-(z))$$

of the graph when λ is in the right half–plane. The domain of the contractive transformation which takes the function

$$F_+(z) + \lambda F_-(z)$$

of z into the function

$$F_+(z) - \lambda^- F_-(z)$$

of z is a closed subspace of the space $\mathcal{F}(W)$. The maximal dissipative property of the adjoint is the requirement that the contractive transformation be everywhere defined for some, and hence every, λ in the right half–plane.

Since $K(w, z)$ belongs to the graph when w is in the half–plane

$$1 < iw^- - iw,$$

an element $H(z)$ of the space $\mathcal{F}(W)$ which is orthogonal to the domain satisfies the identity

$$H(w - \tfrac{1}{2}i) + \lambda H(w + \tfrac{1}{2}i) = 0$$

when w is in the upper half–plane. The function $H(z)$ of z admits an analytic extension to the complex plane which satisfies the identity

$$H(z) + \lambda H(z + i) = 0.$$

A zero of $H(z)$ is repeated with period i. Since

$$H(z)/W(z)$$

is analytic and of bounded type in the upper half–plane, the function $H(z)$ of z vanishes everywhere if it vanishes somewhere.

The space of elements $H(z)$ of the space $\mathcal{F}(W)$ which are solutions of the equation

$$H(z) + \lambda H(z + i) = 0$$

for some λ in the right half–plane has dimension zero or one. The dimension is independent of λ.

If τ is positive, multiplication by

$$\exp(i\tau z)$$

is an isometric transformation of the space $\mathcal{F}(W)$ into itself which takes solutions of the equation for a given λ into solutions of the equation with λ replaced by

$$\lambda \exp(\tau).$$

A solution $H(z)$ of the equation for a given λ vanishes identically since the function

$$\exp(-\tau z)H(z)$$

of z belongs to the space for every positive number τ and has the same norm as the function $H(z)$ of z.

The transformation which takes $F(z)$ into $F(z+i)$ whenever the functions of z belong to the space $\mathcal{F}(W)$ is maximal dissipative since it is the adjoint of its adjoint, which is maximal dissipative.

This completes the proof of the theorem.

An example of an analytic weight function which satisfies the hypotheses of the theorem is obtained when

$$W(z) = \Gamma(\tfrac{1}{2} - iz)$$

since

$$W(z + \tfrac{1}{2}i)/W(z - \tfrac{1}{2}i) = -iz$$

is analytic and has nonnegative real part in the upper half–plane by the recurrence relation for the gamma function. The weight functions which satisfy the hypotheses of the theorem are generalizations of the gamma function in which an identity is replaced by an inequality.

The gamma function is a special solution of a recurrence relation since it defines an analytic weight function with special properties. An Euler weight function is defined as an analytic weight function $W(z)$ such that a maximal dissipative transformation in the weighted Hardy space $\mathcal{F}(W)$ is defined for h in the interval $[0,1]$ by taking $F(z)$ into $F(z+ih)$ whenever the functions of z belong to the space.

An analytic weight function $W(z)$ is an Euler weight function if, and only if, for every h in the interval $[0,1]$ a function $\phi_h(z)$ of z which is analytic and has nonnegative real part in the upper half–plane exists such that the identity

$$W(z + \tfrac{1}{2}ih) = W(z - \tfrac{1}{2}ih)\phi_h(z)$$

holds for z in the upper half–plane. The identity

$$\phi_{h+k}(z) = \phi_h(z - \tfrac{1}{2}ik)\phi_k(z + \tfrac{1}{2}ih)$$

holds for z in the upper half–plane when h, k, and $h+k$ are in the interval $[0,1]$.

Auxiliary Hilbert spaces of analytic functions are applied in the construction of Euler weight functions. The Hilbert space $\mathcal{D}$ is the set of functions $F(z)$ of z analytic in the upper half–plane such that the integral

$$\|f(z)\|^2_{\mathcal{D}} = \int_0^\infty \int_{-\infty}^{+\infty} |f(x+iy)|^2 dxdy$$

converges. When h is positive, the Hilbert space $\mathcal{D}_h$ is the set of functions $f(z)$ of z analytic in the upper half–plane such that the least upper bound

$$\sup \int_{-\infty}^{+\infty} |f(x+iy)|^2 dx$$

taken over all positive y is finite and such that the sum

$$\|f(z)\|_{\mathcal{D}_h}^2 = \sum \int_{-\infty}^{+\infty} |f(x+inh)|^2 h dx$$

taken over all nonnegative integers n converges. When n is zero, the integral

$$\int_{-\infty}^{+\infty} |f(x)|^2 dx = \lim \int_{-\infty}^{+\infty} |f(x+iy)|^2 dx$$

is interpreted as a limit as y decreases to zero.

If the function

$$[f(z+\tfrac{1}{2}ih) - f(z-\tfrac{1}{2}ih)]/h$$

of z belongs to the space $\mathcal{D}_h$ for some function $f(z)$ of z which is analytic in the half–plane

$$iz^- - iz > -h,$$

then

$$f(z) = \int_0^\infty \exp(2\pi itz) F(t) dt$$

is formally the Fourier transform of a function $F(t)$ of positive t since the function

$$-[f(z+\tfrac{1}{2}ih) - f(z-\tfrac{1}{2}ih)]/h$$
$$= \int_{-\infty}^{+\infty} \exp(2\pi itz) F(t)[\exp(\pi ht) - \exp(-\pi ht)] dt$$

of z is the Fourier transform of a square integrable function

$$F(t)[exp(\pi ht) - \exp(-\pi ht)]/h$$

of positive t such that the identity

$$\|[f(z+\tfrac{1}{2}ih) - f(z-\tfrac{1}{2}ih)]/h\|_{\mathcal{D}_h}^2$$
$$= \int_{-\infty}^{+\infty} |F(t)|^2 \exp(\pi ht)[\exp(\pi ht) - \exp(-\pi ht)]/h \, dt$$

is satisfied. The identity reads

$$\|f'(z)\|_{\mathcal{D}}^2 = 2\pi \int_0^\infty |F(t)|^2 t dt$$

in the limit as h decreases to zero.

The function

$$\frac{1}{2\pi(z-w^-)(w^- - ih - z)}$$

of z belongs to the space $\mathcal{D}_h$ and acts as reproducing kernel function for function values at w when w is in the upper half–plane.

The function

$$\frac{1}{2\pi(z-w^-)(w^- - z)}$$

of z belongs to the space $\mathcal{D}$ and acts as reproducing kernel function for function values at w when w is in the upper half–plane.

If a nontrivial function $\phi(z)$ of z is analytic and has nonnegative real part in the upper half–plane, the logarithm

$$\log \phi(z)$$

is defined as an analytic function of z in the upper half–plane whose values lie in a horizontal strip of width π centered on the real axis. An Euler weight function $W(z)$ is constructed which satisfies the identity

$$W(z + \tfrac{1}{2}i) = W(z - \tfrac{1}{2}i)\phi(z).$$

If the function $\log \phi(z)$ of z belongs to the space $\mathcal{D}_1$, the Euler weight function $W(z)$ is defined formally by the Fourier integral

$$\log W(z) = \int_0^\infty \exp(2\pi itz)k(t)dt$$

of a function $k(t)$ of positive t such that

$$-\log \phi(z) = \int_0^\infty \exp(2\pi itz)k(t)[\exp(\pi t) - \exp(-\pi t)]dt.$$

A function $\phi_h(z)$ of z which is analytic and has nonnegative real part in the upper half–plane is defined when h is in the interval $(0,1)$ by the integral

$$-\log \phi_h(z) = \int_0^\infty \exp(2\pi itz)k(t)[\exp(\pi ht) - \exp(-\pi ht)]dt.$$

The logarithm of $\phi_h(z)$ belongs to the space $\mathcal{D}_h$ when the logarithm of $\phi(z)$ belongs to the space $\mathcal{D}_1$ since the integral

$$\|\log \phi_h(z)\|^2_{\mathcal{D}_h} = \int_0^\infty |k(t)\|^2 \exp(\pi ht)[\exp(\pi ht) - \exp(-\pi ht)]hdt$$

converges. The properties of $\phi_h(z)$ are obtained from the integral representation

$$\log \phi_h(z) = \sin(\pi h) \int_{-\infty}^{+\infty} \frac{\log \phi(z-t)dt}{\cos(2\pi it) + \cos(\pi h)}$$

which holds when z is in the upper half–plane since

$$\frac{\sin(\pi h)}{\cos(2\pi iz) + \cos(\pi h)} = \int_{-\infty}^{+\infty} \exp(2\pi itz) \frac{\exp(\pi ht) - \exp(-\pi ht)}{\exp(\pi t) - \exp(-\pi t)} \, dt.$$

The identity implies that $\phi_h(z)$ has nonnegative real part in the upper half–plane since $\phi(z)$ has nonnegative real part in the upper half–plane. The computation of Fourier integrals is an application of the Euler representation

$$\int_0^1 t^{a-1}(1-t)^{b-1} dt = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

of the gamma function, which applies when a and b are complex numbers with positive real part, and of the Euler identity

$$\Gamma(\tfrac{1}{2} + z)\Gamma(\tfrac{1}{2} - z) = \frac{\pi}{\cos(\pi z)}.$$

The integral representation of the logarithm of $\phi_h(z)$ in terms of the logarithm of $\phi(z)$ applies also when the logarithm of $\phi(z)$ does not belong to the space $\mathcal{D}_1$. The identity

$$\phi_{a+b}(z) = \phi_a(z - \tfrac{1}{2}ib)\phi_b(z + \tfrac{1}{2}ia)$$

holds when a and b are positive but not greater than one since

$$\frac{\sin(\pi a + \pi b)}{\cos(2\pi iz) + \cos(\pi a + \pi b)}$$
$$= \frac{\sin(\pi a)}{\cos(2\pi iz + \pi b) + \cos(\pi a)} + \frac{\sin(\pi b)}{\cos(2\pi iz - \pi a) + \cos(\pi b)}.$$

When the logarithm of $\phi(z)$ belongs to the space $\mathcal{D}_1$, the logarithmic derivative

$$iW'(z)/W(z) = \lim h^{-1} \log \phi_h(z)$$

of the desired weight function $W(z)$ is obtained as a limit as h decreases to zero. The logarithmic derivative of the weight function belongs to the space $\mathcal{D}$ since the integral

$$\|W'(z)/W(z)\|_{\mathcal{D}}^2 = 2\pi \int_0^\infty |k(t)|^2 t dt$$

converges. The identity

$$\log W(z + \tfrac{1}{2}ih) - \log W(z - \tfrac{1}{2}ih) = \log \phi_h(z)$$

is obtained by integration with respect to h.

Theorem 2. *If a maximal dissipative transformation is defined in the weighted Hardy space $\mathcal{F}(W)$ by taking $F(z)$ into $F(z+i)$ whenever the functions of z belong to the space, then the analytic weight function $W(z)$ is the product of an Euler weight function and an entire function which is periodic of period i and has no zeros.*

Proof of Theorem 2. Since a maximal dissipative transformation is defined in the weighted Hardy space $\mathcal{F}(W)$ by taking $F(z)$ into $F(z+i)$ whenever the functions of z belong to space, a function $\phi(z)$ of z exists by Theorem 1 which is analytic and has nonnegative real part in the upper half–plane such that

$$W(z+\tfrac{1}{2}i) = W(z-\tfrac{1}{2}i)\phi(z).$$

A function $\phi_h(z)$ of z which is analytic and has nonnegative real part in the upper half–plane is defined when h is in the interval $(0,1)$ by the integral representation

$$\log \phi_h(z) = \sin(\pi h) \int_{-\infty}^{+\infty} \frac{\log \phi(z-t) dt}{\cos(2\pi i t) + \cos(\pi h)}$$

of its logarithm from the logarithm of

$$\phi_1(z) = \phi(z)$$

defined with values in a horizontal strip of width π centered on the real axis. The identity

$$\phi_{h+k}(z) = \phi_h(z-\tfrac{1}{2}ik)\phi_k(z+\tfrac{1}{2}ih)$$

holds when h and k are positive with sum not greater than one. A maximal dissipative transformation T_h with domain and range in the weighted Hardy space $\mathcal{F}(W)$ is constructed by the proof of Theorem 1 to take $F(z)$ into

$$F(z+ih)W(z)\phi_h(z+\tfrac{1}{2}ih)/W(z+ih)$$

whenever the functions of z belong to the space. The identity

$$T_{h+k} = T_h T_k$$

holds when h and k are positive with sum not greater than one. The domain of T_{h+k} is the set of elements of the domain of T_k which are mapped by T_k int the domain of T_h.

A differentiable group of transformations parametrized by real numbers h is obtained in a space of entire functions which are periodic of period i and have no zeros. An entire function

$$\lim\ [W(z)\phi_h(z+\tfrac{1}{2}ih)/W(z+ih) - 1]/(ih)$$

which is periodic of period i is obtained as a uniform limit on compact subsets of the upper half–plane as h decreases to zero.

Since the limit

$$W'(z) = \lim \, [W(z+ih) - W(z)]/(ih)$$

exists uniformly on compact subsets of the complex plane as h decreases to zero, the limit

$$\lim \, [\phi_h(z + \tfrac{1}{2}ih) - 1]/(ih)$$

exists uniformly on compact subsets of the upper half–plane as h decreases to zero. An entire function $S(z)$ of z which is periodic of period i and has no zeros is obtained such that

$$S(z)W(z)$$

is an Euler weight function.

This completes the proof of the theorem.

Entire functions of Hermite class are examples of analytic weight functions which are limits of polynomials having no zeros in the upper half–plane. Such polynomials appear in the Stieltjes representation of positive linear functionals on polynomials.

A linear functional on polynomials with complex coefficients is said to be nonnegative if it has nonnegative values on polynomials whose values on the real axis are nonnegative. A positive linear functional on polynomials is a nonnegative linear functional on polynomials which does not vanish identically. A nonnegative linear functional on polynomials is represented as an integral with respect to a nonnegative measure μ on the Baire subsets of the real line. The linear functional takes a polynomial $F(z)$ into the integral

$$\int F(t) d\mu(t).$$

Stieltjes examines the action of a positive linear functional on polynomials of degree less than r for a positive integer r. A polynomial which as nonnegative values on the real axis is a product

$$F(z)F^*(z)$$

of a polynomial $F(z)$ and the conjugate polynomial

$$F^*(z) = F(z^-)^-.$$

If the positive linear functional does not annihilate

$$F(z)F^*(z)$$

for any nontrivial polynomial $F(z)$ of degree less than r, then a Hilbert space exists whose elements are the polynomials of degree less than r and whose scalar product

$$\langle F(t), G(t) \rangle$$

is defined as the action of the positive linear functional on the polynomial

$$G^*(z)F(z).$$

Stieltjes shows that the Hilbert space of polynomials of degree less than r is contained isometrically in a weighted Hardy space $\mathcal{F}(W)$ whose analytic weight function $W(z)$ is a polynomial of degree r having no zeros in the upper half–plane.

Examples of such spaces are applied by Legendre and Gauss in quadratic approximations of periodic motion and motivate the applications to number theory made precise by the Riemann hypothesis.

An axiomatization of the Stieltjes spaces is stated in a general context [2]. Hilbert spaces are examined whose elements are entire functions and which have these properties:

(H1) Whenever an entire function $F(z)$ of z belongs to the space and has a nonreal zero w, the entire function

$$F(z)(z-w^-)/(z-w)$$

of z belongs to the space and has the same norm as $F(z)$.

(H2) A continuous linear functional on the space is defined by taking a function $F(z)$ of z into its value $F(w)$ at w for every nonreal number w.

(H3) The entire function

$$F^*(z) = F(z^-)^-$$

of z belongs to the space whenever the entire function $F(z)$ of z belongs to the space, and it has the same norm as $F(z)$.

An example of a Hilbert space of entire functions which satisfies the axioms is obtained when an entire function $E(z)$ of z satisfies the inequality

$$|E(x-y)| < |E(x+iy)|$$

for all real x when y is positive. A weighted Hardy space $\mathcal{F}(W)$ is defined with analytic weight function

$$W(z) = E(z).$$

A Hilbert space $\mathcal{H}(E)$ which is contained isometrically in the space $\mathcal{F}(W)$ is defined as the set of entire functions $F(z)$ of z such that the entire functions $F(z)$ of z and $F^*(z)$ of z belong to the space $\mathcal{F}(W)$. The entire function

$$[E(z)E(w)^- - E^*(z)E(w^-)]/[2\pi i(w^- - z)]$$

of z belongs to the space $\mathcal{H}(E)$ for every complex number w and acts as reproducing kernel function for function values at w.

A Hilbert space $\mathcal{H}$ of entire functions which satisfies the axioms (H1), (H2), and (H3) is isometrically equal to a space $\mathcal{H}(E)$ if it contains a nonzero element. The proof applies reproducing kernel functions which exist by the axiom (H2).

For every nonreal number w a unique entire function $K(w, z)$ of z exists which belongs to the space and acts as reproducing kernel function for function values at w. The function does not vanish identically since the axiom (H1) implies that some element of the space has a nonzero value at w when some element of the space does not vanish identically. The scalar self–product $K(w, w)$ of the function $K(w, z)$ of z is positive. The axiom (H3) implies the symmetry

$$K(w^-, z) = K(w, z^-)^-.$$

If λ is a nonreal number, the set of elements of the space which vanish at λ is a Hilbert space of entire functions which is contained isometrically in the given space. The function

$$K(w, z) - K(w, \lambda)K(\lambda, \lambda)^{-1}K(\lambda, z)$$

of z belongs to the subspace and acts as reproducing kernel function for function values at λ. The identity

$$\begin{aligned}&[K(w, z) - K(w, \lambda)K(\lambda, \lambda)^{-1}K(\lambda, z)](z - \lambda^-)(w^- - \lambda)\\ = &[K(w, z) - K(w, \lambda^-)K(\lambda^-, \lambda^-)^{-1}K(\lambda^-, z)](z - \lambda)(w^- - \lambda^-)\end{aligned}$$

is a consequence of the axiom (H1).

An entire function $E(z)$ of z exists such that the identity

$$K(w, z) = [E(z)E(w)^- - E^*(z)E(w^-)]/[2\pi i(w^- - z)]$$

holds for all complex z when w is not real. The entire function can be chosen with a zero at λ when λ is in the lower half–plane and with a zero at λ^- when λ is in the upper half–plane. The function is then unique within a constant factor of absolute value one. A space $\mathcal{H}(E)$ exists and is isometrically equal to the given space $\mathcal{H}$.

Examples of Hilbert spaces of entire functions which satisfy the axioms (H1), (H2), and (H3) are constructed from the analytic weight function

$$W(z) = a^{iz}\Gamma(\tfrac{1}{2} - iz)$$

for every positive number a. The space is contained isometrically in the weighted Hardy space $\mathcal{F}(W)$ and contains every entire function $F(z)$ such that the functions $F(z)$ and $F^*(z)$ of z belong to the space $\mathcal{F}(W)$. The space of entire functions is isometrically equal to a space $\mathcal{H}(E)$ whose defining function $E(z)$ is a confluent hypergeometric series [1]. Selected properties of the space define a class of Hilbert spaces of entire functions.

An Euler space of entire functions is a Hilbert space of entire functions which satisfies the axioms (H1), (H2), and (H3) such that a maximal dissipative transformation is defined in the space for every h in the interval $[0, 1]$ by taking $F(z)$ into $F(z + ih)$ whenever the functions of z belong to the space.

Theorem 3. *A maximal dissipative transformation is defined in a Hilbert space $\mathcal{H}(E)$ of entire functions for a positive number h by taking $F(z)$ into $F(z+ih)$ whenever the functions of z belong to the space if, and only if, a Hilbert space $\mathcal{H}$ of entire functions exists which contains the function*

$$[E(z+\tfrac{1}{2}ih)E(w-\tfrac{1}{2}ih)^- - E^*(z+\tfrac{1}{2}ih)E(w^- + \tfrac{1}{2}ih)]/[2\pi i(w^- - z)]$$
$$+[E(z-\tfrac{1}{2}ih)E(w+\tfrac{1}{2}ih)^- - E^*(z-\tfrac{1}{2}ih)E(w^- - \tfrac{1}{2}ih)]/[2\pi i(w^- - z)]$$

of z as reproducing kernel function for function values at w for every complex number w.

Proof of Theorem 3. The space $\mathcal{H}$ is constructed from the graph of the adjoint of the transformation which takes $F(z)$ into $F(z+ih)$ whenever the functions of z belong to the space. An element

$$F(z) = (F_+(z), F_-(z))$$

of the graph is a pair of entire functions of z, which belong to the space $\mathcal{H}(E)$ such that the adjoint takes $F_+(z)$ into $F_-(z)$. The scalar product

$$\langle F(t), G(t)\rangle = \langle F_+(t), G_-(t)\rangle_{\mathcal{H}(E)} + \langle F_-(t), G_+(t)\rangle_{\mathcal{H}(E)}$$

of elements $F(z)$ and $G(z)$ of the graph is defined as a sum of scalar products in the space $\mathcal{H}(E)$. Scalar self–products are nonnegative since the adjoint of a maximal dissipative transformation is maximal dissipative.

An element

$$K(w,z) = (K_+(w,z), K_-(w,z))$$

of the graph is defined for every complex number w by

$$K_+(w,z) = [E(z)E(w-\tfrac{1}{2}ih)^- - E^*(z)E(w^- + \tfrac{1}{2}ih)]/[2\pi i(w^- + \tfrac{1}{2}ih - z)]$$

and

$$K_-(w,z) = [E(z)E(w+\tfrac{1}{2}ih)^- - E^*(z)E(w^- - \tfrac{1}{2}ih)]/[2\pi i(w^- - \tfrac{1}{2}ih - z)].$$

The identity

$$F_+(w+\tfrac{1}{2}ih) + F_-(w-\tfrac{1}{2}ih) = \langle F(t), K(w,t)\rangle$$

holds for every element

$$F(z) = (F_+(z), F_-(z))$$

of the graph. An element of the graph which is orthogonal to itself is orthogonal to every element of the graph.

A partially isometric transformation of the graph onto a dense subspace of the space $\mathcal{H}$ is defined by taking

$$F(z) = (F_+(z), F_-(z))$$

into the entire function

$$F_+(z + \tfrac{1}{2}ih) + F_-(z - \tfrac{1}{2}ih)$$

of z. The reproducing kernel function for function values at w in the space $\mathcal{H}$ is the function

$$[E(z + \tfrac{1}{2}ih)E(w - \tfrac{1}{2}ih)^- - E^*(z + \tfrac{1}{2}ih)E(w^- + \tfrac{1}{2}ih)]/[2\pi i(w^- - z)]$$
$$+[E(z - \tfrac{1}{2}ih)E(w + \tfrac{1}{2}ih)^- - E^*(z - \tfrac{1}{2}ih)E(w^- - \tfrac{1}{2}ih)]/[2\pi i(w^- - z)]$$

of z for every complex number w.

This completes the construction of a Hilbert space $\mathcal{H}$ of entire functions with the desired reproducing kernel functions when the maximal dissipative transformation exists in the space $\mathcal{H}(E)$. The argument is reversed to construct the maximal dissipative transformation in the space $\mathcal{H}(E)$ when the Hilbert space of entire functions with the desired reproducing kernel functions exists.

A transformation is defined in the space $\mathcal{H}(E)$ by taking $F(z)$ into $F(z + ih)$ whenever the functions of z belong to the space. The graph of the adjoint is a space of pairs

$$F(z) = (F_+(z), F_-(z))$$

such that the adjoint takes the function $F_+(z)$ of z into the function $F_-(z)$ of z. The graph contains

$$K(w, z) = (K_+(w, z), K_-(w, z))$$

with

$$K_+(w, z) = [E(z)E(w - \tfrac{1}{2}ih)^- - E^*(z)E(w^- + \tfrac{1}{2}ih)]/[2\pi i(w^- + \tfrac{1}{2}ih - z)]$$

and

$$K_-(w, z) = [E(z)E(w + \tfrac{1}{2}ih)^- - E^*(z)E(w^- - \tfrac{1}{2}ih)]/[2\pi i(w^- - \tfrac{1}{2}ih - z)]$$

for every complex number w. The elements $K(w, z)$ of the graph span the graph of a restriction of the adjoint. The transformation in the space $\mathcal{H}(E)$ is recovered as the adjoint of its restricted adjoint.

A scalar product is defined on the graph of the restricted adjoint so that an isometric transformation of the graph of the restricted adjoint into the space $\mathcal{H}$ is defined by taking

$$F(z) = (F_+(z), F_-(z))$$

into

$$F_+(z + \tfrac{1}{2}ih) + F_-(z - \tfrac{1}{2}ih).$$

The identity

$$\langle F(t), G(t)\rangle = \langle F_+(t), G_-(t)\rangle_{\mathcal{H}(E)} + \langle F_-(t), G_+(t)\rangle_{\mathcal{H}(E)}$$

holds for all elements

$$F(z) = (F_+(z), F_-(z))$$

of the graph of the restricted adjoint. The restricted adjoint is dissipative since scalar self–products are nonnegative in its graph. The adjoint is dissipative since the transformation in the space $\mathcal{H}(E)$ is the adjoint of its restricted adjoint.

The dissipative property of the adjoint is expressed in the inequality

$$\|F_+(t) - \lambda^- F_-(t)\|_{\mathcal{H}(E)} \leq \|F_+(t) + \lambda F_-(t)\|_{\mathcal{H}(E)}$$

for elements

$$F(z) = (F_+(z), F_-(z))$$

of the graph when λ is in the right half–plane. The domain of the contractive transformation which takes the function

$$F_+(z) + \lambda F_-(z)$$

of z into the function

$$F_+(z) - \lambda^- F_-(z)$$

of z is a closed subspace of the space $\mathcal{H}(E)$. The maximal dissipative property of the adjoint is the requirement that the contractive transformation be everywhere defined for some, and hence every, λ in the right half–plane.

Since $K(w, z)$ belongs to the graph for every complex number w, an entire function $H(z)$ of z which belongs to the space $\mathcal{H}(E)$ and is orthogonal to the domain is a solution of the equation

$$H(z) + \lambda H(z + i) = 0.$$

The function vanishes identically if it has a zero since zeros are repeated periodically with period i and since the function

$$H(z)/E(z)$$

of z is of bounded type in the upper half–plane. The space of solutions has dimension zero or one. The dimension is zero since it is independent of λ.

The transformation which takes $F(z)$ into $F(z + ih)$ whenever the functions of z belong to the space $\mathcal{H}(E)$ is maximal dissipative since it is the adjoint of its adjoint, which is maximal dissipative.

This completes the proof of the theorem.

A construction of Hilbert spaces of entire functions which satisfy the axioms (H1), (H2), and (H3) appears in the spectral theory of ordinary differential equations of second order which are formally self–adjoint. The spectral theory is advantageously reformulated as a spectral theory of first order differential equations for pairs of scalar functions. Differential equations are inverted as integral equations. Formally self–adjoint difference equations are included in the spectral theory.

A canonical form for the integral equation is obtained with a continuous matrix function

$$m(t) = \begin{pmatrix} \alpha(t) & \beta(t) \\ \beta(t) & \gamma(t) \end{pmatrix}$$

of positive t with real entries such that the matrix inequality

$$m(a) \leq m(b)$$

holds when a is less than b. It is assumed that $\alpha(t)$ is positive when t is positive, that

$$\lim \alpha(t) = 0$$

as t decreases to zero, and that the integral

$$\int_0^1 \alpha(t) d\gamma(t)$$

is finite.

The matrix

$$I = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

is applied in the formulation of the integral equation. When a is positive, the integral equation

$$M(a, b, z)I - I = z \int_a^b M(a, b, z) dm(t)$$

admits a unique continuous solution

$$M(a, b, z) = \begin{pmatrix} A(a, b, z) & B(a, b, z) \\ C(a, b, z) & D(a, b, z) \end{pmatrix}$$

as a function of b greater than or equal to a for every complex number z. The entries of the matrix are entire functions of z which are self–conjugate and of Hermite class for every b. The matrix has determinant one. The identity

$$M(a, c, z) = M(a, b, z)M(b, c, z)$$

holds when $a \leq b \leq c$.

A bar is used to denote the conjugate transpose

$$M^- = \begin{pmatrix} A^- & C^- \\ B^- & D^- \end{pmatrix}$$

of a square matrix

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

with complex entries and also for the conjugate transpose

$$C^- = (C_+^-, C_-^-)$$

of a column vector

$$C = \begin{pmatrix} C_+ \\ C_- \end{pmatrix}$$

with complex entries. The space of column vectors with complex entries is a Hilbert space of dimension two with scalar product

$$\langle u, v \rangle = v^- u = v_+^- u_+ + v_-^- u_- .$$

When a and b are positive with a less than or equal to b, a unique Hilbert space $\mathcal{H}(M(a,b))$ exists whose elements are pairs

$$F(z) = \begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix}$$

of entire functions of z such that a continuous transformation of the space into the Hilbert space of column vectors is defined by taking $F(z)$ into $F(w)$ for every complex number w and such that the adjoint takes a column vector c into the element

$$[M(a,b,z)IM(a,b,w)^- - I]c/[2\pi(z - w^-)]$$

of the space.

An entire function

$$E(c,z) = A(c,z) - iB(c,z)$$

of z which is of Hermite class exists for every positive number c such that the self–conjugate entire functions $A(c,z)$ and $B(c,z)$ satisfy the identity

$$(A(b,z), B(b,z)) = (A(a,z), B(a,z))M(a,b,z)$$

when a is less than or equal to b and such that the entire functions

$$E(c,z)\exp[\beta(c)z]$$

of z converge to one uniformly on compact subsets of the complex plane as c decreases to zero.

A space $\mathcal{H}(E(c))$ exists for every positive number c. The space $\mathcal{H}(E(a))$ is contained contractively in the space $\mathcal{H}(E(b))$ when a is less than or equal to b. The inclusion is isometric on the orthogonal complement in the space $\mathcal{H}(E(a))$ of the elements which are linear combinations

$$A(a,z)u + B(a,z)v$$

with complex coefficients u and v. These elements form a space of dimension zero or one since the identity

$$v^- u = u^- v$$

is satisfied.

A positive number b is said to be singular with respect to the function $m(t)$ of t if it belongs to an interval (a, c) such that

$$[\alpha(c) - \alpha(a)][\gamma(c) - \gamma(a)] = [\beta(c) - \beta(a)]^2$$

with $m(b)$ unequal to $m(a)$ and unequal to $m(c)$. A positive number is said to be regular with respect to $m(t)$ if it is not singular with respect to the function of t.

If a and c are positive numbers such that a is less than c and if an element b of the interval (a, c) is regular with respect to $m(t)$, then the space $\mathcal{H}(M(a, b))$ is contained isometrically in the space $\mathcal{H}(M(a, c))$ and multiplication by $M(a, b, z)$ is an isometric transformation of the space $\mathcal{H}(M(b, c))$ onto the orthogonal complement of the space $\mathcal{H}(M(a, b))$ in the space $\mathcal{H}(M(a, c))$.

If a and b are positive numbers such that a is less than b and if a is regular with respect to $m(t)$, then the space $\mathcal{H}(E(a))$ is contained isometrically in the space $\mathcal{H}(E(b))$ and an isometric transformation of the space $\mathcal{H}(M(a, b))$ onto the orthogonal complement of the space $\mathcal{H}(E(a))$ in the space $\mathcal{H}(E(b))$ is defined by taking

$$\begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix}$$

into

$$\sqrt{2}\, [A(a, z)F_+(z) + B(a, z)F_-(z)].$$

A function $\tau(t)$ of positive t with real values exists such that the function

$$m(t) + Iih(t)$$

of positive t with matrix values is nondecreasing for a function $h(t)$ of t with real values if, and only if, the functions

$$\tau(t) - h(t)$$

and

$$\tau(t) + h(t)$$

of positive t with real values are nondecreasing. The function $\tau(t)$ of t, which is continuous and nondecreasing and which is unique within an added constant, is called the greatest nondecreasing function such that

$$m(t) + Ii\tau(t)$$

is nondecreasing.

If a and b are positive numbers such that a is less than b, multiplication by

$$\exp(ihz)$$

is a contractive transformation of the space $\mathcal{H}(E(a))$ into the space $\mathcal{H}(E(b))$ for a real number h, if, and only if, the inequalities

$$\tau(a) - \tau(b) \leq h \leq \tau(b) - \tau(a)$$

are satisfied. The transformation is isometric when a is regular with respect to $m(t)$.

An analytic weight function $W(z)$ may exist such that multiplication by

$$\exp[i\tau(c)z]$$

is an isometric transformation of the space $\mathcal{H}(E(c))$ into the weighted Hardy space $\mathcal{F}(W)$ for every positive number c which is regular with respect to $m(t)$. The analytic weight function is unique within a constant factor of absolute value one if the function

$$\alpha(t) + \gamma(t)$$

of positive t is unbounded in the limit of large t. The function

$$W(z) = \lim E(c, z) \exp[i\tau(c)z]$$

can be chosen as a limit uniformly on compact subsets of the upper half–plane.

If multiplication by

$$\exp(i\tau z)$$

is an isometric transformation of a space $\mathcal{H}(E)$ into the weighted Hardy space $\mathcal{F}(W)$ for some real number τ and if the space $\mathcal{H}(E)$ contains an entire function $F(z)$ of z whenever the function

$$(z - w)F(z)$$

of z belongs to the space for a real number w, then the space $\mathcal{H}(E)$ is isometrically equal to the space $\mathcal{H}(E(c))$ for some positive number c which is regular with respect to $m(t)$.

The construction of Hilbert spaces of entire functions associated with an analytic weight function applied to every Euler weight function.

Theorem 4. *If $W(z)$ is an Euler weight function, then for some real number τ a Hilbert space of entire functions which satisfies the axioms (H1), (H2), and (H3) and which contains a nonzero element exists such that multiplication by*

$$\exp(i\tau z)$$

is an isometric transformation of the space into the weighted Hardy space $\mathcal{F}(W)$ and such that the space contains every entire function $F(z)$ of z such that the functions

$$\exp(i\tau z)F(z)$$

and

$$\exp(i\tau z)F^*(z)$$

of z belong to the weighted Hardy space. A Hilbert space of entire functions which satisfies the axioms (H1), (H2), and (H3) and which contains a nonzero element is an Euler space of entire functions if multiplication by

$$\exp(i\tau z)$$

is an isometric transformation of the space into the weighted Hardy space and if an entire function $F(z)$ of z belongs to the space whenever the function

$$(z - w)F(z)$$

of z belongs to the space for some real number w.

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