On the Finite Volume Element Method*

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ABSTRACT

The finite volume element method (FVE) is a discretization technique for partial differential equations. It uses a volume integral formulation of the problem with a finite partitioning set of volumes to discretize the equations, then restricts the admissible functions to a finite element space to discretize the solution. This paper develops discretization error estimates for general self-adjoint elliptic boundary value problems with FVE based on triangulations with linear finite element spaces and a general type of control volume. We establish $O(h)$ estimates of the error in a discrete $\boldsymbol{H}^1$ semi-norm. Under an additional assumption of local uniformity of the triangulation the estimate is improved to $O(h^2)$. Results on the effects of numerical integration are also included.

Key words. finite volume, finite element, discretization, second order elliptic equation

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1. Introduction

The finite volume element method (FVE) is a discretization technique for partial differential equations, especially those that arise from physical conservation laws. FVE uses a volume integral formulation of the problem with a finite partitioning set of volumes to discretize the equations, then restricts the admissible functions to a finite element space to discretize the solution. FVE is closely related to the control volume finite element method (CVFE), which was introduced several years earlier in the mechanical engineering literature [2]. However, there are important differences in how each method treats composite grids, especially in that FVE allows more general construction of the control volumes. The methods are the same for simple elliptic equations and volumes based on the Voronoi mesh. Thus, the theory here applies directly to both techniques.

The classical finite volume method (FV) is in common use for discretizing computational fluid dynamics equations. Reasons for its popularity include its ability to be faithful to the physics in general and conservation in particular, to capture shocks, to produce simple stencils, to apply to a fairly wide range of fluid flow equations, to effectively treat Neumann boundary conditions and nonuniform grids, and to facilitate multigrid solution. Yet the FV approach is not fully systematic: use of FV requires a scheme for approximating certain fluxes, which is often done in an effective but rather ad hoc and restrictive way that depends upon truncation error analysis. The limitations of truncation error analysis were treated in depth in [13, 14], which, among other things, demonstrated that truncation errors can be very large (e.g., $O(1)$) even in cases where the actual errors are small (e.g., $O(h)$ or $O(h^2)$).

FVE was developed as an attempt to use finite element ideas to create a more systematic FV methodology. The basic idea is to approximate the discrete fluxes needed in FV by replacing the unknown PDE solution by a finite element approximation. This means that the discretization design process can pay more attention to the local character of the solution (to choose accurate finite element spaces), and less to the equations. Furthermore, it provides a very effective discretization process for multilevel adaptive methods (see [15]).

In [7], we established accuracy estimates for FVE for diffusion equations on simple composite grids; however, the proofs appeal to the resulting stencil entries and are therefore quite complex. We developed a simple theory of FVE for diffusion equations for general triangulations in [6], but it applies only to a special choice of control volumes. The present paper develops error estimates for general self-adjoint elliptic boundary value problems on triangulations with linear finite element spaces, and with a general type of control volume. Here we also incorporate the effects of numerical integration. In Section 2, we briefly describe the FVE method. The general construction is suggested in Section 3. Section 4 is devoted to error estimates of the FVE method under the assumption of uniform ellipticity of the FVE operator, a sufficient condition for which is the use of the control volumes constructed in terms of the circumcenters of element triangles. This we describe in Section 5. Finally, in Section 6 we discuss the effects of numerical integration.

2. FVE Method

For simplicity, assume that $\Omega \subset \boldsymbol{R}^2$ is a polygonal domain. Consider the self-adjoint elliptic boundary value problem

$$
\begin{aligned}
-\bigtriangledown \cdot (A \bigtriangledown u) &= f \text{ in } \Omega \\
u &= 0 \text{ on } \Gamma_0 \\
(A \bigtriangledown u) \cdot \overrightarrow{n} &= g \text{ on } \Gamma_1
\end{aligned}
\tag{2.1}
$$

where Γ_0 and Γ_1 partition the boundary of Ω, the ds-measure of Γ_0 is strictly positive, $f \in \boldsymbol{L}^2(\Omega)$ and $g \in \boldsymbol{L}^2(\Gamma_1)$ are given real-valued functions, $A = (a_{ij})_{2\times 2}$ is a given real-valued matrix function, and $a_{ij} \in \boldsymbol{L}^\infty(\Omega)$, $1 \leq i, j \leq 2$. Assume henceforth that the following *ellipticity condition* holds: there exists a constant $\alpha_1 > 0$ such that

$$
\xi^T A(x,y)\xi \geq \alpha_1 \xi^T \xi \tag{2.2}
$$

for all $\xi \in \boldsymbol{R}^2$ and $(x, y) \in \bar{\Omega}$.

By taking the integral of (2.1) over any control volume $V \subset \bar{\Omega}$ with a Lipschitz boundary and using the Gauss Divergence Theorem (see [5, 16]) on the left-hand side, (2.1) may be transformed to the following *Primitive Form* (or *Surface Integral Form*):

Find $u \in \boldsymbol{H}_0^{(2)}(\Omega;\Gamma_0) \equiv \{v \in \boldsymbol{H}^1(\Omega) : v = 0 \text{ on } \Gamma_0\} \cap \boldsymbol{H}^2(\Omega)$ such that, for any volume $V \subset \bar{\Omega}$ with Lipschitz boundary,

$$-\int_{\partial V} (A \bigtriangledown u) \cdot \overrightarrow{n} ds = \int_V f dz, \tag{2.3}$$

where $\overrightarrow{n}$ is the unit outward normal vector on ∂V.

In general, discretizations based on the Primitive Form can in some sense preserve this conservation law. This is quite important in computational flow dynamics (CFD), and is one of the reasons for its popularity.

There are two ways to discretize the problem (2.3): while both use a finite set of volumes on Ω, one method, FV, uses divided differences to appproximate the fluxes in (2.3) (see [11]); another, FVE, restricts the unknowns to be in a finite-dimensional space. In particular, let $\mathcal{V}^h$ be a finite set of control volumes and $\boldsymbol{S}_0^h$ be a finite-dimensional space. Assume that the dimension of $\boldsymbol{S}_0^h$ equals the cardinality of $\mathcal{V}^h$. The discrete FVE problem is then written as follows:

Find $u^h \in \boldsymbol{S}_0^h$ such that, for all $V^h \in \mathcal{V}^h$,

$$-\int_{\partial V^h} (A \bigtriangledown u^h) \cdot \overrightarrow{n} ds = \int_{V^h} f dz. \tag{2.4}$$

¿From the FVE method, treatment of the Neumann boundary condition is straightforward: we do not need to impose the Neumann boundary condition directly on the finite element space $\boldsymbol{S}_0^h$; instead, we incorporate it in (2.4) by specifying $A \bigtriangledown u^h$ whenever ∂V^h coincides with a Neumann boundary. However, the Dirichlet condition is imposed directly on $\boldsymbol{S}_0^h$. Therefore, we may say that the Neumann boundary condition and the Dirichlet boundary condition are *natural* and *essential,* respectively, as in the finite element method case. In the next section, we will discuss two basic FVE design choices, the finite element space $\boldsymbol{S}_0^h$ and the finite set of control volumes $\mathcal{V}^h$.

3. Finite Element Space $\boldsymbol{S}_0^h$, Control Volumes $\mathcal{V}^h$, and Notation

Here, as in [6], we restrict ourselves to piecewise linear finite element spaces defined on triangulations. To describe $\boldsymbol{S}_0^h$, first consider a family of piecewise linear finite element spaces

$$\boldsymbol{S}^h = \{v \in \boldsymbol{C}^0(\Omega) : \quad v|_K \quad \text{is linear for all} \quad K \in \mathcal{T}^h\}.$$

Here, $\mathcal{T}^h$ is a triangulation of the domain $\bar{\Omega}$ so that each $K \in \mathcal{T}^h$ is a triangle and $\bar{\Omega} = \cup_{\mathcal{T}^h} K$. We assume that $\mathcal{T}^h$ is *regular* (see [8]): there exists a constant σ such that, for all $K \in \cup_h \mathcal{T}^h$,

$$\frac{h_K}{\rho_K} \leq \sigma, \tag{3.1}$$

where

$$h_K = \text{diam}(K)$$

and

$$\rho_K = \sup\{\text{diam}(C) : C \text{ is a circle contained in } K\}.$$

It is known that (3.1) is equivalent to Zlámal's condition (see [18]): there exists a constant $\theta_0 > 0$ such that, $\forall K \in \cup_h \mathcal{T}^h$,

$$\theta_K \geq \theta_0, \tag{3.2}$$

where θ_K denotes the smallest interior angle of K. Under these assumptions, we choose

$$\boldsymbol{S}_0^h = \{v \in \boldsymbol{S}^h : \quad v|_{\Gamma_0} = 0\}.$$

In general, we may construct the control volumes as follows: choose any interior point or median, z_K, of $K \in \cup_h \mathcal{T}^h$ and connect it with the medians, of K (see Figure 3.1). In this paper, we are particularly interested in the case that is the circumcenter, orthocenter, incenter, or centroid (i.e., center of gravity) of the triangle K, which are the respective midpoint of the circumscribed circle of K, intersection of its altitudes, midpoint of its inscribed circle, and intersection of its medians. The control volume associated with the circumcenter of the triangle was considered in [6] and forms the so-called Voronoi mesh;

that related to the centroid is in common use in CFD and forms the so-called Donald mesh. In order to keep the circumcenter and orthocenter from lying outside of the triangle, we assume throughout this paper that no interior angle of any triangle in $\mathcal{T}^h$ is larger than 90^0.

Suppose that the $m \geq n$ grid points (i.e., vertices of $\mathcal{T}^h$) are singly subscripted so that

$$\bar{\Omega}^h = \{z_i^h : \quad 1 \leq i \leq m\}$$

denotes the set of nodes of $\mathcal{T}^h$ and

$$\Omega^h = \{z_i^h : \quad 1 \leq i \leq n\} = \bar{\Omega}^h \cap (\Omega \cup \Gamma_1)$$

denotes nodes of both its interior and Neumann boundary $\Gamma_1^h = \bar{\Omega}^h \cap \Gamma_1$. Let $\Gamma_0^h = \bar{\Omega}^h \setminus \Omega^h$. Here and henceforth, we drop the superscript h when there is no danger of ambiguity. For each i=1,...,m, let $N(i)$ denote the set of neighbors of z_i in $\bar{\Omega}^h$ (i.e., points z_j such that z_i and z_j are distinct vertices of a common element $K \in \mathcal{T}^h$). Let

$$\omega_i = \{j : z_j \in N(i),\ 1 \leq j \leq m\}$$

denote the subscript set for neighbors of z_i in $\bar{\Omega}^h$ and let

$$\omega = \{\{i,j\} : 1 \leq i,j \leq m,\ j \in \omega_i\}.$$

Note that the unordered pairs $\{i,j\} = \{j,i\}$ in ω are in one-to-one correspondence to the edges in $\mathcal{T}^h$. Now given $1 \leq i,j \leq m$ such that $\{i,j\} \in \omega$, let

$$\gamma_{ij} = V_i \cap V_j$$

where V_i and V_j are the control volumes associated with points z_i and z_j, respectively. Let $\overrightarrow{n}_{ij}$ to be the unit outward normal vector on γ_{ij} (outward with respect to V_i^h). Let Z_{ij} denote the line segment connecting z_i and z_j. For each $\{i,j\} \in \omega$, let $|\gamma_{ij}|$ and $|Z_{ij}|$ denote the Euclidean lengths of γ_{ij} and Z_{ij}, respectively.

Now each $\{i,j\}\in\omega$ corresponds to two triangles, with a common face Z_{ij}, which we denote by K' and K''. We say that the control volumes are *symmetric* if $\gamma_{ij}\cap K'$ and $\gamma_{ij}\cap K''$ are perpendicular to Z_{ij} and $|\gamma_{ij}\cap K'|=|\gamma_{ij}\cap K''|$ for all $\{i,j\}\in\omega$; they are *essentially symmetric* if they are symmetric except for some volumes lying in a subregion, $\tilde{\Omega}$, that consists of a fixed number of strips in Ω with width $O(h)$; otherwise, they are *nonsymmetric* . Note that the control volumes associated with either the orthocenter, the incenter, or the centroid are symmetric if each triangle $K\in\mathcal{T}^h$ is equilateral; the control volumes related to the circumcenter are symmetric if $\mathcal{T}^h$ consists of equilateral triangles or triangles which are obtained by bisecting rectangles of the same shape.

By connecting the point z_K with the vertices of $K\in\mathcal{T}^h$, we obtain a new triangulation $\tilde{\mathcal{T}}^h$. We say that the control volumes are *regular* if the triangulation $\tilde{\mathcal{T}}^h$ is regular. For this, we have the following simple facts from geometry.

Proposition 3.1. If $\mathcal{T}^h$ is regular, then the control volumes associated with the incenter and centroid are regular.

Denote the angles of K' and K'' opposite the common face Z_{ij} by the respective $\bar{\theta}_{K'}$ and $\bar{\theta}_{K''}$ (see Figure 3.2).

Proposition 3.2. Assume that $\mathcal{T}^h$ is regular.

(1) If there exists a constant $\tilde{\theta}>0$ such that, for all K', $K''\in\mathcal{T}^h$,

$$\bar{\theta}_{K'}\geq\tilde{\theta}\quad\text{and}\quad\bar{\theta}_{K''}\geq\tilde{\theta},\tag{3.3}$$

then the control volumes related to the orthocenter are regular.

(2) If there exists a constant $\tilde{\theta}>0$ such that, for all K', $K''\in\mathcal{T}^h$,

$$\text{either}\quad\bar{\theta}_{K'}=\frac{\pi}{2}\quad\text{or}\quad\bar{\theta}_{K'}\leq\frac{\pi}{2}-\tilde{\theta}\tag{3.4}$$

and

$$\text{either}\quad\bar{\theta}_{K''}=\frac{\pi}{2}\quad\text{or}\quad\bar{\theta}_{K''}\leq\frac{\pi}{2}-\tilde{\theta},\tag{3.5}$$

then the control volumes related to the circumcenter are regular.

Define the discrete $\boldsymbol{H}^1$ semi-norm by

$$|v|_{1,\bar{\Omega}^h} = \left(\sum_{\{i,j\}\in\omega^h} (v(z_i) - v(z_j))^2 \right)^{\frac{1}{2}}. \tag{3.6}$$

For each $\{i, j\} \in \omega$, define the linear functional b_{ij} by

$$b_{ij}(v) = -\int_{\gamma_{ij}} (A \bigtriangledown v) \cdot \overrightarrow{n}_{ij} ds \tag{3.7}$$

and the linear operator B by

$$Bv = \left(\sum_{j\in\omega_i} b_{ij}(v) \right)_{i=1}^{n}. \tag{3.8}$$

We call B *uniformly elliptic* on $\boldsymbol{S}_0^h$ if there exists a constant $\alpha_2 > 0$ independent of the space $\boldsymbol{S}_0^h$ such that, for all v in $\boldsymbol{S}_0^h$,

$$\sum_{i=1}^{n} v(z_i)(Bv)_i \geq \alpha_2 |v|^2_{1,\bar{\Omega}^h}. \tag{3.9}$$

Conditions which guarantee uniform ellipticity of B will be given in the Sections 5 and 6.

Let $\overrightarrow{n}_i$ be the unit outward normal vector on ∂V_i $(i = 1, ..., n)$. Since

$$(Bv)_i = \sum_{j\in\omega_i} b_{ij}(v) = -\int_{\partial V_i} (A \bigtriangledown v) \cdot \overrightarrow{n}_i ds, \tag{3.10}$$

then we can rewrite (2.4) as follows:

Find $u^h \in \boldsymbol{S}_0^h$ such that

$$Bu^h = f^h, \tag{3.11}$$

where f^h is the n-dimensional vector with components $f_i^h = \int_{V_i} f dz$.

4. Error Estimates

Let $u \in \boldsymbol{H}_0^{(2)}(\Omega;\Gamma_0)$ and $u^h \in \boldsymbol{S}_0^h$ denote the solutions of (2.3) and (2.4), respectively. Since $\boldsymbol{H}_0^{(2)}(\Omega;\Gamma_0) \subset \boldsymbol{C}^0(\Omega)$, we may define the linear interpolant u_I of u in $\boldsymbol{S}_0^h$ as follows: $u_I \in \boldsymbol{S}_0^h$ such that $u_I(z_i) = u(z_i), 1 \le i \le m$. Our central aim is to estimate the discrete $\boldsymbol{H}^1$ semi-norm of the *discretization error*

$$e = u - u^h.$$

To do this, we will make use of its *discrete counterpart*

$$e^h = u_I - u^h$$

and the *interpolation error*

$$e_I = u - u_I.$$

Denote

$$\omega_0 = \{\{i,j\} \in \omega : \ |\gamma_{ij}| \neq 0\}.$$

Our first lemma develops a basic error estimate which establishes convergence of the FVE discretization.

Theorem 4.1. Assume that B satisfies (3.9). Then

$$|e|_{1,\bar{\Omega}^h} \le \frac{1}{\alpha_2}\left(\sum_{\{i,j\}\in\omega_0}(b_{ij}(e_I))^2\right)^{\frac{1}{2}}. \tag{4.1}$$

Proof. From (2.3) and (2.4) we have

$$Bu = Bu^h.$$

Hence, from the linearity of B, we have

$$Be^h = -Be_I.$$

Note that $e(z_i) = e^h(z_i)$, $1 \le i \le m$, and that $b_{ij}(e_I) = -b_{ji}(e_I)$ for all $\{i,j\} \in \omega$. Thus, by (3.9), it follows that

$$\begin{aligned}
\alpha_2 |e|^2_{1,\bar\Omega^h} &= \alpha_2 |e^h|^2_{1,\bar\Omega^h} \\
&\le \sum_{i=1}^n e^h(z_i)(Be^h)_i \\
&= -\sum_{i=1}^n e^h(z_i) \sum_{j\in\omega_i} b_{ij}(e_I) \\
&= \sum_{\{i,j\}\in\omega_0} (e^h(z_j) - e^h(z_i)) b_{ij}(e_I) \\
&\le |e^h|_{1,\bar\Omega^h} \left(\sum_{\{i,j\}\in\omega_0} b^2_{ij}(e_I) \right)^{\frac{1}{2}}.
\end{aligned}$$

This proves (4.1) and, hence, the lemma. ∎

Let $E_{ij} \subset \boldsymbol{R}^2$ denote the closed convex hull of $\gamma_{ij} \cup Z_{ij}$(see Figure 4.1). For each $\{i,j\} \in \omega$ and $K \in \mathcal{T}^h$ such that $E_{ij} \cap K \neq \emptyset$, assume that $E_{ij} \cap K$ is *affine-equivalent* to a reference triangle $\hat{E} \subset \boldsymbol{R}^2$ as follows: there exists an invertible mapping $F_{ij,K} : \boldsymbol{R}^2 \longrightarrow \boldsymbol{R}^2$ of the affine form $F_{ij,K}(z) = D_{ij,K} z + d_{ij,K}$ such that $E_{ij} \cap K = F_{ij,K}(\hat{E})$. Let $\hat\gamma = F^{-1}_{ij,K}(\gamma_{ij} \cap K)$ and define the linear functional $f_{ij,K}$ by

$$f_{ij,K}(v) = -\int_{\gamma_{ij}\cap K} (A \bigtriangledown (v - v_I)) \cdot \overrightarrow{n}_{ij} ds, \tag{4.2}$$

where v_I is the linear interpolant of v in $\boldsymbol{S}^h_0$. Let

$$\boldsymbol{P}_k(\boldsymbol{R}^2) = span\{p : p(x,y) = x^q y^r, \quad q, r \ge 0, \quad 0 \le q + r \le k\},$$

the set of polynomial of degree k on $E \subset \bar\Omega$. Then it is obvious that the linear functional $f_{ij,K}$ vanishes on $\boldsymbol{P}_1(\boldsymbol{R}^2)$:

$$f_{ij,K}(v) = 0, \qquad \forall v \in \boldsymbol{P}_1(\boldsymbol{R}^2). \tag{4.3}$$

For the symmetric control volume case, E_{ij} is rhombus (as in Figure 4.1). Assume that each E_{ij} is affine-equivalent to a reference rhombus $\hat{E} \subset \boldsymbol{R}^2$ as follows: there exists an invertible mapping $F_{ij} : \boldsymbol{R}^2 \longrightarrow \boldsymbol{R}^2$ of the affine form $F_{ij}(z) = D_{ij}z + d_{ij}$ such that $E_{ij} = F_{ij}(\hat{E})$. Let $\hat{\gamma} = F_{ij}^{-1}(\gamma_{ij})$ and $\hat{Z} = F_{ij}^{-1}(Z_{ij})$. Without loss of generality, suppose that $\hat{\gamma}$ and $\hat{Z}$ lie on the $\hat{x}$ and $\hat{y}$ axes, respectively. Define the linear functional f_{ij} by

$$f_{ij}(v) = b_{ij}(v - v_I) = -\int_{\gamma_{ij}} (A \bigtriangledown (v - v_I)) \cdot \overrightarrow{n}_{ij} ds, \tag{4.4}$$

where v_I is the linear interpolant of v in $\boldsymbol{S}_0^h$. We then have the following stronger result.

Lemma 4.1. Assume that the matrix A is constant on γ_{ij} and that E_{ij} is affine-equivalent to the reference rhombus $\hat{E}$. Then for the case of symmetric volumes, the linear functional $\hat{f}_{ij}$ defined by $\hat{f}_{ij}(\hat{v}) \equiv f_{ij}(v)$ vanishes on $\boldsymbol{P}_2(\boldsymbol{R}^2)$.

Proof. Let $v(z) = \hat{v}(\hat{z})$ and $v_I(z) = \hat{v}_I(\hat{z})$. Then

$$\hat{f}_{ij}(\hat{v}) = -\int_{\hat{\gamma}} \|\hat{n}_{ij}\|_2 (AD_{ij}^{-T} \hat{\bigtriangledown}(\hat{v} - \hat{v}_I)) \cdot (D_{ij}\bar{n}_{ij}) \frac{|\gamma_{ij}|}{|\hat{\gamma}|} d\hat{s},$$

where $\hat{n}_{ij} = D_{ij}^{-1} \overrightarrow{n}_{ij}$ and $\bar{n}_{ij} = \frac{\hat{n}_{ij}}{\|\hat{n}_{ij}\|_2}$. Apparently, $\hat{f}_{ij}(\hat{v})$ vanishes on $\boldsymbol{P}_1(\boldsymbol{R}^2)$. For any $\hat{v} \in \boldsymbol{P}_2(\boldsymbol{R}^2)$, $\hat{v}$ may be represented by

$$\hat{v} = \tilde{v} + (A_1\hat{z}, \hat{z}),$$

where A_1 is a 2×2 symmetric constant matrix and $\tilde{v} \in \boldsymbol{P}_1(\boldsymbol{R}^2)$. By the linearity of $\hat{f}_{ij}$, $\hat{\bigtriangledown}\hat{v} = 2A_1\hat{z}$ and $\int_{\hat{\gamma}} \hat{\bigtriangledown}\hat{v}_I = 0$ for $\hat{v} = (A_1\hat{z}, \hat{z})$. We thus have, for any $\hat{v} \in \boldsymbol{P}_2(\boldsymbol{R}^2)$, that

$$\begin{aligned} \hat{f}_{ij}(\hat{v}) &= -\int_{\hat{\gamma}} \|\hat{n}_{ij}\|_2 (2AD_{ij}^{-T} A_1\hat{z}) \cdot (D_{ij}\bar{n}_{ij}) \frac{|\gamma_{ij}|}{|\hat{\gamma}|} d\hat{s} \\ &= -2\|\hat{n}_{ij}\|_2 \frac{|\gamma_{ij}|}{|\hat{\gamma}|} (AD_{ij}^{-T} A_1 \int_{\hat{\gamma}} \hat{z} d\hat{s}) \cdot (D_{ij}\bar{n}_{ij}) \\ &= 0. \end{aligned}$$

Hence, the lemma is proved. ∎

Let E be any triangle (or rhombus), the respective γ and Z be any one side and its median (or diagonal) of E, and $\overrightarrow{n}$ be the unit vector which is orthogonal to γ. Let

$$h_E = \text{diam}(E)$$

and

$$\rho_E = \sup\{\text{diam}(C) : \quad C \quad \text{is a circle contained in} \quad E\}.$$

Assume that there exists a constant σ_E such that

$$h_E \leq \sigma_E |Z| \quad \text{and} \quad h_E \leq \sigma_E \rho_E. \tag{4.5}$$

Define the linear functional f by

$$f(v) = -\int_\gamma (A \bigtriangledown (v - v_I)) \cdot \overrightarrow{n} ds,$$

where v_I is interpolant of v in $\boldsymbol{S}_0^h$. We then have the following estimate of $|f(v)|$.

Lemma 4.2. Assume that $\|A\|_2$ is bounded on E, E satisfies (4.5), and E is affine-equivalent to a reference element $\hat{E} \subset \boldsymbol{R}^2$. Suppose that the linear functional $\hat{f}$ defined by $\hat{f}(\hat{v}) = f(v)$ vanishes on $\boldsymbol{P}_{\nu-1}(\boldsymbol{R}^2)$ for some $\nu \geq 2$. Then, for all $v \in \boldsymbol{H}_0^{(\nu)}(\Omega; \Gamma_0)$, we have

$$|f(v)| \leq c' \max{}_{z\in\gamma} \|A\|_2 h_E^{\nu-1} |v|_{\nu,E}, \tag{4.6}$$

where c' is a constant independent of h_E.

Proof. Let $v(z) = \hat{v}(\hat{z})$, $v_I(z) = \hat{v}_I(\hat{z})$, and $A(z) = \hat{A}(\hat{z})$. Then

$$f(v) = \hat{f}(\hat{v}) = -\int_{\hat{\gamma}} \|\hat{n}\|_2 (AD^{-T} \hat{\bigtriangledown}(\hat{v} - \hat{v}_I)) \cdot (D\bar{n}) \frac{|\gamma|}{|\hat{\gamma}|} d\hat{s},$$

where $\hat{n} = D^{-1}\overrightarrow{n}$ and $\bar{n}$ is the unit normal vector with the same direction as $\hat{n}$. Since $\|\bar{n}\|_2 = 1$, by using the Cauchy-Schwarz inequality and the Sobolev trace imbedding theorem (see [1]), we have

$$\begin{aligned}
|\hat{f}(\hat{v})| &\leq \|D^{-1}\|_2^2 \max{}_{\hat{z}\in\hat{\gamma}} \|\hat{A}(\hat{z})\|_2 \|D\|_2 \frac{|\gamma|}{|\hat{\gamma}|} \int_{\hat{\gamma}} \|\hat{\bigtriangledown}(\hat{v} - \hat{v}_I)\|_2 d\hat{s} \\
&\leq \max{}_{z\in\gamma} \|A(z)\|_2 \|D^{-1}\|_2^2 \|D\|_2 \frac{|\gamma|}{|\hat{\gamma}|} |\hat{\gamma}|^{\frac{1}{2}} \Big(\int_{\hat{\gamma}} \|\hat{\bigtriangledown}(\hat{v} - \hat{v}_I)\|_2^2 d\hat{s}\Big)^{\frac{1}{2}} \\
&\leq c_1 |\hat{\gamma}|^{-\frac{1}{2}} \max{}_{z\in\gamma} \|A(z)\|_2 \|D^{-1}\|_2^2 \|D\|_2 |\gamma| \cdot \|\hat{v} - \hat{v}_I\|_{\nu,\hat{E}},
\end{aligned}$$

where c_1 is a constant independent of $\hat{v}$. From the Bramble-Hilbert lemma (see [4]), one has

$$
\begin{aligned}
|\hat{f}(\hat{v})| &\le c_1 c_2 |\hat{\gamma}|^{-\frac{1}{2}} \max_{z\in\gamma} \|A(z)\|_2 \|D^{-1}\|_2^2 \|D\|_2 |\gamma| \cdot |\hat{v} - \hat{v}_I|_{\nu,\hat{E}} \\
&\le c_1 c_2 c_3 |\hat{\gamma}|^{-\frac{1}{2}} \max_{z\in\gamma} \|A(z)\|_2 \|D^{-1}\|_2^2 \|D\|_2^{\nu+1} |\gamma| |\det(D)|^{-\frac{1}{2}} |v|_{\nu,E}.
\end{aligned}
$$

Let $\hat{\rho} = \sup\{\mathrm{diam}(\hat{C}) : \hat{C}$ is a circle contained in $\hat{E}\}$, $\hat{h} = \mathrm{diam}(\hat{E})$, and $|E|$ and $|\hat{E}|$ be the areas of E and $\hat{E}$, respectively. We have (see [8])

$$
\|D\|_2 \le \frac{h_E}{\hat{\rho}}, \qquad \|D^{-1}\|_2 \le \frac{\hat{h}}{\rho}, \qquad |\det(D)| = \frac{|E|}{|\hat{E}|}, \quad \text{and} \quad |E| = c_4 |Z||\gamma|,
$$

where $c_4 = \frac{1}{2}\sin\theta_0$ or $\frac{1}{2}$, and θ_0 is the interior angle of E between two sides Z and γ. These facts and (4.5) imply that

$$
\begin{aligned}
|f(v)| &= |\hat{f}(\hat{v})| \\
&\le c_1 c_2 c_3 |\hat{\gamma}|^{-\frac{1}{2}} \max_{z\in\gamma} \|A(z)\|_2 \left(\frac{\hat{h}}{\rho}\right)^2 \left(\frac{h_E}{\hat{\rho}}\right)^{\nu+1} |\gamma| \left(\frac{|\hat{E}|}{c_4 |Z||\gamma|}\right)^{\frac{1}{2}} |v|_{\nu,E} \\
&\le c' \max_{z\in\gamma} \|A(z)\|_2 h_E^{\nu-1} |v|_{\nu,E},
\end{aligned}
$$

where

$$
c' = c_1 c_2 c_3 c_4^{-\frac{1}{2}} \sigma_E^{-(2+\frac{1}{2})} |\hat{\gamma}|^{-\frac{1}{2}} \hat{h}^2 |\hat{E}|^{\frac{1}{2}} \hat{\rho}^{-(\nu+1)}.
$$

This proves the lemma. ∎

Theorem 4.2. Assume that $a_{ij}(z) \in \boldsymbol{W}^1_\infty(\Omega)$, $1 \le i,j \le 2$, and $u \in \boldsymbol{H}^{(\nu)}(\Omega)$, $\nu = 2$ or 3. Suppose that B is uniformly elliptic on $\boldsymbol{S}^h_0$ and that the triangulation $\mathcal{T}^h$ and the control volumes, $\mathcal{V}^h$, are regular. Then we have:

(1) for the general case,

$$
|e|_{1,\bar{\Omega}^h} \le ch|u|_{2,\Omega}; \tag{4.7}
$$

(2) for the case of essentially symmetric control volumes,

$$
|e|_{1,\bar{\Omega}^h} \le ch^{\frac{3}{2}} \|u\|_{3,\Omega}; \tag{4.8}
$$

(3) for the case of symmetric control volumes,

$$|e|_{1,\bar{\Omega}^h} \le ch^2 \|u\|_{3,\Omega}. \tag{4.9}$$

Here, c is a constant independent of the mesh size $h = \max_{K\in\mathcal{T}^h} h_K$.

Proof. (1) By Theorem 4.1 and Lemma 4.2, we have

$$\begin{aligned}
|e|_{1,\bar{\Omega}^h} &\le \frac{1}{\alpha_2}\left(\sum_{\{i,j\}\in\omega_0} (b_{ij}(e_I)^2\right)^{\frac{1}{2}} \\
&\le \frac{\sqrt{2}}{\alpha_2}\left(\sum_{\{i,j\}\in\omega_0}\sum_{K\in\mathcal{T}^h} (\int_{\gamma_{ij}\cap K} (A\bigtriangledown e_I)\cdot \overrightarrow{n}_{ij} ds)^2\right)^{\frac{1}{2}} \\
&\le \frac{\sqrt{2}}{\alpha_2}\left(\sum_{\{i,j\}\in\omega_0} [(c'\max_{z\in\gamma'_{ij}}\|A\|_2 h_{E'_{ij}}|u|_{2,E'_{ij}})^2 + (c'\max_{z\in\gamma''_{ij}}\|A\|_2 h_{E''_{ij}}|u|_{2,E''_{ij}})^2]\right)^{\frac{1}{2}}.
\end{aligned}$$

(4.7) is thus proved with $c = \frac{\sqrt{2}c'}{\alpha_2}\max_{z\in\Omega}\|A\|_2$.

(3) For convenience, we drop the subscripts ij. Let $\bar{A} = A(\bar{z})$ where $\bar{z}$ is the midpoint of Z. Then

$$f(u) = I_1 + I_2,$$

where

$$I_1 = -\int_\gamma ((A-\bar{A})\bigtriangledown(u-u_I))\cdot\overrightarrow{n} ds$$

and

$$I_2 = -\int_\gamma (\bar{A}\bigtriangledown(u-u_I))\cdot\overrightarrow{n} ds.$$

Since $a_{ij}(z)\in \boldsymbol{W}^1_\infty(\Omega),\ 1\le i,j\le 2$, then

$$\max_{z\in\gamma}\|A-\bar{A}\|_2 \le c'' h_E,$$

where c'' is a constant independent of h_E. It follows from Lemma 4.2 that

$$|I_1| \le c'c'' h_E^2 |u|_{2,E}$$

and

$$|I_2| \le c' \|\bar{A}\|_2 h_E^2 |u|_{3,E}.$$

(4.9) is now proved with $c = \frac{1}{\alpha_2}\max\{c'',\ \max_{\hat{z}\in\Omega}\|\bar{A}\|_2\}$.

(2) Let $\omega_1 = \{\{i,j\} \in \omega :\ E_{ij} \cap K \subset \tilde{\Omega}$ for some $K \in \mathcal{T}^h\}$. By (1) and (3) we know that

$$\begin{aligned} |e|_{1,\bar{\Omega}^h} &\le \frac{1}{\alpha\, 2}\left(\sum_{\{i,j\}\in\omega_1} + \sum_{\{i,j\}\in\omega\setminus\omega_1} (b_{ij}(e_I))^2\right)^{\frac{1}{2}} \\ &\le \frac{c}{\alpha_2}(h^2|u|^2_{2,\tilde{\Omega}} + h^4\|u\|^2_{3,\Omega\setminus\tilde{\Omega}})^{\frac{1}{2}}. \end{aligned}$$

(4.8) thus follows from the observation that

$$\|u\|_{3,\Omega\setminus\tilde{\Omega}} \le \|u\|_{3,\Omega}$$

and that (see [5, 17])

$$|u|_{2,\tilde{\Omega}} \le \tilde{c} h^{\frac{1}{2}} \|u\|_{3,\Omega},$$

where $\tilde{c}$ is a constant independent of the mesh size h. ∎

5. Sufficient Conditions for Uniform Ellipticity of B

In the previous section, we established error estimates for FVE under the assumption that B is uniformly elliptic. Uniform ellipticity of B is established in [6] for the case that control volumes are defined by circumcenters and A is diagonal. Here we first give a sufficient condition for uniform ellipticity of B for the case that control volumes which are not defined by the circumcenters and $A = a(z)I$. We will then consider the circumcenter case for general A.

Proposition 5.1. Assume that $A = a(z)I$ and that it satisfies the condition (2.2). Then there exists an $h_0 > 0$ and a constant $\alpha > 0$, dependent only on h_0 and α_1, such that, for all $h \le h_0$, (3.9) holds with constant $\alpha_2 = \alpha$. Furthermore, if $a(z) \equiv 1$, then (3.9) holds with $\alpha_2 = 1$ for any mesh size $h > 0$.

Proof. The proof is just a straightforward consequence of the Lemma 3 in [3] and its analogue. $\blacksquare$

We now give a sufficient condition for the case that A is not diagonal. However, for this theory we restrict our attention to the case that the control volume are associated with circumcenters.

For each element $K \in \cup_h \mathcal{T}^h$, denote the vertices, side segments, and control volume segments of K by a_i, Z_i and γ_i, respectively, and let $\overrightarrow{n}_i$ be the unit normal vector on γ_i, $i = 1, 2, 3$, (see Figure 5.1). We first establish (3.5) for the case that A is constant.

Lemma 5.1. Suppose that the matrix A is constant on each $K \in \cup_h \mathcal{T}^h$ and that it satisfies the uniformly ellipticity condition (2.2). If each element is either a right or isosceles triangle, then B is uniformly elliptic with constant $\alpha_2 = \alpha_1 \sigma^{-1}$.

Proof. Note that

$$\begin{aligned}
\sum_{i=1}^{n} v(z_i)(Bv)_i &= \sum_{\{i,j\}\in\omega_0} (v(z_j) - v(z_i)) b_{ij}(v) \\
&= \sum_{\{i,j\}\in\omega_0} |Z_{ij}| (\bigtriangledown v, \overrightarrow{n}_{ij}) \sum_{K\in\cup_h\mathcal{T}^h} \int_{\gamma_{ij}\cap K} (A \bigtriangledown v, \overrightarrow{n}_{ij}) ds \\
&= \sum_{K\in\cup_h\mathcal{T}^h} w_K,
\end{aligned}$$

where

$$w_K = \sum_{i=1}^{3} |Z_i||\gamma_i| (\bigtriangledown v, \overrightarrow{n}_i)(A \bigtriangledown v, \overrightarrow{n}_i).$$

Suppose that the element K is isosceles. Without loss of generality, assume that the vertex angle, which is the angle formed by two equal sides, is θ. Since

$$\overrightarrow{n}_3 = -\frac{\overrightarrow{n}_1 + \overrightarrow{n}_2}{\|\overrightarrow{n}_1 + \overrightarrow{n}_2\|},$$

then

$$
\begin{aligned}
w_K &= (A \bigtriangledown v, \sum_{i=1}^{3} |Z_i||\gamma_i|(\bigtriangledown v, \overrightarrow{n}_i)\overrightarrow{n}_i) \\
&= (A \bigtriangledown v, [(|Z_1||\gamma_1| + \frac{|Z_3||\gamma_3|}{a})(\bigtriangledown v, \overrightarrow{n}_1) + \frac{|Z_3||\gamma_3|}{a}(\bigtriangledown v, \overrightarrow{n}_2)]\overrightarrow{n}_1 \\
&\quad + [\frac{|Z_3||\gamma_3|}{a}(\bigtriangledown v, \overrightarrow{n}_1) + (|Z_2||\gamma_2| + \frac{|Z_3||\gamma_3|}{a})(\bigtriangledown v, \overrightarrow{n}_2)]\overrightarrow{n}_2),
\end{aligned}
$$

where $a = \|\overrightarrow{n}_1 + \overrightarrow{n}_2\|^2 = 2(1 + cos\langle \overrightarrow{n}_1, \overrightarrow{n}_2 \rangle) = 2(1 - cos\theta)$. We now proceed to obtain a lower bound on w_K. First note that, since $\overrightarrow{n}_1$ and $\overrightarrow{n}_2$ are linearly independent in $\boldsymbol{R}^2$, we may let

$$\bigtriangledown v = v_1 \overrightarrow{n}_1 + v_2 \overrightarrow{n}_2.$$

But then

$$(\bigtriangledown v, \overrightarrow{n}_1) = v_1 - v_2 cos\theta \quad \text{and} \quad (\bigtriangledown v, \overrightarrow{n}_2) = -v_1 cos\theta + v_2.$$

Solving these two equations for v_1 and v_2 yields

$$v_1 = \frac{1}{sin^2\theta}[(\bigtriangledown v, \overrightarrow{n}_1) + (\bigtriangledown v, \overrightarrow{n}_2)cos\theta]$$

and

$$v_2 = \frac{1}{sin^2\theta}[(\bigtriangledown v, \overrightarrow{n}_1)cos\theta + (\bigtriangledown v, \overrightarrow{n}_2)].$$

Note that $|Z_3||\gamma_3| = 2cos\theta|Z_1||\gamma_1| = 2cos\theta|Z_2||\gamma_2|$. Then $w_K = (A \bigtriangledown v, q)$ where

$$
\begin{aligned}
q &= [(|Z_1||\gamma_1| + \frac{|Z_3||\gamma_3|}{a})(\bigtriangledown v, \overrightarrow{n}_1) + \frac{|Z_3||\gamma_3|}{a}(\bigtriangledown v, \overrightarrow{n}_2)]\overrightarrow{n}_1 1 \\
&\quad + [\frac{|Z_3||\gamma_3|}{a}(\bigtriangledown v, \overrightarrow{n}_1) + (|Z_2||\gamma_2| + \frac{|Z_3||\gamma_3|}{a})(\bigtriangledown v, \overrightarrow{n}_2)]\overrightarrow{n}_2 \\
&= |Z_1||\gamma_1|\{[(1 + \frac{2cos\theta}{2(1-cos\theta)})(\bigtriangledown v, \overrightarrow{n}_1) + \frac{2cos\theta}{2(1-cos\theta)}(\bigtriangledown v, \overrightarrow{n}_2)]\overrightarrow{n}_1 \\
&\quad + [\frac{2cos\theta}{2(1-cos\theta)}(\bigtriangledown v, \overrightarrow{n}_1) + (1 + \frac{2cos\theta}{2(1-cos\theta)})(\bigtriangledown v, \overrightarrow{n}_2)]\overrightarrow{n}_2\} \\
&= \frac{|Z_1||\gamma_1|}{1-cos\theta}\{[(\bigtriangledown v, \overrightarrow{n}_1) + cos\theta(\bigtriangledown v, \overrightarrow{n}_2)]\overrightarrow{n}_1 + [cos\theta(\bigtriangledown v, \overrightarrow{n}_1) + (\bigtriangledown v, \overrightarrow{n}_2)]\overrightarrow{n}_2\} \\
&= |Z_1||\gamma_1|\frac{sin^2\theta}{1-cos\theta} \bigtriangledown v.
\end{aligned}
$$

Hence,

$$\begin{aligned} w_K &= (A \bigtriangledown v, q) \\ &= |Z_1||\gamma_1| \frac{sin^2\theta}{1 - cos\theta}(A \bigtriangledown v, \bigtriangledown v) \\ &\geq \alpha_1 |Z_1||\gamma_1| \frac{sin^2\theta}{1 - cos\theta}(\bigtriangledown v, \bigtriangledown v). \end{aligned}$$

In other words, we can bound w_K from below by replacing A by α_1. Returning to the definition of w_K, this means that

$$\begin{aligned} w_K &\geq \alpha_1 \sum_{i=1}^{3} (\bigtriangledown v, \overrightarrow{n}_i)(\bigtriangledown v, \overrightarrow{n}_i)|Z_i||\gamma_i| \\ &= \alpha_1 [(v(z_2) - v(z_1))^2 \frac{|\gamma_1|}{|Z_1|} + (v(z_3) - v(z_2))^2 \frac{|\gamma_2|}{|Z_2|} + (v(z_1) - v(z_3))^2 \frac{|\gamma_3|}{|Z_3|}] \\ &\geq \alpha_1 \sigma^{-1} [(v(z_2) - v(z_1))^2 + (v(z_3) - v(z_2))^2 + (v(z_1) - v(z_3))^2], \end{aligned}$$

where σ is from (3.1). For the case of that the element K is a right triangle, the above inequality of w_K follows from observations that

$$|\gamma_3| = 0, \qquad |Z_1||\gamma_1| = |Z_2||\gamma_2|, \quad \text{and} \quad q = |Z_1||\gamma_1| \bigtriangledown v.$$

These prove the lemma. ∎

Theorem 5.1. Consider the case that the control volumes are formed from element circumcenters. Assume that $a_{ij}(z) \in \boldsymbol{W}^1_\infty(\Omega)$, $1 \leq i, j \leq 2$, and that each triangle $K \in \mathcal{T}^h$ is either right or isosceles. Then there exists an $h_0 > 0$ and a constant $\alpha > 0$, dependent only on h_0, such that, for all $h \leq h_0$, B is uniformly elliptic, i.e., B satisfies (3.9) with constant $\alpha_2 = \alpha$.

Proof. Let $\bar{A}(z) = A(z_K)$ for all $z \in K \subset \mathcal{T}^h$, where z_K is the circumcenter of triangle K. Since $a_{ij} \in \boldsymbol{W}^1_\infty(\Omega)$, $1 \leq i, j \leq 2$, then there exists a constant c_1 such that, for all $K \in \mathcal{T}^h$,

$$\max_{z \in K} \|\bar{A}(z) - A(z)\| \leq c_1 h.$$

Let

$$\tilde{w}_K = \sum_{i=1}^{3} |Z_i| (\bigtriangledown v, \overrightarrow{n}_i) \int_{\gamma_i} ((A - \bar{A}) \bigtriangledown v, \overrightarrow{n}_i) ds,$$

then

$$
\begin{aligned}
|\tilde{w}_K| &\leq \sum_{i=1}^{3} |Z_i||\gamma_i|\| \bigtriangledown v\|^2 \max_{z\in K} \|A - \bar{A}\| \\
&\leq c_1 h \sum_{i=1}^{3} |Z_i||\gamma_i|\| \bigtriangledown v\|^2 \\
&\leq c_1 c_2 h \sum_{i=1}^{3} |Z_i||\gamma_i|(\bigtriangledown v, \overrightarrow{n}_i)^2.
\end{aligned}
$$

The last inequality follows from the fact that there exists a constant $c_2 > 0$ such that, for any $K \in \mathcal{T}^h$,

$$
\sum_{i=1}^{3} |Z_i||\gamma_i| \leq c_2 \sum_{i=1}^{3} |Z_i||\gamma_i| cos^2\langle \bigtriangledown v, \overrightarrow{n}_i\rangle.
$$

We then have

$$
\begin{aligned}
\sum_{i=1}^{n} v(z_i)(Bv)_i = & \sum_{\{i,j\}\in\omega_0} |Z_{ij}|(\bigtriangledown v, \overrightarrow{n}_{ij}) \sum_{K\in\mathcal{T}^h} \int_{\gamma_{ij}} (\bar{A} \bigtriangledown v, \overrightarrow{n}_{ij}) ds \\
& + \sum_{\{i,j\}\in\omega_0} |Z_{ij}|(\bigtriangledown v, \overrightarrow{n}_{ij}) \int_{\gamma_{ij}} ((A - \bar{A}) \bigtriangledown v, \overrightarrow{n}_{ij}) ds \\
= & \sum_{K\in\mathcal{T}^h} (w_K + \tilde{w}_K) \\
\geq & \alpha_1 \sigma^{-1} |v|^2_{1,\bar{\Omega}^h} - c_1 c_2 h |v|^2_{1,\bar{\Omega}^h}.
\end{aligned}
$$

Hence, the theorem follows from chosing h_0 such that, for all $h \leq h_0$,

$$
\alpha = \alpha_1 \sigma_{-1} - c_1 c_2 h > 0. \quad \blacksquare
$$

6. The Effects of Numerical Integration

Denote the basis of the space $\boldsymbol{S}_0^h$ by $\{\phi_l(z)\}_{l=1}^n$, where $\phi_l(z)$ is the hat function associated with z_l, that is, $\phi_l(z)$ is linear on each triangle element $K \in \mathcal{T}^h$ and $\phi_l(z_i) = \delta_{li}$ for all $1 \leq l, i \leq n$. Then solving the corresponding discrete problem amounts to finding the coefficients $u_l,\ 1 \le l \le n$, of the expansion $u^h = \sum_{l=1}^n u_l \phi_l$. These coefficients are solution of the linear system

$$\sum_{l=1}^{n} (B\phi_l)_i u_l = f_i, \qquad 1 \le i \le n, \tag{6.1}$$

where

$$f_i = \int_{V_i} f dz = \sum_{K \in \mathcal{T}^h} \int_{V_i \cap K} f dz. \tag{6.2}$$

Let $\overrightarrow{n}_{ij}$ be the unit outward normal vector on γ_{ij}, which is associated with V_i. Then, for any $v \in \boldsymbol{S}_0^h$, we have

$$\begin{aligned} -\int_{\partial V_i} (A \bigtriangledown v) \cdot \overrightarrow{n}_i ds &= -\sum_{j \in \omega_i} \int_{\gamma_{ij}} (A \bigtriangledown v) \cdot \overrightarrow{n}_{ij} ds \\ &= -\sum_{j \in \omega_i} \sum_{K \in \mathcal{T}^h} \int_{\gamma_{ij} \cap K} (A \bigtriangledown v) \cdot \overrightarrow{n}_{ij} ds \\ &= -\sum_{j \in \omega_i} \sum_{K \in \mathcal{T}^h} (\int_{\gamma_{ij} \cap K} A ds) \bigtriangledown v \cdot \overrightarrow{n}_{ij}. \end{aligned}$$

The last equality follows from the fact that $\bigtriangledown v$ and $\overrightarrow{n}_{ij}$ are constant on $\gamma_{ij} \cap K$. Therefore, (3.10) can be rewritten as

$$(Bv)_i = -\sum_{j \in \omega_i} \sum_{K \in \mathcal{T}^h} (\int_{\gamma_{ij} \cap K} A ds) \bigtriangledown v \cdot \overrightarrow{n}_{ij}. \tag{6.3}$$

Note that the integral of a matrix $M = (m_{ij})_{n \times n}$ is defined componentwise by

$$\int M d\mu = (\int m_{ij} d\mu)_{n \times n}.$$

In practice, even if the functions a_{ij} and f have simple analytical expressions, the integrals $\int_{\gamma_{ij} \cap K} A(z) ds$ and $\int_{V_i \cap K} f(z) dz$ which appear in (6.3) and (6.2), respectively, are seldom computed exactly. To study the effects of numerical integration, we will assume that $a_{ij}(z)$ and $f(z)$ are defined everywhere on the respective $\{\partial V_i : i = 1, ..., m\}$ and $\bar{\Omega}$. For each $1 \le i \le n$, $j \in \omega_i$, and $K \in \mathcal{T}^h$, let

$$F_{ij,K} : \hat{z} \in \hat{K} \longrightarrow F_{ij,K}(\hat{z}) = D_{ij,K} \hat{z} + d_{ij,K} \tag{6.4}$$

be the invertible affine mapping from $\hat{K}$ onto K. Assume without loss of generality that the Jacobian of the mapping $F_{ij,K}$ is positive. Let $\hat{\gamma}$ be such that $F_{ij,K}(\hat{\gamma}) = \gamma_{ij} \cap K$ and $\hat{V}$ be such that $F_{ij,K}(\hat{V}) = V_i \cap K$. Then

$$\int_{\gamma_{ij}\cap K} A(z)ds = \frac{|\gamma_{ij} \cap K|}{|\hat{\gamma}|} \int_{\hat{\gamma}} \hat{A}(\hat{z})d\hat{s} \tag{6.5}$$

and

$$\int_{V_i\cap K} f(z)dz = det(D_{ij,K}) \int_{\hat{V}} \hat{f}(\hat{z})d\hat{z}, \tag{6.6}$$

where the hat quantities are defined by the usual correspondence, i.e., $A(z) = \hat{A}(\hat{z})$ and $f(z) = \hat{f}(\hat{z})$ for all $z = F_{ij,K}(\hat{z})$, $\hat{z} \in \hat{K}$. In other words, computing the integrals $\int_{\gamma_{ij}\cap K} Ads$ and $\int_{V_i\cap K} fdz$ amounts to computing the respective integrals $\int_{\hat{\gamma}} \hat{A}(\hat{z})d\hat{s}$ and $\int_{\hat{V}} \hat{f}(\hat{z})d\hat{z}$. Let $\hat{\beta}_{l,\hat{\gamma}}$ $(l = 1, ..., L_1)$ and $\hat{\beta}_{l,\hat{V}}$ $(l = 1, ..., L_2)$ be the quadrature coefficients, and $\hat{\eta}_{l,\hat{\gamma}}$ $(l = 1, ..., L_1)$ and $\hat{\eta}_{l,\hat{V}}$ $(l = 1, ..., L_2)$ be the quadrature points. Assume that $\hat{\beta}_{l,\hat{\gamma}}$ and $\hat{\beta}_{l,\hat{V}}$ are strictly positive, and $\hat{\eta}_{l,\hat{\gamma}} \in \hat{\gamma}$, $\hat{\eta}_{l,\hat{V}} \in \hat{V}$. Then the integrals $\int_{\hat{\gamma}} \hat{A}(\hat{z})d\hat{s}$ and $\int_{\hat{V}} \hat{f}(\hat{z})d\hat{z}$ are approximated by quatratures as follows:

$$\int_{\hat{\gamma}} \hat{A}(\hat{z})d\hat{s} \sim \sum_{l=1}^{L_1} \hat{\beta}_{l,\hat{\gamma}} \hat{A}(\hat{\eta}_{l,\hat{\gamma}}) \tag{6.7}$$

and

$$\int_{\hat{V}} \hat{f}(\hat{z})d\hat{z} \sim \sum_{l=1}^{L_2} \hat{\beta}_{l,\hat{V}} \hat{f}(\hat{\eta}_{l,\hat{V}}). \tag{6.8}$$

For $1 \le l \le L_1$, define

$$\beta_{l,\gamma_{ij}\cap K} = \frac{|\gamma_{ij} \cap K|}{|\hat{\gamma}|} \hat{\beta}_{l,\hat{\gamma}} \quad \text{and} \quad \eta_{l,\gamma_{ij}\cap K} = F_{ij,K}(\hat{\eta}_{l,\hat{\gamma}}) \tag{6.9}$$

and, for $1 \le l \le L_2$, define

$$\beta_{l,V_i\cap K} = det(D_{ij,K}) \hat{\beta}_{l,\hat{V}} \quad \text{and} \quad \eta_{l,V_i\cap K} = F_{ij,K}(\hat{\eta}_{l,\hat{V}}). \tag{6.10}$$

Then the quadrature approximations over element K are given by

$$\int_{\gamma_{ij}\cap K} A(z)ds \sim \sum_{l=1}^{L_1} \beta_{l,\gamma_{ij}\cap K} A(\eta_{l,\gamma_{ij}\cap K}) \tag{6.11}$$

and

$$\int_{V_i\cap K} f(z)dz \sim \sum_{l=1}^{L_2} \beta_{l,V_i\cap K} f(\eta_{l,V_i\cap K}). \tag{6.12}$$

Accordingly, we introduce the following quadrature error functionals: for any $1 \le i \le n$, $j \in \omega_i$ and each $K \in \mathcal{T}^h$, define

$$R_{\gamma_{ij}\cap K}(A) = \int_{\gamma_{ij}\cap K} A(z)ds - \sum_{l=1}^{L_1} \beta_{l,\gamma_{ij}\cap K} A(\eta_{l,\gamma_{ij}\cap K}), \tag{6.13}$$

$$\hat{R}_{\hat{\gamma}}(\hat{A}) = \int_{\hat{\gamma}} \hat{A}(\hat{z})d\hat{s} - \sum_{l=1}^{L_1} \hat{\beta}_{l,\hat{\gamma}} \hat{A}(\hat{\eta}_{l,\hat{\gamma}}), \tag{6.14}$$

$$R_{V_i\cap K}(f) = \int_{V_i\cap K} f(z)dz - \sum_{l=1}^{L_2} \beta_{l,V_i\cap K} f(\eta_{l,V_i\cap K}), \tag{6.15}$$

and

$$\hat{R}_{\hat{V}}(\hat{f}) = \int_{\hat{V}} \hat{f}(\hat{z})d\hat{z} - \sum_{l=1}^{L_2} \hat{\beta}_{l,\hat{V}} \hat{f}(\hat{\eta}_{l,\hat{V}}). \tag{6.16}$$

These are related by

$$R_{\gamma_{ij}\cap K}(A) = \frac{|\gamma_{ij} \cap K|}{|\hat{\gamma}|} \hat{R}_{\hat{\gamma}}(\hat{A}) \tag{6.17}$$

and

$$R_{V_i\cap K}(f) = det(D_{ij,K}) \hat{R}_{\hat{V}}(\hat{f}). \tag{6.18}$$

Let

$$\tilde{f}_i^h \equiv \sum_{K\in\mathcal{T}^h} \tilde{f}_{i,K}^h = \sum_{K\in\mathcal{T}^h} \sum_{l=1}^{L_2} \beta_{l,V_i\cap K} f(\eta_{l,V_i\cap K}) \tag{6.19}$$

and define the linear operator $\tilde{B} : \boldsymbol{S}_0^h \longrightarrow \boldsymbol{R}^n$ by its elements:

$$(\tilde{B}v)_i = -\sum_{j\in\omega_i} \sum_{K\in\mathcal{T}^h} \sum_{l=1}^{L_1} \beta_{l,\gamma_{ij}\cap K} A(\eta_{l,\gamma_{ij}\cap K}) \bigtriangledown v \cdot \overrightarrow{n}_{ij}, 1 \le i \le n. \tag{6.20}$$

As in Section 3, we say $\tilde{B}$ *uniformly elliptic* on $\boldsymbol{S}_0^h$ if there exists a constant $\tilde{\alpha} > 0$ independent of the space $\boldsymbol{S}_0^h$ such that, for all $v \in \boldsymbol{S}_0^h$,

$$\sum_{i=1}^{n} v(z_i)(\tilde{B}v)_i \ge \tilde{\alpha}|v|^2_{1,\bar{\Omega}^h}. \tag{6.21}$$

Now, instead of solving the linear system (6.1) with the coefficients (6.3) and right-hand sides (6.2), we solve the modified linear system

$$\sum_{i=1}^{n} (\tilde{B}\phi_k)_i u_k = \tilde{f}_i^h, \qquad 1 \leq i \leq n. \tag{6.22}$$

For our subsequent analysis, rather than working with the matrix system (6.22), it will be more convenient to consider the following equivalent formulation of the discrete problem:

Find $u^h \in \boldsymbol{S}_0^h$ such that

$$\tilde{B}u^h = \tilde{f}^h \tag{6.23}$$

where $\tilde{f}^h$ is the vector in $\boldsymbol{R}^n$ with components $\tilde{f}_i^h$, $1 \leq i \leq n$.

For any $v \in \boldsymbol{S}_0^h$, define the discrete $\boldsymbol{L}^2$ norm by

$$|v|_{0,\bar{\Omega}^h} = (\sum_{\{i,j\}\in\omega} v^2(z_i))^{\frac{1}{2}}, \tag{6.24}$$

For any vector $w \in \boldsymbol{R}^n$, define the $\boldsymbol{l}^2$ norm by

$$|w|_{0,\bar{\Omega}^h} = (\sum_{i=1}^{n} w_i^2)^{\frac{1}{2}}. \tag{6.25}$$

We now have the following basic error estimate.

Theorem 6.1. Assume that $\tilde{B}$ satisfies (6.21). Then

$$|e|_{1,\bar{\Omega}^h} \leq \frac{1}{\tilde{\alpha}}\{c'|f^h - \tilde{f}^h|_{0,\bar{\Omega}^h} + (\sum_{\{i,j\}\in\omega_0} b_{ij}^2(e_I))^{\frac{1}{2}} + [\sum_{\{i,j\}\in\omega_0} (\sum_{K\in\mathcal{T}^h} (R_{\gamma_{ij}\cap K}(A) \bigtriangledown u_I) \cdot \overrightarrow{n}_{ij})^2]^{\frac{1}{2}}\}, \tag{6.26}$$

where c' is a constant dependent only on the domain Ω.

Proof. By the linearity of B and $\tilde{B}$, (2.3), and (6.23), we have

$$\begin{aligned}(\tilde{B}e^h)_i &= (\tilde{B}u_I)_i - (Bu_I)_i + (Bu_I)_i - (Bu)_i + (Bu)_i - (\tilde{B}u^h)_i \\ &= ((\tilde{B}-B)u_I)_i - (Be_I)_i + (f_i^h - \tilde{f}_i^h).\end{aligned}$$

¿From this, (6.21), and the fact that $e(z_i) = e^h(z_i)$, $1 \le i \le m$, it follows that

$$\begin{aligned}&\tilde{\alpha}|e|^2_{1,\bar{\Omega}^h} = \tilde{\alpha}|e^h|^2_{1,\bar{\Omega}^h} \\ &\le \sum_{i=1}^n e^h(z_i)(\tilde{B}e^h)_i \\ &\le |\sum_{i=1}^n e^h(z_i)((\tilde{B}-B)u_I)_i| + |\sum_{i=1}^n e^h(z_i)(Be_I)_i| + |\sum_{i=1}^n e^h(z_i)(f_i^h - \tilde{f}_i^h)|. \qquad (6.27)\end{aligned}$$

According to (6.20) and (6.3) and since $\overrightarrow{n}_{ij} = -\overrightarrow{n}_{ji}$, we then have

$$\begin{aligned}&|\sum_{i=1}^n e^h(z_i)((\tilde{B}-B)u_I)_i| \\ &= |\sum_{i=1}^n e^h(z_i) \sum_{j\in\omega_i} \sum_{K\in\mathcal{T}^h} (\int_{\gamma_{ij}\cap K} A ds - \sum_{l=1}^{L_1} \beta_{l,\gamma_{ij}\cap K} A(\eta_{l,\gamma_{ij}\cap K})) \bigtriangledown u_I \cdot \overrightarrow{n}_{ij}| \\ &= |\sum_{\{i,j\}\in\omega_0} (e^h(z_i) - e^h(z_j)) \sum_{K\in\mathcal{T}^h} R_{\gamma_{ij}\cap K}(A) \bigtriangledown u_I \cdot \overrightarrow{n}_{ij}| \\ &\le |e^h|_{1,\bar{\Omega}^h} [\sum_{\{i,j\}\in\omega_0} (\sum_{K\in\mathcal{T}^h} R_{\gamma_{ij}\cap K}(A) \bigtriangledown u_I \cdot \overrightarrow{n}_{ij})^2]^{\frac{1}{2}}. \qquad (6.28)\end{aligned}$$

The last inequality follows from the Cauchy-Schwarz inequality. ¿From the proof of Theorem 4.1, we know that

$$|\sum_{i=1}^n e^h(z_i)(Be_I)_i| \le |e^h|_{1,\bar{\Omega}^h} (\sum_{\{i,j\}\in\omega_0} b_{ij}^2(e_I))^{\frac{1}{2}}. \qquad (6.29)$$

By the Cauchy-Schwarz inequality and the Poincaré-Friedrichs inequality (see [8]), we have

$$\begin{aligned}|\sum_{i=1}^n e^h(z_i)(f_i^h - \tilde{f}_i^h)| &\le |e^h|_{0,\bar{\Omega}^h} |f^h - \tilde{f}^h|_{0,\bar{\Omega}^h} \\ &\le c'|e^h|_{1,\bar{\Omega}^h} |f^h - \tilde{f}^h|_{0,\bar{\Omega}^h}, \qquad (6.30)\end{aligned}$$

where c' is a constant dependent only on the domain Ω. (6.17) now follows from inequalities (6.27), (6.28), (6.29), and (6.30). Hence, the theorem is proved. ∎

Estimating the discretization error now amounts to bounding the three terms on the right-hand side of inequality (6.27). We can use the assumptions and proof of Theorem 4.2 to show that the second term, $\left(\sum_{\{i,j\}\in\omega_0} b_{ij}^2(e_I)\right)^{\frac{1}{2}}$, is $O(h^\nu)$, where $\nu = 1$ in general, $\nu = \frac{3}{2}$ for essentially symmetric control volumes, and $\nu = 2$ for symmetric control volumes. The other two terms are due to numerical quadrature. Thus, our present objective is to give sufficient conditions on the quadratures which ensure that the effect of numerical integration does not decrease these orders of convergence.

Lemma 6.1. Assume that, for some integer $\nu \geq 1$, the quadrature error functional $\hat{R}_{\hat{V}}(\hat{v})$ vanishes on $\boldsymbol{P}_{\nu-1}(\hat{V})$. Then, for any $q > \frac{2}{\nu}$ and any $f \in \boldsymbol{W}_q^\nu(V_i \cap K)$,

$$|R_{V_i\cap K}(f)| \leq c h_K^\nu |V_i \cap K|^{1-\frac{1}{q}} |f|_{\nu,q,V_i\cap K}, \tag{6.31}$$

where $|V_i \cap K|$ is the measure of $V_i \cap K$ and c is a constant independent of $K \in \mathcal{T}^h$ and h_K.

Proof. $f \in \boldsymbol{W}_q^\nu(V_i \cap K)$ implies that $\hat{f} \in \boldsymbol{W}_q^\nu(\hat{V})$ and $q > \frac{2}{\nu}$ implies that $\boldsymbol{W}_q^\nu(\hat{V})$ is imbedded into $\boldsymbol{C}^0(\hat{V})$. It follows from the Sobolev imbedding theorem that

$$|\hat{R}_{\hat{V}}(\hat{f})| \leq c_1 |\hat{f}|_{0,\infty,\hat{V}} \leq c_2 \|\hat{f}\|_{\nu,q,\hat{V}}.$$

By the Bramble-Hilbert lemma, we have

$$|\hat{R}_{\hat{V}}(\hat{f})| \leq c_3 |\hat{f}|_{\nu,q,\hat{V}}.$$

Now (6.31) follows from the facts that

$$|\hat{f}|_{\nu,q,\hat{V}} \leq c_4 \|D_{ij,K}\|^\nu |det(D_{ij,K})|^{-\frac{1}{q}} |f|_{\nu,q,V_i\cap K},$$

$$\|D_{ij,K}\| \leq \frac{h_K}{\hat{\rho}}, \qquad |det(D_{ij,K})| = \frac{|V_i \cap K|}{|\hat{V}|},$$

and

$$R_{V_i \cap K}(f) = det(D_{ij,K}) \hat{R}_{\hat{V}}(\hat{f}),$$

where $|\hat{V}|$ is the measure of $\hat{V}$. Hence, the lemma is proved. ∎

Without loss of generality, assume that the line segment $\hat{\gamma}$ lies on $\hat{x}$ axis. We omit the proof of the next lemma since it is virtually the same as that for Lemma 6.1.

Lemma 6.2. Assume that, for some integer $\nu \geq 1$, the quadrature error functional $\hat{R}_{\hat{\gamma}}(\hat{v})$ vanishes on $\boldsymbol{P}_{\nu-1}(\hat{\gamma})$. Then, for any $v \in \boldsymbol{W}^{\nu}_{\infty}(\gamma_{ij} \cap K)$ $(1 \leq i \leq n, j \in \omega_i)$, we have

$$|R_{\gamma_{ij} \cap K}(v)| \leq c h_K^{\nu+1} |v|_{\nu, \gamma_{ij} \cap K}, \tag{6.32}$$

where c is a constant independent of $K \in \mathcal{T}^h$ and h_K.

Note that $|\frac{\partial u_I}{\partial x}|$ and $|\frac{\partial u_I}{\partial y}|$ are both bounded by $c h_K^{-\frac{1}{2}} \|u\|_{2, E_{ij} \cap K}$ for all $z \in K$. According to Lemma 6.2, we then have that

$$\begin{aligned} |R_{\gamma_{ij} \cap K}(A) \bigtriangledown u_I \cdot \overrightarrow{n}_{ij}| &\leq \sum_{l=1}^{2} (|R_{\gamma_{ij} \cap K}(a_{l1})| |\frac{\partial u_I}{\partial x}| + |R_{\gamma_{ij} \cap K}(a_{2l})| |\frac{\partial u_I}{\partial y}|) \\ &\leq c h_K^{\nu + \frac{1}{2}} \sum_{l,k=1}^{2} |a_{lk}|_{\nu, \infty, \gamma_{ij} \cap K} \|u\|_{2, E_{ij} \cap K}, \end{aligned}$$

where c absorbs various constants. Therefore, application of the Cauchy-Schwarz inequality implies that

$$\begin{aligned} &\left(\sum_{K \in \mathcal{T}^h} (R_{\gamma_{ij} \cap K}(A) \bigtriangledown u_I) \cdot \overrightarrow{n}_{ij} \right)^2 \\ &\leq c \sum_{K \in \mathcal{T}^h} ((R_{\gamma_{ij} \cap K}(A) \bigtriangledown u_I) \cdot \overrightarrow{n}_{ij})^2 \\ &\leq c h^{2\nu+1} \sum_{K \in \mathcal{T}^h} (\sum_{l,k=1}^{2} |a_{lk}|_{\nu, \infty, \gamma_{ij} \cap K})^2 \|u\|^2_{2, E_{ij} \cap K} \\ &\leq c h^{2\nu+1} \sum_{K \in \mathcal{T}^h} \sum_{l,k=1}^{2} |a_{lk}|^2_{\nu, \infty, \gamma_{ij} \cap K} \|u\|^2_{2, E_{ij} \cap K} \\ &\leq c h^{2\nu+1} \sum_{l,k=1}^{2} |a_{lk}|^2_{\nu, \infty, \gamma_{ij}} \|u\|^2_{2, E_{ij}}. \end{aligned}$$

Hence, the third term on the right-hand side of inequality (6.26) is bounded according to

$$\left(\sum_{\{i,j\}\in\omega} (\sum_{K\in\mathcal{T}^h} (R_{\gamma_{ij}\cap K}(A) \bigtriangledown u_I) \cdot \overrightarrow{n}_{ij})^2 \right)^{\frac{1}{2}}$$
$$\leq ch^{\nu+\frac{1}{2}} \|u\|_{2,\Omega} \sum_{l,k=1}^{2} |a_{lk}|_{\nu,\infty,\Omega}. \tag{6.33}$$

To estimate the first term on the right-hand side of (6.26), note from the definitions of f_i^h, $\tilde{f}_i^h$, and $R_{V_i\cap K}(f)$ that

$$f_i^h - \tilde{f}_i^h = \sum_{K\in\mathcal{T}^h} R_{V_i\cap K}(f).$$

Since $|V_i \cap K| \leq h^2$, then by Lemma 6.1 and the Cauchy- Schwarz inequality we have

$$\begin{aligned} |f_i^h - \tilde{f}_i^h| &\leq \sum_{K\in\mathcal{T}^h} |R_{V_i\cap K}(f)| \\ &\leq ch^{\nu+2(1-\frac{1}{q})} \sum_{K\in\mathcal{T}^h} |f|_{\nu,q,V_i\cap K} \\ &\leq ch^{\nu+2(1-\frac{1}{q})} (\sum_{K\in\mathcal{T}^h} |f|^q_{\nu,q,V_i\cap K})^{\frac{1}{q}} \\ &= ch^{\nu+2(1-\frac{1}{q})} |f|_{\nu,q,V_i}. \end{aligned}$$

Thus, since $n \leq ch^{-2}$, we have

$$\begin{aligned} (\sum_{i=1}^{n} |f_i^h - \tilde{f}_i^h|^2)^{\frac{1}{2}} &\leq ch^{\nu+2(1-\frac{1}{q})} (\sum_{i=1}^{n} |f|^2_{\nu,q,V_i})^{\frac{1}{2}} \\ &\leq ch^{\nu+2(1-\frac{1}{q})} |f|_{\nu,q,\Omega} n^{\frac{q-2}{2q}} \\ &\leq ch^{\nu+1} |f|_{\nu,q,\Omega}. \end{aligned}$$

Hence, the first term on the right-hand side of (6.26) is bounded according to

$$|f^h - \tilde{f}^h|_{0,\bar{\Omega}^h} \leq ch^{\nu+1} |f|_{\nu,q,\Omega}. \tag{6.34}$$

Combining inequalities (6.26), (6.33), and (6.34), we have our final result.

Theorem 6.2. Assume that $a_{ij}(z) \in \boldsymbol{W}^{\nu}_{\infty}(\Omega)$, $1 \leq i, j \leq 2$, $\nu \geq 1$; $f(z) \in \boldsymbol{W}^1_q(\Omega)$ with $q > 2$; and $u \in \boldsymbol{H}^{(\nu)}(\Omega)$, $\nu = 2$ or 3. Suppose that $\tilde{B}$ is uniformly elliptic on $\boldsymbol{S}^h_0$ and that $\mathcal{T}^h$ and $\mathcal{V}^h$ are regular. Suppose also that the quadrature error functionals $\hat{R}_{\hat{\gamma}}(\hat{v})$ and $\hat{R}_{\hat{V}}(\hat{v})$ vanish on the respective $\boldsymbol{P}_0(\hat{\gamma})$ and $\boldsymbol{P}_0(\hat{V})$. Then we have:

(1) for the general case,

$$|e|_{1,\bar{\Omega}^h} \leq ch(h|f|_{1,q,\Omega} + |u|_{2,\Omega} + h^{\frac{1}{2}}\|u\|_{2,\Omega} \sum_{l,k=1}^{2} |a_{lk}|_{1,\infty,\Omega}); \tag{6.35}$$

(2) for the case of essentially symmetric control volumes,

$$|e|_{1,\bar{\Omega}^h} \leq ch^{\frac{3}{2}}(h^{\frac{1}{2}}|f|_{1,q,\Omega} + \|u\|_{3,\Omega} + \|u\|_{2,\Omega} \sum_{l,k=1}^{2} |a_{lk}|_{1,\infty,\Omega}); \tag{6.36}$$

(3) for the case of symmetric control volumes,

$$|e|_{1,\bar{\Omega}^h} \leq ch^2(|f|_{1,q,\Omega} + \|u\|_{3,\Omega} + h^{\frac{1}{2}}\|u\|_{2,\Omega} \sum_{l,k=1}^{2} |a_{lk}|_{2,\infty,\Omega}). \tag{6.37}$$

Here, c is a constant independent of h.

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Figure 3.1. Sample control volume V^h (dotted lines).

Figure 3.2. Opposing angles $\bar{\theta}_{K'}$ and $\bar{\theta}_{K''}$ of the respective triangles K' and K'' with respect to the common face.

Figure 4.1. Sample rhombus E_{ij} (dotted lines).

Figure 5.1. Vertices z_i, side segments Z_i, control volume segments γ_i (dotted lines), and unit normal vectors $\overrightarrow{n}_i$.