A Least-Squares Finite Element Approximation for the Compressible Stokes Equations

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Received January 24, 1999; accepted July 20, 1999

This article studies a least-squares finite element method for the numerical approximation of compressible Stokes equations. Optimal order error estimates for the velocity and pressure in the H^1 are established. The choice of finite element spaces for the velocity and pressure is not subject to the inf-sup condition. © 2000 John Wiley & Sons, Inc. Numer Methods Partial Differential Eq 16: 62–70, 2000

I. INTRODUCTION

In this article, we consider a least-squares finite element method for the "compressible" Stokes equations of the form

$$\begin{cases} -\mu\Delta\mathbf{u} - (\mu+\lambda)\nabla\nabla\cdot\mathbf{u} + \rho(\mathbf{U}\cdot\nabla\mathbf{u}^t)^t + \nabla p = \mathbf{f}, & \text{in } \Omega, \\ \rho'\mathbf{U}\cdot\nabla p + \rho\nabla\cdot\mathbf{u} = f_3, & \text{in } \Omega, \\ \mathbf{u} = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\mathbf{u} = (u,v)^t$ and p are the dependent variables; the symbols Δ, ∇, and $\nabla\cdot$ denote the Laplacian, gradient, and divergence operators, respectively ($\Delta\mathbf{u} = (\Delta u, \Delta v)^t$ and $\nabla\mathbf{u}^t = (\nabla u, \nabla v)$); μ and λ, the two coefficients of viscosity, are given constants satisfying $\mu > 0$ and $2\mu + \lambda > 0$; the $\mathbf{U} = (U,V)^t$ and P are given functions described the "ambient flow"; density $\rho(P)$ is a given positive increasing function of P; $\rho' = \frac{d\rho}{dP}$; $\mathbf{f} = (f_1, f_2)^t$ and f_3 are given functions; and Ω is a

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Contract grant sponsor: National Science Foundation; contract grant number: DMS-961972
 CCC 0749-159X/00/010062-09

bounded, open, connected domain in R^2. Note that $\mathbf{U} \cdot \nabla \mathbf{u}^t = (\mathbf{U} \cdot \nabla u, \mathbf{U} \cdot \nabla v)$. This system is obtained by linearizing the steady-state, compressible, viscous, Navier–Stokes equations around the "ambient flow," (U, V, P). Assume that $\mathbf{U}$ and P are C^1 functions in $\bar{\Omega}$, and $\mathbf{U} = 0$ on the boundary $\partial\Omega$.

In [1], Kellogg and Liu discretize (1.1) by using finite element methods. Conforming finite elements are used to approximate both velocity and pressure. The error bound is derived, if the finite element subspaces for velocity and pressure satisfy the inf-sup condition. Such error bound does not have optimal order of accuracy. Some other results about viscous compressible flow can be found in [2–5] and the references therein.

Recently, there has been substantial interest in the use of least-squares principles for numerical approximation of the second-order elliptic problems, incompressible Stokes and Navier–Stokes equations, and linear elasticity (for example, see [6–12] and references therein). In this article, we develop a least-squares finite element discretization for (1.1). Conforming finite element is used for velocity, $\mathbf{u}$, and pressure, p. An optimal convergence rate is obtained, and the choice of finite element spaces is not subject to the inf-sup condition. The order $O(h)$ established in Theorem 3.2 is optimal with respect to the assumed regularity of the solution, and is optimal with respect to the degree of the finite element spaces, if both variables are approximated by piecewise linear elements. In Section II, we introduce a least-squares formulation for (1.1) and establish its ellipticity. The corresponding finite element approximation is discussed in Section III.

II. LEAST-SQUARES FORMULATION

We use the standard notation and definition for the Sobolev spaces $H^s(\Omega)$ for $s \geq 0$; the standard associated inner products are denoted by $(\cdot,\, \cdot)_s$ and their respective norms by $\| \cdot \|_s$. For $s = 0$, $H^s(\Omega)^2$ coincides with $L^2(\Omega)^2$. In this case, the norm and inner product is denoted by $\| \cdot \|$ and $(\cdot,\, \cdot)$, respectively. As usual, define

$$H_0^1(\Omega) = \{v \in H^1(\Omega) \,:\, v = 0 \text{ on } \partial\Omega\}$$

and denote its dual by $H^{-1}(\Omega)$ with norm defined by

$$\|\psi\|_{-1} = \sup_{0 \neq \phi \in H_0^1(\Omega)} \frac{(\psi, \phi)}{\|\phi\|_1}.$$

Define the product spaces $H^s(\Omega)^2 = \prod_{i=1}^2 H^s(\Omega)$ with standard product norms. Finally, define

$$L_0^2(\Omega) = \{v \in L^2(\Omega) \,:\, \int_\Omega v \, dz = 0\}.$$

In this section, we consider a least-squares functional based on system (1.1). Our primary objective here is to establish ellipticity of this least-squares functional in the appropriate Sobolev space.

Dividing both sides of the second equation in (1.1) by ρ gives

$$\bar{\mathbf{U}} \cdot \nabla p + \nabla \cdot \mathbf{u} = \bar{f}_3, \tag{2.1}$$

where $\bar{f}_3 = \frac{f_3}{\rho}$ and $\bar{\mathbf{U}} = (\bar{U}, \bar{V})^t$ with $\bar{U} = \frac{\rho' U}{\rho}$ and $\bar{V} = \frac{\rho' V}{\rho}$. We will make use of the following lemma (see [1]).

Lemma 2.1. *For any $\mathbf{u} \in H_0^1(\Omega)^2$ and any $p \in L_0^2(\Omega)$, there exist positive constants K and α such that if $\|\partial_x U\|_\infty + \|\partial_y V\|_\infty + \|\nabla P\|_\infty < K$, then we have that*

$$(\mu\nabla\mathbf{u}, \nabla\mathbf{u}) + ((\mu+\lambda)\nabla\cdot\mathbf{u}, \nabla\cdot\mathbf{u}) + (\rho(\mathbf{U}\cdot\nabla\mathbf{u}^t)^t,\ \mathbf{u}) \geq \alpha\|\mathbf{u}\|_1^2. \tag{2.2}$$

Also, there exists a constant $\gamma(K) > 0$ such that

$$(\bar{\mathbf{U}}\cdot\nabla p, p) \geq -\gamma(K)\|p\|^2, \tag{2.3}$$

where $\gamma(K)$ can be made arbitrarily small by making K small.

We define the least-squares functional in terms of norms of the residuals for system (1.1):

$$\begin{aligned} G(\mathbf{u}, p;\ \mathbf{f}, \bar{f}_3) \ &= \ \| -\mu\Delta\mathbf{u} - (\mu+\lambda)\nabla(\nabla\cdot\mathbf{u}) + \rho(\mathbf{U}\cdot\nabla\mathbf{u}^t)^t + \nabla p - \mathbf{f}\|_{-1}^2 \\ &\quad + \|\bar{\mathbf{U}}\cdot\nabla p + \nabla\cdot\mathbf{u} - \bar{f}_3\|^2. \end{aligned} \tag{2.4}$$

Let

$$\mathcal{V} = H_0^1(\Omega)^2 \times \left(H^1(\Omega)/R\right).$$

The least-square formulation for system (1.1) is to minimize the quadratic functional $G(\mathbf{u}, p;\ \mathbf{f}, \bar{f}_3)$ with given $\mathbf{f}$ and $\bar{f}_3$ over $\mathcal{V}$: find $(\mathbf{u}, p) \in \mathcal{V}$ such that

$$G(\mathbf{u}, p;\ \mathbf{f}, \bar{f}_3) = \inf_{(\mathbf{v},q)\in\mathcal{V}} G(\mathbf{v}, q;\ \mathbf{f}, \bar{f}_3). \tag{2.5}$$

We establish the ellipticity and continuity of the homogeneous functional G in the following theorem. Below, we will use C with or without subscripts to denote a generic positive constant, possibly different at different occurrences, which is independent of the mesh size h introduced in the subsequent section, but may depend on the domain Ω, μ, and λ.

Theorem 2.1. *For sufficiently small $\gamma(K) \geq 0$, there exist the constants C_1 and C_2 such that for any $(\mathbf{u}, p) \in \mathcal{V}$ we have*

$$C_1\ \left(\|\mathbf{u}\|_1^2 + \|p\|^2 + \|\bar{\mathbf{U}}\cdot\nabla p\|^2\right) \leq G(\mathbf{u}, p;\ \mathbf{0}, 0) \tag{2.6}$$

and

$$G(\mathbf{u}, p;\ \mathbf{0}, 0) \leq C_2\ \left(\|\mathbf{u}\|_1^2 + \|p\|^2 + \|\bar{\mathbf{U}}\cdot\nabla p\|^2\right). \tag{2.7}$$

Proof. Upper bound (2.7) follows from the triangle inequality and from the easily established bounds

$$\|\Delta\mathbf{u}\|_{-1} \leq \|\mathbf{u}\|_1, \quad \|\nabla(\nabla\cdot\mathbf{u})\|_{-1} \leq \|\mathbf{u}\|_1, \quad \text{and} \quad \|\nabla p\|_{-1} \leq \|p\|. \tag{2.8}$$

We proceed to show the validity of lower bound (2.6) for $(\mathbf{u}, p) \in \mathcal{V}$, satisfying that $\mathbf{u} \in H^2(\Omega)^2$. Then (2.6) follows for $(\mathbf{u}, p) \in \mathcal{V}$ by continuity. For any $p \in L_0^2(\Omega)$, note first that (see, e.g., [13])

$$\|p\| \leq C_1\|\nabla p\|_{-1}. \tag{2.9}$$

It then follows from the triangle inequality, (2.8), and the boundedness of ρ and $\mathbf{U}$ that

$$\begin{aligned} \|p\| \ &\leq \ C_1\Big(\| -\mu\Delta\mathbf{u} - (\mu+\lambda)\nabla(\nabla\cdot\mathbf{u}) + \rho(\mathbf{U}\cdot\nabla\mathbf{u}^t)^t + \nabla p\|_{-1} \\ &\qquad + \| -\mu\Delta\mathbf{u} - (\mu+\lambda)\nabla(\nabla\cdot\mathbf{u}) + \rho(\mathbf{U}\cdot\nabla\mathbf{u}^t)^t\|_{-1}\Big) \\ &\leq \ C_1\, G^{\frac{1}{2}}(\mathbf{u}, p;\ \mathbf{0}, 0) + C_1C_2\|\mathbf{u}\|_1. \end{aligned} \tag{2.10}$$

By using Lemma 2.1 and integrating by parts, we have that

$$\begin{aligned}\alpha\|\mathbf{u}\|_1^2 &\le (\mu\nabla\mathbf{u},\nabla\mathbf{u}) + ((\mu+\lambda)\nabla\cdot\mathbf{u},\nabla\cdot\mathbf{u}) + (\rho(\mathbf{U}\cdot\nabla\mathbf{u}^t)^t,\mathbf{u})\\ &= (-\mu\Delta\mathbf{u} - (\mu+\lambda)\nabla(\nabla\cdot\mathbf{u}) + \rho(\mathbf{U}\cdot\nabla\mathbf{u}^t)^t + \nabla p,\ \mathbf{u}) + (p,\nabla\cdot\mathbf{u}) \qquad (2.11)\end{aligned}$$

and that

$$\begin{aligned}(p,\ \nabla\cdot\mathbf{u}) &= (p,\ \bar{\mathbf{U}}\cdot\nabla p + \nabla\cdot\mathbf{u}) - (p,\ \bar{\mathbf{U}}\cdot\nabla p)\\ &\le \|\bar{\mathbf{U}}\cdot\nabla p + \nabla\cdot\mathbf{u}\|\,\|p\| + \gamma(K)\|p\|^2. \qquad (2.12)\end{aligned}$$

The last inequality used the Cauchy–Schwarz inequality and Lemma 2.1. It now follows from (2.11), the definition of H^{-1}-norm, (2.12), (2.10), and the arithmetic-geometric mean inequality that

$$\begin{aligned}\alpha\|\mathbf{u}\|_1^2 &\le C\left(G(\mathbf{u},p;\,\mathbf{0},0) + \|\mathbf{u}\|_1 G^{\frac{1}{2}}(\mathbf{u},p;\,\mathbf{0},0)\right) + 2(C_1C_2)^2\gamma(K)\|\mathbf{u}\|_1^2\\ &\le C\,G(\mathbf{u},p;\,\mathbf{0},0) + \left(\frac{\alpha}{2} + 2(C_1C_2)^2\gamma(K)\right)\|\mathbf{u}\|_1^2.\end{aligned}$$

Hence, for sufficiently small $\gamma(K)$, i.e., $\gamma(K) < \frac{1}{4}\alpha(C_1C_2)^{-2}$, we have that

$$\|\mathbf{u}\|_1^2 \le CG(\mathbf{u},p;\mathbf{0},0).$$

Now, upper bounds in (2.6) for the terms $\|p\|^2$ and $\|\bar{\mathbf{U}}\cdot\nabla p\|^2$ are immediate consequences of (2.10) and the triangle inequality. This completes the proof of the validity of (2.6) and, hence, the theorem.

Remark 2.1. *The restriction $\gamma(K) < \frac{1}{4}\alpha(C_1C_2)^{-2}$ is similar to that in [1]. Also, as in [1], discontinuous pressure finite element spaces are not included here.*

III. FINITE ELEMENT APPROXIMATIONS

In this section, we present a discrete H^{-1} least-squares finite element approximation for the compressible Stokes equation based on (2.5). We first discuss the well-posedness of the discrete problem, and then establish optimal error estimates for the velocity in H^1 and for the pressure in L^2.

We use a Rayleigh–Ritz type finite element method to approximate the minimum of the least-squares functional $G(\mathbf{u},p;\mathbf{f},\bar{f}_3)$ defined in (2.4). Let $\mathcal{T}_h$ be a partition of the Ω into finite elements, i.e., $\Omega = \cup_{K\in\mathcal{T}_h}K$ with $h = \max\{\mathrm{diam}(K) : K \in \mathcal{T}_h\}$. Assume that the triangulation $\mathcal{T}_h$ is quasi-uniform, i.e., it is regular and satisfies the inverse assumption (see [14]). Let $\mathcal{V}^h = \mathcal{U}^h \times \mathcal{P}^h$ be a finite dimensional subspace of $\mathcal{V}$ such that, for any $(\mathbf{v},q) \in (H^2(\Omega)^2 \times H^2(\Omega)) \cap \mathcal{V}$, there exists an interpolant of $(\mathbf{v},q)$, denoted by $(\mathbf{v}^I,q^I)$, in $\mathcal{V}^h$ satisfying

$$\|\mathbf{v}-\mathbf{v}^I\| + h\|\mathbf{v}-\mathbf{v}^I\|_1 \le Ch^{r+1}\|\mathbf{v}\|_2, \qquad (3.1)$$

$$\sum_{K\in\mathcal{T}^h} h_K(\|\Delta(\mathbf{v}-\mathbf{v}^I)\|_{0,K} + \|\nabla(\nabla\cdot(\mathbf{v}-\mathbf{v}^I))\|_{0,K}) \le Ch^r\|\mathbf{v}\|_2 \qquad (3.2)$$

$$\|q-q^I\| + h\|q-q^I\|_1 \le Ch^{r+1}\|q\|_2, \qquad (3.3)$$

where r is an integer with $r \geq 1$ and, $(\cdot, \cdot)_{0,K}$ and $\|\cdot\|_{0,K}$ indicate the inner product and norm in $L^2(K)$. It is well known that (3.1)–(3.3) hold for typical finite element spaces consisting of continuous piecewise polynomials with respect to quasi-uniform triangulations (cf. [14]).

We need to replace the H^{-1}-norm in (2.4) by a computationally feasible discrete H^{-1}-norm that ensures the equivalence on $\mathcal{V}^h$ between the standard norm in $\mathcal{V}$ and that induced by the discrete homogeneous functional (see [15]). So, let A: $H^{-1}(\Omega)^2 \to H_0^1(\Omega)^2$ be the solution operator for the Poisson problem

$$\begin{cases} -\Delta\boldsymbol{\psi} + \boldsymbol{\psi} = \mathbf{v} & \text{in} \quad \Omega, \\ \boldsymbol{\psi} = 0 & \text{on} \quad \partial\Omega; \end{cases} \tag{3.4}$$

i.e., $A\mathbf{v} = \boldsymbol{\psi}$ for a given $\mathbf{v} \in H^{-1}(\Omega)^2$ is the solution of (3.4). It is well known that $(A\cdot, \cdot)^{\frac{1}{2}}$ defines a norm that is equivalent to the H^{-1}-norm. Let A_h: $H^{-1}(\Omega)^2 \to \mathcal{U}^h$ be the discrete solution operator $\boldsymbol{\psi} = A_h\mathbf{v} \in \mathcal{U}^h$ for the Poisson problem (3.4) defined by

$$\int_\Omega (\nabla\boldsymbol{\psi} \cdot \nabla\mathbf{w} + \boldsymbol{\psi} \cdot \mathbf{w}) = (\mathbf{v}, \mathbf{w}), \quad \mathbf{w} \in \mathcal{U}^h.$$

It is easy to check that $(A_h\cdot, \cdot)^{\frac{1}{2}}$ defines a semi-norm on $H^{-1}(\Omega)^2$, which is equivalent to the discrete H^{-1} seminorm:

$$\|\cdot\|_{-1,h} \equiv \sup_{\mathbf{w}\in\mathcal{U}^h} \frac{(\cdot, \mathbf{w})}{\|\mathbf{w}\|_1}.$$

Assume that there is a preconditioner $B_h : H^{-1}(\Omega)^2 \to \mathcal{U}^h$ that is symmetric with respect to the $L^2(\Omega)^2$-inner product and spectrally equivalent to A_h; i.e., there exists a positive constant C, independent of the mesh size h such that

$$\frac{1}{C}(A_h\mathbf{v}, \mathbf{v}) \leq (B_h\mathbf{v}, \mathbf{v}) \leq C(A_h\mathbf{v}, \mathbf{v}), \quad \mathbf{v} \in \mathcal{U}^h. \tag{3.5}$$

Finally, we define "discrete" Laplacian and gradient operators: the "discrete" Laplacian operator, Δ_h: $H_0^1(\Omega)^2 \to \mathcal{U}^h$, for a given $\mathbf{v} \in H_0^1(\Omega)^2$ is defined by $\Delta_h\mathbf{v} = \boldsymbol{\psi}$ satisfying

$$(\boldsymbol{\psi}, \mathbf{w}) = -(\nabla\mathbf{v}, \nabla\mathbf{w}), \quad \forall\ \mathbf{w} \in \mathcal{U}^h;$$

and the "discrete" gradient operator, ∇_h: $L^2(\Omega) \to \mathcal{U}^h$, for a given $q \in L^2(\Omega)$ is defined by $\nabla_h q = \mathbf{v}$ satisfying

$$(\mathbf{v}, \mathbf{w}) = -(q, \nabla \cdot \mathbf{w}), \quad \forall\ \mathbf{w} \in \mathcal{U}^h.$$

The implementation of computing the discrete gradient and Laplace operators can be found in [8] and [15]. Now, we are ready to define the discrete counterparts of the least-squares functional G as follows:

$$\begin{aligned} &G_h(\mathbf{u}, p; \mathbf{f}, \bar{f}_3) \\ &\quad = |-\mu\Delta_h\mathbf{u} - (\mu+\lambda)\nabla_h(\nabla\cdot\mathbf{u}) + \rho(\mathbf{U}\cdot\nabla\mathbf{u}^t)^t + \nabla p - \mathbf{f}|^2_{-1,h} \\ &\quad\quad + \sum_{K\in\mathcal{T}_h} h_K^2\| - \mu\Delta\mathbf{u} - (\mu+\lambda)\nabla(\nabla\cdot\mathbf{u}) + \rho(\mathbf{U}\cdot\nabla\mathbf{u}^t)^t + \nabla p - \mathbf{f}\|^2_{0,K} && (3.6) \\ &\quad\quad + \|\bar{\mathbf{U}}\cdot\nabla p + \nabla\cdot\mathbf{u} - \bar{f}_3\|^2, && (3.7) \end{aligned}$$

where $|\cdot|_{-1,h} \equiv (B_h\cdot,\,\cdot)^{\frac{1}{2}}$ defines a seminorm on $H^{-1}(\Omega)^2$, which is equivalent to $\|\cdot\|_{-1,h}$, by (3.5). Then the least-squares finite element approximation to (2.5) is to find $(\mathbf{u}_h, p_h) \in \mathcal{V}^h$ such that

$$G_h(\mathbf{u}_h, p_h;\, \mathbf{f}, \bar{f}_3) = \inf_{V \in \mathcal{V}^h} G_h(\mathbf{v}, q;\, \mathbf{f}, \bar{f}_3). \tag{3.8}$$

Theorem 3.1. *For sufficiently small $\gamma(K) \geq 0$, there exist positive constants C_1 and C_2 such that*

$$C_1\left(\|\mathbf{u}\|_1^2 + \|p\|^2 + \|\bar{\mathbf{U}}\cdot\nabla p\|^2\right) \leq G_h(\mathbf{u}, p; \mathbf{0}, 0) \tag{3.9}$$

and

$$G_h(\mathbf{u}, p; \mathbf{0}, 0) \leq C_2\left(\|\mathbf{u}\|_1^2 + \|p\|^2 + \|\bar{\mathbf{U}}\cdot\nabla p\|^2\right) \tag{3.10}$$

for any $(\mathbf{u}, p) \in \mathcal{V}^h$.

Proof. Let $Q_h\colon L^2(\Omega)^2 \to \mathcal{U}^h$ be the $L^2(\Omega)^2$ projection onto $\mathcal{U}^h$, then (3.1) implies that

$$\|\mathbf{v} - Q_h\mathbf{v}\| \leq Ch\|\mathbf{v}\|_1 \quad \text{and} \quad \|Q_h\mathbf{v}\|_1 \leq C\|\mathbf{v}\|_1 \tag{3.11}$$

for any $\mathbf{v} \in H_0^1(\Omega)^2$. Since B_h and A_h are symmetric with respect to the $L^2(\Omega)^2$ inner product, we have that $B_h = B_hQ_h$ and $A_h = A_hQ_h$. These further imply that the spectral equivalence, (3.5), between B_h and A_h holds for all $\mathbf{v}$ in $L^2(\Omega)^2$. Now, the upper bound in (3.10) follows from the triangle and inverse inequalities and from the easily established bounds

$$|\Delta_h\mathbf{u}|_{-1,h} \leq \|\nabla\mathbf{u}\| \quad \text{and} \quad |\nabla_h(\nabla\cdot\mathbf{u})|_{-1,h} \leq \|\nabla\mathbf{u}\|. \tag{3.12}$$

To prove the lower bound in (3.9), note that a standard duality argument implies that

$$\|\mathbf{v} - Q_h\mathbf{v}\|_{-1} \leq Ch\|\mathbf{v}\| \leq C\left(\sum_{K\in\mathcal{T}_h} h_K^2\|\mathbf{v}\|_{0,K}^2\right)^{\frac{1}{2}}$$

and that

$$\|Q_h\mathbf{v}\|_{-1} = \sup_{\mathbf{w}\in H_0^1(\Omega)^2}\frac{(Q_h\mathbf{v},\,\mathbf{w})}{\|\mathbf{w}\|_1} \leq \sup_{\mathbf{w}\in H_0^1(\Omega)^2}\frac{(\mathbf{v},\,Q_h\mathbf{w})}{\|Q_h\mathbf{w}\|_1} \leq C\|\mathbf{v}\|_{-1,h}$$

for any $\mathbf{v} \in L^2(\Omega)^2$. Therefore, we have

$$\|\mathbf{v}\|_{-1}^2 \leq C\left(\sum_{K\in\mathcal{T}_h} h_K^2\|\mathbf{v}\|_{0,K}^2 + \|\mathbf{v}\|_{-1,h}^2\right),$$

which, together with the choice $\mathbf{v} = \nabla p$ and the inequality (2.9), gives that

$$\|p\|^2 \leq C\left(\sum_{K\in\mathcal{T}_h} h_K^2\|\nabla p\|_{0,K}^2 + \|\nabla p\|_{-1,h}^2\right) \leq C_1\left(\sum_{K\in\mathcal{T}_h} h_K^2\|\nabla p\|_{0,K}^2 + |\nabla p|_{-1,h}^2\right)$$

for any $p \in P^h$. The last inequality used (3.5). Now, it follows from the triangle and inverse inequalities, the boundedness of ρ and $\mathbf{U}$, and (3.12) that

$$\|p\|^2 \;\leq\; C\bigg(G_h(\mathbf{u}, p; 0, 0) + \sum_{K\in\mathcal{T}_h} h_K^2\left(\|\Delta\mathbf{u}\|_{0,K}^2 + \|\nabla(\nabla\cdot\mathbf{u})\|_{0,K}^2 + \|(\mathbf{U}\cdot\nabla\mathbf{u}^t)^t\|_{0,K}^2\right)$$

$$
\begin{aligned}
&\quad + |\Delta_h \mathbf{u}|^2_{-1,h} + |\nabla_h(\nabla\cdot\mathbf{u})|^2_{-1,h} + |(\mathbf{U}\cdot\nabla\mathbf{u}^t)^t|^2_{-1,h}\Big) \\
&\le\; C\,G_h(\mathbf{u},p;\mathbf{0},0) + C_3\|\mathbf{u}\|_1^2. \qquad (3.13)
\end{aligned}
$$

By using Lemma 2.0 and the definitions of the discrete Laplacian and gradient operators, integrating by parts, and using the definition of the discrete H^{-1}-norm, (2.12), and (3.5), we have that, for any $\mathbf{u}\in\mathcal{U}^h$,

$$
\begin{aligned}
\alpha\|\mathbf{u}\|_1^2 \;&\le\; (\mu\nabla\mathbf{u},\nabla\mathbf{u}) + ((\mu+\lambda)\nabla\cdot\mathbf{u},\nabla\cdot\mathbf{u}) + (\rho(\mathbf{U}\cdot\nabla\mathbf{u}^t)^t,\mathbf{u}) \\
&=\; \big(-\mu\Delta_h\mathbf{u} - (\mu+\lambda)\nabla_h(\nabla\cdot\mathbf{u}) + \rho(\mathbf{U}\cdot\nabla\mathbf{u}^t)^t,\;\; \mathbf{u}\big) \\
&=\; \big(-\mu\Delta_h\mathbf{u} - (\mu+\lambda)\nabla_h(\nabla\cdot\mathbf{u}) + \rho(\mathbf{U}\cdot\nabla\mathbf{u}^t)^t + \nabla p,\;\; \mathbf{u}\big) + (p,\nabla\cdot\mathbf{u}) \\
&\le\; \|-\mu\Delta_h\mathbf{u} - (\mu+\lambda)\nabla_h(\nabla\cdot\mathbf{u}) + \rho(\mathbf{U}\cdot\nabla\mathbf{u}^t)^t + \nabla p\|_{-1,h}\|\mathbf{u}\|_1 \\
&\quad + \|\bar{\mathbf{U}}\cdot\nabla p + \nabla\cdot\mathbf{u}\|\|p\| + \gamma\|p\|^2 \\
&\le\; CG_h(\mathbf{u},p;\mathbf{0},0) + CG_h(\mathbf{u},p;\mathbf{0},0)\|\mathbf{u}\|_1 + \gamma(K)C_3\|\mathbf{u}\|_1.
\end{aligned}
$$

Hence, the arithmetic-geometric mean inequality implies that

$$\left(\frac{\alpha}{2} - \gamma(K)C_3^2\right)\|\mathbf{u}\|_1 \le C\,G_h(\mathbf{u},p;\mathbf{0},0).$$

For sufficiently small $\gamma(K) < \frac{1}{2}\alpha C_3^{-2}$, we then have that

$$\|\mathbf{u}\|_1^2 \le CG_h(\mathbf{u},p;\mathbf{0},0),$$

which, together with (3.13) and the triangle inequality, implies

$$\|p\|^2 \le CG_h(\mathbf{u},p;\mathbf{0},0) \quad\text{and}\quad \|\bar{\mathbf{U}}\cdot\nabla p\| \le CG_h(\mathbf{u},p;\mathbf{0},0).$$

This completes the proof of (3.9) and, hence, the theorem.

Denote by $b_h(\cdot\,;\cdot)$ the bilinear form induced by the quadratic form $G_h(\mathbf{u},p;\mathbf{0},0)$, i.e.,

$$
\begin{aligned}
b_h(\mathbf{u},p;\mathbf{v},q) \;&=\; (B_h\mathcal{L}_h(\mathbf{u},p),\,\mathcal{L}_h(\mathbf{v},q)) + \sum_K h_K^2\,(\mathcal{L}(\mathbf{u},p),\,\mathcal{L}(\mathbf{v},q))_{0,K} \\
&\quad + (\bar{\mathbf{U}}\cdot\nabla p + \nabla\cdot\mathbf{u},\;\bar{\mathbf{U}}\cdot\nabla q + \nabla\cdot\mathbf{v}),
\end{aligned}
$$

where operators $\mathcal{L}$ and $\mathcal{L}_h$ are given by

$$
\begin{aligned}
\mathcal{L}(\mathbf{v},q) \;&=\; \mu\Delta\mathbf{v} - (\mu+\lambda)\nabla(\nabla\cdot\mathbf{v}) + \rho(\mathbf{U}\cdot\nabla\mathbf{v}^t)^t + \nabla q \\
\text{and}\qquad \mathcal{L}_h(\mathbf{v},q) \;&=\; \mu\Delta_h\mathbf{v} - (\mu+\lambda)\nabla_h(\nabla\cdot\mathbf{v}) + \rho(\mathbf{U}\cdot\nabla\mathbf{v}^t)^t + \nabla q.
\end{aligned}
$$

Then the corresponding variational form of (3.8) is to find $(\mathbf{u}_h,p_h)$ such that

$$b_h(\mathbf{u}_h,p_h;\mathbf{v},q) = f(\mathbf{v},q), \quad \forall\ (\mathbf{v},q)\in\mathcal{V}^h, \qquad (3.14)$$

where the linear form $f(\cdot)$ is given by

$$f(\mathbf{v},q) = (B_h\mathbf{f},\;\mathcal{L}_h(\mathbf{v},q)) + \sum_K h_K^2\,(\mathbf{f},\;\mathcal{L}(\mathbf{v},q))_{0,K} + (\bar{f}_3,\bar{\mathbf{U}}\cdot\nabla q + \nabla\cdot\mathbf{v}).$$

Theorem 3.2. *Let $(\mathbf{u}_h, p_h) \in \mathcal{V}^h$ be the solution of—(3.14), and let $(\mathbf{u}, p) \in (H^2(\Omega)^2 \times H^2(\Omega)) \cap \mathcal{V}$ be the solution of—(2.5). Then, under assumptions of—Theorem 3.0, we have that*

$$\begin{aligned} \|\mathbf{u} - \mathbf{u}_h\|_1 + \|p - p_h\| + \|\bar{\mathbf{U}} \cdot \nabla(p - p_h)\| &\leq Ch(\|\mathbf{u}\|_2 + \|p\|_2) \\ &\leq Ch(\|\mathbf{f}\|_1 + \|f_3\|_2). \end{aligned} \tag{3.15}$$

Proof. It is easy to see that the following error equation holds:

$$b_h(\mathbf{u}_h - \mathbf{u}, p_h - p; \mathbf{v}, q) = 0$$

for all $(\mathbf{v}, q) \in \mathcal{V}^h$. Let $(\mathbf{u}^I, p^I) \in \mathcal{V}^h$ be the interpolant of $(\mathbf{u}, p)$ satisfying (3.1–3.3), we then have that

$$b_h(\mathbf{u}_h - \mathbf{u}^I, p_h - p^I; \mathbf{u}_h - \mathbf{u}^I, p_h - p^I) = b_h(\mathbf{u} - \mathbf{u}^I, p - p^I; \mathbf{u}_h - \mathbf{u}^I, p_h - p^I).$$

Since B_h is symmetric positive definite, using the Cauchy–Schwarz inequality and dividing $b_h^{\frac{1}{2}}(\mathbf{u}_h - \mathbf{u}^I, p_h - p^I; \mathbf{u}_h - \mathbf{u}^I, p_h - p^I)$ on the both sides give

$$b_h(\mathbf{u}_h - \mathbf{u}^I, p_h - p^I; \mathbf{u}_h - \mathbf{u}^I, p_h - p^I) \leq Cb_h(\mathbf{u} - \mathbf{u}^I, p - p^I; \mathbf{u} - \mathbf{u}^I, p - p^I).$$

It then follows from Theorem 3.1, the above error equation, the Cauchy–Schwarz and triangle inequalities, and approximation properties (3.1)–(3.3) that

$$\begin{aligned} &\|\mathbf{u}_h - \mathbf{u}^I\|_1^2 + \|p_h - p^I\|^2 + \|\bar{\mathbf{U}} \cdot \nabla(p_h - p^I)\|^2 \\ &\leq C(\|\mathbf{u} - \mathbf{u}^I\|_1^2 + \|p - p^I\|^2 + \|\bar{\mathbf{U}} \cdot \nabla(p - p^I)\|^2), \end{aligned}$$

which, together with the triangle inequality and (3.1–3.3), imply the first inequality in (3.14). The second inequality is a direct consequence of the following regularity result (see [1]):

$$\|\mathbf{u}\|_3 + \|p\|_2 \leq K(\|\mathbf{f}\|_1 + \|f_3\|_2).$$

This completes the proof of the theorem.

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