Norm Estimates of Product Operators with Application to Domain Decomposition*

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ABSTRACT

In this paper, we consider norm estimates for operators of the product form $E = (I - T_J) \cdots (I - T_1)(I - T_0)$ in an abstract Hilbert space. By imposing appropriate assumptions on T_j, explicit norm estimates for E are established. We then apply the abstract theory to convergence analysis of domain decomposition for nonsymmetric and indefinite second order elliptic problems.

1. INTRODUCTION

In [6], Bramble, Pasciak, Wang, and Xu considered a multiplicative iterative method for the solution of symmetric positive definite problems on an abstract finite dimensional Hilbert space that are defined in terms of product operators based on a number of subspaces. They developed a technique to estimate the norm of the error propagation operator of the form $E = (I - T_J) \cdots (I - T_1)(I - T_0)$ for the multiplicative iterative method. Their assumptions were that the operators T_j are symmetric, semidefinite, and bounded above by a constant less than two, and that the smallest eigenvalue

* This work was performed while the author was at the Courant Institute of Mathematical Sciences in New York University and supported under the National Science Foundation Grant No. DMS88-06727.
** I would like to thank Professor Steve McCormick for valuable suggestions.

655 Avenue of the Americas, New York, NY 10010 0096-3003/93/$5.00

of the sum of these operators T_j is bounded below by a constant greater than zero. Applications of their results include convergence analysis for domain decomposition and multigrid methods for the second order selfadjoint elliptic boundary value problems. Based on this technique with slight modification, Bramble, Pasciak, Wang, and Xu in [5] and Wang in [19] provided convergence estimates for certain multigrid algorithms without full regularity assumptions, and Bramble and Pasciak in [2] analyzed smoothing operators for use in multigrid algorithms. These analyses strongly depend upon symmetry and semidefiniteness of the operators.

In this paper, we develop norm estimates of the product operator E in a finite dimensional abstract Hilbert space under different assumptions. Essentially, we assume that the linear operators T_j are nonnegative definite up to a small perturbation and bounded above by a constant less than 2, and we require the smallest eigenvalue of the sum of the $T_j^t T_j$ to be bounded below by a constant greater than zero. Applications of our abstract analysis include convergence analysis for domain decomposition and multigrid methods for nonsymmetric and indefinite second order elliptic boundary value problems.

The outline of the remainder of this paper is as follows. Section 2 provides an abstract analysis for estimating the norms of the product operator E under appropriate assumptions. Nonsymmetric and indefinite second order elliptic problems are briefly described in Section 3. In Section 4, we apply the abstract theory developed in Section 2 to domain decomposition methods with exact and inexact subproblem solvers whose uniform convergence are established provided the coarse grid mesh size is sufficiently small.

2. NORM ESTIMATES OF PRODUCT OPERATOR IN HILBERT SPACE

Let $\mathscr{H}$ be an abstract Hilbert space with inner product $(\cdot,\cdot)$, and let $\|\cdot\|$ denote its induced norm. For each $j = 1, 2, \ldots, J$, let $\mathscr{H}_j$ be a closed subspace of $\mathscr{H}$ with inner product $(\cdot,\cdot)_j$ and induced norm $\|\cdot\|_j$. Assume that $\|v\|_j \leqslant \|v\|$ for any $v \in \mathscr{H}$ and $\|v\|_j = \|v\|$ for any $v \in \mathscr{H}_j$. Let T_j be a linear operator mapping $\mathscr{H}$ into $\mathscr{H}_j$. In this section, we will give explicit upper bounds for the norm of the product operator

$$E \equiv (I - T_J)(I - T_{J-1}) \cdots (I - T_1). \tag{2.1}$$

To do this, we will make assumptions imposed upon the family of linear operators $\{T_j\}$ and closed subspaces $\{\mathscr{H}_j\}$. A fundamental assumption for our analysis in this section is Assumption 2.2, which involves an inequality regarding the sum of the operators $\{T_j\}$. The main results consist of the norm estimates of the product operator E defined in (2.1) in the case that the family of linear operators $\{T_j\}$ is either nonnegative definite or 'almost' negative definite (see Theorems 2.1 and 2.3). When the family of subspaces $\{\mathscr{H}_j\}$ is band orthogonal (see Definition 2.1), these norm estimates of the product operator E are improved (see Theorems 2.2 and 2.4 as well as Remarks 2.2 and 2.3).

The most important application of the results in this section is to the computation of the solutions of the discrete equations that arise from the numerical approximation of elliptic partial differential equations. In the case of finite element approximations, the inner products $(\cdot,\cdot)$ and $(\cdot,\cdot)_j$ can be the bilinear forms associated with the symmetric positive definite part of the differential operator, and $\mathscr{H}$ the finite element approximation space. The family of subspaces $\{\mathscr{H}_j\}$ can be related either to subdomains in overlapping domain decomposition applications or to coarser grids in multigrid applications.

To begin our analysis, for each $j = 1, 2, \dots, J$, set

$$E_j = (I - T_j)(I - T_{j-1}) \cdots (I - T_1). \tag{2.2}$$

For convenience, let

$$E_0 = I,$$

where I is the identity operator on $\mathscr{H}$. Evidently,

$$E_J = E.$$

It is easy to see (see [6]) that, for any $j \in \{1, 2, \cdots, J\}$,

$$E_{j-1} - E_j = T_j E_{j-1}, \tag{2.3}$$

from which follows

$$I = E_i + \sum_{j=1}^{i} T_j E_{j-1} \tag{2.4}$$

$i = 1, 2, \dots, J$.

ASSUMPTION 2.1. *There exists a constant* $\omega \in (0,2)$ *such that*

$$(T_j v, T_j v) \leqslant \omega (T_j v, v)$$

for any $v \in \mathscr{H}$ *and* $j = 1, 2, \ldots, J$.

Essentially, Assumption 2.1 means that the family of linear operators $\{T_j\}$ is nonnegative definite and bounded above by ω. The proof of our first lemma is slightly different from one in [19].

LEMMA 2.1. *Under Assumption* 2.1, *we have*

$$(2-\omega)\sum_{j=1}^{J}(T_j E_{j-1}v, E_{j-1}v) \leqslant \|v\|^2 - \|Ev\|^2. \tag{2.5}$$

PROOF. For any $v \in \mathscr{H}$, it follows from (2.3) and Assumption 2.1 that

$$\begin{aligned}\|E_{j-1}v\|^2 - \|E_j v\|^2 &= \|T_j E_{j-1}v\|^2 + 2(T_j E_{j-1}v, E_j v)\\ &= 2(T_j E_{j-1}v, E_{j-1}v) - (T_j E_{j-1}v, T_j E_{j-1}v)\\ &\geqslant (2-\omega)(T_j E_{j-1}v, E_{j-1}v).\end{aligned}$$

(2.5) now follows from summing the above inequality. ∎

Next, we introduce nonnegative symmetric matrice

$$\varepsilon = \begin{pmatrix} \sqrt{\varepsilon_{11}} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ \sqrt{\varepsilon_{J1}} & \cdots & \sqrt{\varepsilon_{JJ}} \end{pmatrix} \quad \text{and} \quad \varepsilon^k = \begin{pmatrix} \varepsilon_{11}^k & \cdots & \varepsilon_{1J}^k \\ \vdots & \ddots & \vdots \\ \varepsilon_{J1}^k & \cdots & \varepsilon_{JJ}^k \end{pmatrix}$$

where ε_{ij}, $\varepsilon_{ij}^k \in [0, 1]$ are the cosines of the angle between the subspaces $\mathscr{H}_i$ and $\mathscr{H}_j$ with respect to the inner products $(\cdot,\cdot)$ and $(\cdot,\cdot)_k$, respectively, i.e.,

ε_{ij} and ε_{ij}^k are the smallest constants satisfying the respective inequalities

$$(v_i, v_j) \leqslant \varepsilon_{ij}\|v_i\| \, \|v_j\|, \quad (v_i, v_j)_k \leqslant \varepsilon_{ij}^k \|v_i\|_k \|v_j\|_k, \quad \forall v_i \in \mathscr{H}_i, \quad \forall v_j \in H_j. \tag{2.6}$$

These are known as the *strengthened Cauchy-Schwarz inequalities*. Clearly, $\varepsilon_{ii} = \varepsilon_{ii}^i = 1$, and $\varepsilon_{ij} = \varepsilon_{ij}^k = 0$ if $\mathscr{H}_i$ is orthogonal to $\mathscr{H}_j$ with respect to the inner products $(\cdot,\cdot)$ and $(\cdot,\cdot)_k$, respectively. The relevant quantity for these matrice ε and ε^k are their 2-norms, $|\varepsilon|$ and $|\varepsilon^k|$.

LEMMA 2.2. *Under Assumption* 2.1, *we have*

$$\sum_{j=1}^{J} (T_j v, T_j v) \leqslant \omega \max\{1, \omega\} |\varepsilon|^2 \sum_{j=1}^{J} (T_j E_{j-1} v, E_{j-1} v). \tag{2.7}$$

PROOF. For any $j \in \{1, 2, \dots, J\}$ and $v \in \mathscr{H}$, let

$$b_j = (T_j E_{j-1} v, E_{j-1} v)^{1/2} \geqslant 0$$

so that

$$\sum_{j=1}^{J} (T_j E_{j-1} v, E_{j-1} v) = \sum_{j=1}^{J} b_j^2.$$

By (2.4) and the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
(T_j v, T_j v) &= (T_j v, T_j E_{j-1} v) + \sum_{i=1}^{j-1} (T_j v, T_j T_i E_{i-1} v) \\
&\leqslant \|T_j v\| \left(\|T_j E_{j-1} v\| + \sum_{i=1}^{j-1} \|T_j T_i E_{i-1} v\| \right) \\
&\leqslant \omega^{1/2} \|T_j v\| \left(b_j + \sum_{i=1}^{j-1} (T_j T_i E_{i-1} v, T_i E_{i-1} v)^{1/2} \right).
\end{aligned}$$

The last inequality follows from Assumption 2.1, which also implies that the linear operator T_j is bounded by ω. Cancelling $\|T_j v\|$ from both sides of the above inequality and squaring, it thus follows from (2.6) and Assumption 2.1 again that

$$
\begin{aligned}
(T_j v, T_j v) &\leqslant \omega \left(b_j + \sum_{i=1}^{j-1} \varepsilon_{ij}^{1/2} \|T_j T_i E_{i-1} v\|^{1/2} \|T_i E_{i-1} v\|^{1/2} \right)^2 \\
&\leqslant \omega \left(b_j + \omega^{1/2} \sum_{i=1}^{j-1} \varepsilon_{ij}^{1/2} (T_i E_{i-1} v, T_i E_{i-1} v)^{1/2} \right)^2 \\
&= \omega \left(b_j + \omega^{1/2} \sum_{i=1}^{j-1} \varepsilon_{ij}^{1/2} b_i \right)^2 \\
&\leqslant \omega \max\{1, \omega\} \left(\sum_{i=1}^{j} \varepsilon_{ij}^{1.2} b_i \right)^2.
\end{aligned}
$$

The lemma now follows from summing the above inequality and using the fact that

$$
\sum_{j=1}^{J} \left(\sum_{i=1}^{j} \varepsilon_{ij}^{1/2} b_i \right)^2 = \sum_{j=1}^{J} (\varepsilon b)_j^2 \leqslant |\varepsilon|^2 \sum_{j=1}^{J} b_j^2. \qquad \blacksquare
$$

The strengthened Cauchy-Schwarz inequality was used by Widlund [20] and Xu [21] to bound $\sum_{j=1}^{J} (T_j v, v)$ in terms of $\sum_{j=1}^{J} (T_j E_{j-1} v, E_{j-1} v)$ in the case that the family of linear operators $\{T_j\}$ is symmetric, semidefinite, and bounded above by a constant less than two. Their analyses, and a similar approach by Bramble, Pasciak, Wang, and Xu in [6], strongly depend upon symmetry and semidefiniteness of the linear operators. They do, however lead to simpler upper bounds and possibly better results when the family of subspaces $\{\mathscr{H}_j\}$ satisfy certain properties. For other uses of the strengthened Cauchy-Schwarz inequality, see also [15] and [18].

A fundamental assumption for the analysis in this section involves an inequality regarding the sum of the operators $\{T_j\}$.

ASSUMPTION 2.2. *There exists a positive constant C_0 satisfying*

$$\|v\|^2 \leqslant C_0 \sum_{j=1}^{J} (T_j v, T_j v), \quad \forall v \in \mathscr{H}.$$

This inequality was used by Cai and Widlund [7] in their treatment of additive overlapping domain decomposition methods for nonsymmetric and indefinite elliptic problems. See also [5, 6, 11, 12, 18, 19, 21].

REMARK 2.1. Suppose T_j is the orthogonal projection operator P_j from $\mathscr{H}$ onto $\mathscr{H}_j$, defined by

$$(P_j v, u) = (v, u), \quad \forall v \in \mathscr{H}, \quad \forall u \in \mathscr{H}_j.$$

Then the verification of Assumption 2.2 reduces to the construction of a decomposition of every $v \in \mathscr{H}$ of the form

$$v = \sum_{j=1}^{J} v_j$$

where $v_j \in \mathscr{H}_j$ satisfies

$$\sum_{j=1}^{J} \|v_j\|^2 \leqslant C_0 \|v\|^2.$$

This observation is due to Lions [13] (see also [6]).

As a direct result of (2.5), (2.7), and Assumption 2.2, we have

THEOREM 2.1. *Under Assumptions* 2.1 *and* 2.2, *we have*

$$\|E\| \leqslant \left(1 - \frac{2 - \omega}{C_0 \, \omega \max\{1, \omega\} |\varepsilon|^2}\right)^{1/2}. \tag{2.9}$$

PROOF. For any $v \in \mathscr{H}$, (2.5), (2.7), and Assumption 2.2 immediately imply that

$$\|Ev\|^2 \leqslant \left(1 - \frac{2-\omega}{C_0\,\omega \max\{1,\,\omega\}|\varepsilon|^2}\right)\|v\|^2,$$

which established the theorem. ∎

The above theorem has applications to convergence analysis without regularity assumptions for multigrid methods that use nonsymmetric smoothers (see [2] and [19]).

The bound on the norm of the product operator E provided in the Theorem 2.1 approaches one as $|\varepsilon|$, the 2-norm of the matrix ε, increases. The worst case is that $|\varepsilon|$ is order of J, which is the number of subspaces in the family $\{\mathscr{H}_j\}$. However, the norm estimate of the product operator E may be improved when the family of subspaces $\{\mathscr{H}_j\}$ satisfies the following orthogonality property.

DEFINITION 2.1. The family of subspaces $\{\mathscr{H}_j\}$ is called band orthogonal if there exists a positive integer $m < J$ independent of J such that, for any fixed $j \in \{1, 2, \dots, J\}$, $\mathscr{H}_j$ is orthogonal to at least $J - m$ other subspaces in the family with respect to the inner products $(\cdot,\cdot)$ and $(\cdot,\cdot)_i$ $(i = 1, 2, \dots, J)$.

THEOREM 2.2. *Assume that the family of subspaces* $\{\mathscr{H}_j\}$ *is band orthogonal and that Assumptions* 2.1 *and* 2.2 *hold. Then*

$$\|E\| \leqslant \left(1 - \frac{2-\omega}{C_0 m^2 \omega \max\{1,\,\omega\}}\right)^{1/2}. \tag{2.10}$$

PROOF. (2.10) follows from Theorem 2.1 and the fact that $|\varepsilon| \leqslant m$ when the family of subspaces $\{\mathscr{H}_j\}$ is band orthogonal. ∎

REMARK 2.2. Assume that $\mathscr{H}_0$ is a closed subspace of $\mathscr{H}$ and that T_0 is a linear operator mapping $\mathscr{H}$ to $\mathscr{H}_0$ that satisfies Assumption 2.1. Then, under the assumptions of Theorem 2.2, a similar argument shows that

$$\|E(I - T_0)\| \leqslant \left(1 - \frac{2 - \omega}{C_0(m+1)^2 \max\{1, \omega\}}\right)^{1/2}. \tag{2.11}$$

Theorem 2.2 can be applied to convergence analysis of overlapping domain decomposition with inexact subproblem solvers; band orthogonality of the family of linear subspaces, defined as in the Definition 2.1, would correspond to the intersection property of the subdomains for this application.

The above discussions are based on the assumption that the family of linear operators $\{T_j\}$ is nonnegative definite and bounded above by a constant less than two. In order to discuss the applications to overlapping domain decomposition and multigrid methods for nonsymmetric and indefinite elliptic problems, we make the following assumption on the linear operators T_j, which amounts to assuming that these operators are nonnegative definite and bounded above by a constant less than 2 up to a small perturbation.

Assumption 2.3. *There exists a constant* $\omega \in (0, 2)$ *such that, for any fixed* $j \in \{1, 2, \dots, J\}$, *we have*

$$(T_j v, T_j v) \leqslant \omega(T_j v, v) + \tau_j(v, v)_j,$$

where $\{\tau_j\}$ *are (nonnegative) real numbers for which there exist constants* C *independent of* J *and* $\tau', \tau'' > 0$ *such that*

1. $\sum_{j=1}^{J-1} j\tau_{j+1} \leqslant C$;
2. $\sum_{j=1}^{J} \tau_j \|v\|_j^2 \leqslant \tau' \|v\|^2 \ \forall v \in \mathscr{H}$;
3. $\sum_{j=1}^{J} \tau_j \|v\|_j^2 \leqslant \tau'' \|v\|_i^2 \ \forall i \in \{1, 2, \dots, J\}$ *and* $\forall v \in \mathscr{H}_i$.

For any $j \in \{1, 2, \dots, J\}$, it is easy to see that Assumption 2.3 implies that

$$\|T_j\| \leqslant \xi_1 = \frac{\omega}{2} + \sqrt{\tau + \frac{\omega^2}{4}} \tag{2.12}$$

with $\tau = \max_{1 \leqslant j \leqslant J} \tau_j$.

LEMMA 2.3. *Under Assumption 2.3, we have*

$$(2-\omega)\sum_{j=1}^{J}\left(T_jE_{j-1}v,\, E_{j-1}v\right) \leqslant \|v\|^2 + \sum_{j=1}^{J}\tau_j\|E_{j-1}v\|_j^2 - \|Ev\|^2. \quad (2.13)$$

PROOF. (2.13) follows from the proof of Lemma 2.1, but using Assumption 2.3 instead of Assumption 2.1. ∎

LEMMA 2.4. *Under Assumption 2.3, we have*

$$\sum_{j=1}^{J}\left(T_jv, T_jv\right) \leqslant C_1\sum_{j=1}^{J}\left(T_jE_{j-1}v, T_jE_{j-1}v\right), \quad (2.14)$$

where

$$C_1 = 2\max\{1,\, \omega\xi_1\}|\varepsilon|^2 + C. \quad (2.15)$$

PROOF. For any $j \in \{1, 2, \dots, J\}$ and $v \in \mathscr{H}$, let

$$b_j = \|T_jE_{j-1}v\|$$

so that

$$\sum_{j=1}^{J}\left(T_jE_{j-1}v, T_jE_{j-1}v\right) = \sum_{j=1}^{J}b_j^2.$$

By (2.4), the Cauchy-Schwarz inequality, Assumption 2.3, and (2.6), we have

$$\begin{aligned}\left(T_jv, T_jv\right) &= \left(T_jv, T_jE_{j-1}v\right) + \sum_{i=1}^{j-1}\left(T_jv, T_jT_iE_{i-1}v\right)\\ &\leqslant \|T_jv\|\left(b_j + \sum_{i=1}^{j-1}\|T_jT_iE_{i-1}v\|\right)\end{aligned}$$

$$\leqslant \|T_j v\| \left(b_j + \sum_{i=1}^{j-1} \left[\omega(T_j T_i E_{i-1} v, T_i E_{i-1} v) + \tau_j b_i^2 \right]^{1/2} \right)$$

$$\leqslant \|T_j v\| \left(b_j + \sum_{i=1}^{j-1} \left[\omega \varepsilon_{ij} \|T_j T_i E_{i-1} v\| \, \|T_i E_{i-1} v\| + \tau_j b_i^2 \right]^{1/2} \right).$$

Cancelling $\|T_j v\|$ from both sides of the above inequality, squaring it, and using (2.12), it then follows that

$$(T_j v, T_j v) \leqslant \left(b_j + \sum_{i=1}^{j-1} (\omega \varepsilon_{ij} \xi_1 + \tau_j)^{1/2} b_i \right)^2$$

$$\leqslant \left(b_j + \omega^{1/2} \xi_1^{1/2} \sum_{i=1}^{j-1} \varepsilon_{ij}^{1/2} b_i + \tau_j^{1/2} \sum_{i=1}^{j-1} b_i \right)^2$$

$$\leqslant 2 \left(\max\{1, \omega \xi_1\} \left(\sum_{i=1}^{j} \varepsilon_{ij}^{1/2} b_i \right)^2 + \tau_j \left(\sum_{i=1}^{j-1} b_i \right)^2 \right).$$

By summing the above inequality, (2.15) now follows from (2.8) and the fact that

$$\sum_{j=1}^{J} \tau_j \left(\sum_{i=1}^{j-1} b_i \right)^2 \leqslant C \sum_{j=1}^{J} b_j^2.$$

This completes the proof of the lemma. ■

LEMMA 2.5. *Under Assumption* 2.3, *we have*

$$\sum_{j=1}^{J} (T_j E_{j-1} v, T_j E_{j-1} v) \leqslant C_2 ((\omega + 4\tau')\|v\|^2 - \omega \|Ev\|^2) \qquad (2.16)$$

where

$$C_2 = \frac{1}{2 - \omega - 4\xi_2 \tau''} \quad \textit{and} \quad \xi_2 = \max_{1 \leqslant j \leqslant J} |\varepsilon^j|. \qquad (2.17)$$

PROOF. It follows from Assumption 2.3 and Lemma 2.3 that

$$\sum_{j=1}^{J}(T_jE_{j-1}v, T_jE_{j-1}v) \leqslant \omega\sum_{j=1}^{J}(T_jE_{j-1}v, E_{j-1}v) + \sum_{j=1}^{J}\tau_j(E_{j-1}v, E_{j-1}v)_j$$

$$\leqslant \frac{\omega}{2-\omega}\|v\|^2 + \frac{2}{2-\omega}\sum_{j=1}^{J}\tau_j(E_{j-1}v, E_{j-1}v)_j$$

$$-\frac{\omega}{2-\omega}\|Ev\|^2.$$

By (2.4), (2.6), Assumption 2.3, and the fact that $\|T_iw\|_i^2 = \|T_iw\|^2 \ \forall w \in \mathscr{H}$, we have

$$\sum_{j=1}^{J}\tau_j(E_{j-1}v, E_{j-1}v)_j \leqslant 2\sum_{j=1}^{J}\tau_j\|v\|_j^2 + 2\sum_{j=1}^{J}\tau_j\left\|\sum_{i=1}^{j-1}T_iE_{i-1}v\right\|_j^2$$

$$\leqslant 2\tau'\|v\|^2 + 2\sum_{j=1}^{J}\tau_j\sum_{i,k=1}^{j-1}\varepsilon_{ik}^j\|T_iE_{i-1}v\|_j\|T_kE_{k-1}v\|_j$$

$$\leqslant 2\tau'\|v\|^2 + 2\xi_2\sum_{j=1}^{J}\tau_j\sum_{i=1}^{j-1}\|T_iE_{i-1}v\|_j^2$$

$$\leqslant 2\tau'\|v\|^2 + 2\xi_2\tau''\sum_{i=1}^{J}\|T_iE_{i-1}v\|_i^2$$

$$= 2\tau'\|v\|^2 + 2\xi_2\tau''\sum_{j=1}^{J}(T_jE_{j-1}v, T_jE_{j-1}v).$$

The above two inequalities imply (2.16). ∎

The inequalities in Lemmas 2.4 and 2.5 and Assumption 2.2 can be combined to establish the following result.

THEOREM 2.3. *Assume that Assumptions 2.2 and 2.3 hold. Then there exists a positive constant τ_0 such that, if τ', $\tau'' \leqslant \tau_0$, then*

$$\|E\| \leqslant (1 - \delta)^{1.2}, \tag{2.18}$$

where

$$\delta = \frac{1 - 4\tau' C_0 C_1 C_2}{C_0 C_1 C_2 \omega} \in (0, 1). \tag{2.19}$$

PROOF. Assume τ_0 is so small that $\delta \in (0, 1)$ for $\tau', \tau'' \leqslant \tau_0$. (If $\tau_0 < 2 - \omega/4\xi_2$ then C_2 in (2.17) is defined. From the following argument we can then conclude that $\delta < 1$. In order to guarantee $\delta > 0$, we must choose τ_0 possibly smaller so that $4\tau' C_0 C_1 C_2 < 1$ when $\tau', \tau'' \leqslant \tau_0$.) For any $v \in \mathscr{H}$, Lemmas 2.4 and 2.5 imply that

$$\sum_{j=1}^{J} \|T_j v\|^2 \leqslant C_1 C_2 \big((\omega + 4\tau') \|v\|^2 - \omega \|Ev\|^2 \big),$$

which, together with Assumption 2.2 yields

$$\|Ev\|^2 \leqslant (1 - \delta) \|v\|^2. \qquad \blacksquare$$

THEOREM 2.4. *Assume that the family of subspaces $\{\mathscr{H}_j\}$ is band orthogonal and that Assumptions 2.2 and 2.3 hold. Then there exists a positive constant τ_0 such that, if $\tau', \tau'' \leqslant \tau_0$, then*

$$\|E\| \leqslant (1 - \tilde{\delta})^{1/2}, \tag{2.20}$$

where

$$\tilde{\delta} = \frac{1 - 4\tau' C_0 \tilde{C}_1 \tilde{C}_2}{C_0 \tilde{C}_1 \tilde{C}_2 \omega} \in (0, 1), \tag{2.21}$$

and

$$\tilde{C}_1 = 2m^2 \max\{1, \omega\xi_1\} + C \quad and \quad \tilde{C}_2 = \frac{1}{2 - \omega - 4m\tau''}. \tag{2.22}$$

PROOF. This theorem follows from Theorem 2.3 and the fact that $\xi_2 \leqslant m$ when the family of subspaces $\{\mathscr{H}_j\}$ is band orthogonal. ■

REMARK 2.3. As in Remark 2.2, let $\mathscr{H}_0$ be a closed subspace of $\mathscr{H}$ and let T_0 be a linear operator mapping $\mathscr{H}$ to $\mathscr{H}_0$ that satisfies Assumption 2.3. Then, under the assumptions of Theorem 2.4, (2.20) holds with $\tilde{\delta}$ defined in (2.21) and

$$\tilde{C}_1 = 2(m+1)^2 \max\{1, \omega\xi_1\} + C, \quad \tilde{C}_2 = \frac{1}{2 - \omega - 4(m+1)\tau''}.$$

3. ELLIPTIC BOUNDARY VALUE PROBLEMS

For simplicity, assume that the domain Ω is an open, bounded, and polygonal subset in R^2. Extensions of the results in subsequent sections to higher dimensions are straightforward. Consider the elliptic boundary value problem

$$\begin{cases} -\nabla \cdot (D \nabla u) + \vec{b} \cdot \nabla u + du = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{3.1}$$

It is convenient to write (3.1) in the weak form: Find $u \in H_0^1(\Omega)$ such that

$$a(u,v) = (f,v) \quad \forall v \in H_0^1(\Omega). \tag{3.2}$$

Here, $(\cdot,\cdot)$ denotes the inner product in $L^2(\Omega)$ whose induced norm is denoted by $\|\cdot\|$, and, for any $u, v \in H_0^1(\Omega)$,

$$a(u,v) = a^s(u,v) + b(u,v) \tag{3.3}$$

where

$$a^s(u,v) = \int_\Omega (D\nabla u)^T \nabla v\, dx \quad \text{and} \quad b(u,v) = \int_\Omega (\vec{b}\cdot\nabla u + du) v\, dx. \tag{3.4}$$

We make the standard assumption that $a^s(u,v)$ is bounded and uniformly elliptic in $H_0^1(\Omega)$, so the energy norm $|||\cdot||| = \sqrt{a^s(\cdot,\cdot)}$ is equivalent to the H^1 seminorm. Also, we assume that, for any $u, v \in H_0^1(\Omega)$,

$$|b(u,v)| \leqslant \begin{cases} c\,|||\,u\,|||\,\|v\|, \\ c\|u\|\,|||\,v\,|||, \end{cases} \tag{3.5}$$

and

$$|||\,v\,|||^2 - c\|v\|^2 \leqslant a(v,v) \quad \forall v \in H_0^1(\Omega), \tag{3.6}$$

which is Gårding's inequality. Here and henceforth, we use c to denote generic positive constant independent of the number of the subspaces J and the mesh parameter h. We also assume the following regularity result: Given $g \in L^2(\Omega)$, then the solution $u \in H_0^1(\Omega)$ of the adjoint problem of (3.2)

$$a(v,u) = (g,v) \quad \forall v \in H_0^1(\Omega)$$

satisfies

$$\|u\|_{1+\alpha} \leqslant c\|g\|, \tag{3.7}$$

where $\alpha \in (0,1)$ is independent of the data g but dependent on the shape of the domain. Here, $\|\cdot\|_{1+\alpha}$ is the usual $H^{1+\alpha}(\Omega)$ norm.

Let $\mathcal{S}^h$ be the piecewise linear finite element space associated with a regular triangulation $\mathcal{T}^h$ (see [10]). The finite element approximation to the solution of problem (3.2) is now defined as the solution of the following problem: Find $u \in \mathcal{S}^h$ such that

$$a(u,v) = (f,v) \quad \forall v \in \mathcal{S}^h. \tag{3.8}$$

In general, the existence and uniqueness of the solution u in (3.8) does not immediately follow. It turns out, however, that if h is sufficiently small, then the above problem does have an unique solution.

4. APPLICATION TO SCHWARZ ALTERNATING PROCEDURE

In this section, we apply the abstract theory developed in Section 2 to the Schwarz alternating procedure (see [8, 9, 13, 17]) (also called the overlapping domain decomposition method). The iterative operators for the Schwarz alternating procedure may be written in the form of the product operator of Section 2. Therefore, the results of Section 2 show that, in order to bound the convergence rate of Schwarz alternating procedure, it suffices to verify Assumption 2.2 and either Assumption 2.1 or 2.3, and apply either Theorem 2.2 or 2.3, or Remark 2.2 or 2.3. The verification of Assumption 2.2 is given by Cai and Widlund [7] for application of the Schwarz alternating procedure using exact subproblem and coarse problem solvers. The main results of this section include uniform convergence of the Schwarz alternating procedure with both exact and inexact subproblem solvers (see Propositions 4.1 and 4.2).

To define the Schwarz alternating procedure, assume that we are given a regular coarse triangulation $\mathscr{T}^H = \{K_j\}_{j=1}^J$, and a regular fine triangulation $\mathscr{T}^h$ obtained by further partitioning all of the elements in $\mathscr{T}^H$. Associated with the coarse triangulation $\mathscr{T}^H$, we construct a set of overlapping subdomains $\{\Omega_j\}_{j=1}^J$ by extending each element $K_j \in \mathscr{T}^H$ to a larger subdomain Ω_j, whose diameter is denoted by H_j, such that $\overline{\Omega} = \bigcup_{j=1}^J \overline{\Omega}_j$. Assume that the maximum number of subdomain overlaps is bounded, so that the associated family of subdomain spaces will be band orthogonal with fixed m. Assume further that the distance between the boundaries ∂K_j and $\partial \Omega_j$ is bounded below by $\zeta_1 H$ and above by $\zeta_2 H$, i.e., for all $j \in \{1, 2, \dots, J\}$, there exist constants $\zeta_1, \zeta_2 > 0$ such that

$$\zeta_1 H \leqslant \operatorname{dist}(\partial K_j, \partial \Omega_j) \leqslant \zeta_2 H.$$

Also assume that the boundaries of the Ω_j do not cut through any element in $\mathscr{T}^h$, i.e., they must coincide with boundaries of elements of $\mathscr{T}^h$.

Let $\mathscr{S}^H$ and $\mathscr{S}^h$ be the respective piecewise linear finite element subspaces associated with $\mathscr{T}^H$ and $\mathscr{T}^h$, and denote the respective $\mathscr{S}^H$ and $\mathscr{S}^h$ equipped with the energy norm $|||\cdot|||$ by $\mathscr{V}_0$ and $\mathscr{V}$. With the set of overlapping subdomains $\{\Omega_j\}_{j=1}^J$, we define the family of the finite element approximation subspaces $\{\mathscr{V}_j\}_{j=1}^J$ by

$$\mathscr{V}_j = \{v \in \mathscr{V} | \operatorname{supp} v \subset \Omega_j\}$$

for all $j \in \{1, 2, \dots, J\}$, where supp v means the support of v. The norm for $\mathscr{V}_j$ is defined by $|||\cdot|||_j = a_j^s(\cdot,\cdot)$, where $a_j^s(\cdot,\cdot)$ is a bilinear form defined on $\mathscr{V}_j \times \mathscr{V}_j$ by

$$a_j^s(u,v) = \int_{\Omega_j} (D\nabla u)^T \nabla v \, dx \quad \forall u, v \in \mathscr{V}_j.$$

For any $v \in \mathscr{V}$, we define linear operators P_j and P_j^s from $\mathscr{V}$ onto $\mathscr{V}_j$ by

$$a(P_j v, \varphi) = a(v, \varphi) \quad \forall \varphi \in \mathscr{V}_j, \quad j \in \{0, 1, \dots, J\} \tag{4.1}$$

and

$$a^s(P_j^s v, \varphi) = a(v, \varphi) \quad \forall \varphi \in \mathscr{V}_j, \quad j \in \{1, 2, \dots, J\}, \tag{4.2}$$

respectively. Associated with these subspaces $\{\mathscr{V}_j\}_{j=0}^J$, we consider the following Schwarz alternating procedure for solving problem (3.8).

ALGORITHM 4.1. *Given an approximation $u^l \in \mathscr{V}$ to the solution u of* (3.8), *define the next approximation $u^{l+1} \in \mathscr{V}$ as follows*:

1. *Set* $W_{-1} = u^l$.
2. *For* $j = 0, 1, \dots, J$ *in turn, define* W_j *by*

$$W_j = W_{j-1} + P_j(u - W_{j-1}).$$

3. *Set* $u^{l+1} = W_J$.

This algorithm requires exact solution of each nonsymmetric and indefinite subproblem on $\mathscr{V}_j$ $(j = 1, 2, \dots, J)$ and the coarse grid problem on $\mathscr{V}_0$. Alternatively, we may replace the linear operators P_j in the above algorithm by linear operators P_j^s, which means that the symmetric positive definite part of the nonsymmetric indefinite operator is used as a preconditioner for each subproblem.

A simple calculation implies that the error propagation or iteration operator for Algorithm 4.1 is

$$E_1 = (I - P_J) \cdots (I - P_1)(I - P_0). \tag{4.3}$$

Alternatively, we have iteration operator

$$E_2 = (I - P_J^s) \cdots (I - P_1^s)(I - P_0) \tag{4.4}$$

for Algorithm 4.1 with P_j^s replacing P_j.

By noting that the family of subspaces $\{\mathscr{V}_j\}_{j=1}^J$ is generally band orthogonal, the results of Section 2 show that, in order to bound the rate of convergence for Algorithm 4.1 and its alternative, it suffices to verify Assumptions 2.2 and 2.3 for the operators P_j and P_j^s, and apply Remark 2.3. To do this, we first note the following three lemmas that were stated and proved in [11] and [7].

LEMMA A. *For any $v \in \mathscr{V}$, there exists a decomposition of v with $v = \sum_{j=0}^J v_j$ and $v_j \in \mathscr{V}_j$ such that*

$$\sum_{j=0}^{J} a^s(v_j, v_j) \leqslant C_0 a^s(v, v), \tag{4.5}$$

where C_0 is a constant independent of h and H.

This lemma was proved by Dryja and Widlund [9]. See also [11].

LEMMA B. *There exists a constant $H_0 > 0$ such that, if $H \leqslant H_0$, then*

$$|||P_0 v||| \leqslant c\,|||v||| \quad \text{and} \quad \|(I - P_0)v\| \leqslant cH^{\alpha}\,|||v|||\,. \ \forall v \in \mathscr{V}. \tag{4.6}$$

This is a straightforward consequence of Schatz's work [14], which is obtained by replacing the exact solution and finite element approximation by the fine and coarse grid finite element approximation, respectively. Assumption 2.2 follows from

LEMMA C. *There exists a constant $H_0 > 0$ such that, if $H \leqslant H_0$, then*

$$c\,|||v|||^2 \leqslant \sum_{j=0}^{J} a^s(P_j v, P_j v) \tag{4.7}$$

and

$$c\,|||\,v\,|||^2 \leqslant a^s(P_0 v, P_0 v) + \sum_{j=1}^{J} a^s(P_j^s v, P_j^s v) \tag{4.8}$$

for any $v \in \mathscr{V}$.

This lemma may be proved by perturbation argument using Gårding's inequality and Lemmas A and B. Next, we need to verify Assumption 2.3.

LEMMA 4.1. *For any* $v \in \mathscr{V}$, *we have*

$$a^s(P_0 v, P_0 v) \leqslant \frac{4}{3} a^s(P_0 v, v) + cH^{2\alpha} a^s(v, v), \tag{4.9}$$

$$a^s(P_j v, P_j v) \leqslant \frac{4}{3} a^s(P_j v, v) + cH^2 a_j^s(v, v), \tag{4.10}$$

and

$$a^s(P_j^s v, P_j^s v) \leqslant \frac{4}{3} a^s(P_j^s v, v) + cH^2 a_j^s(v, v), \tag{4.11}$$

for all $j \in \{1, 2, \ldots, J\}$.

PROOF. For any $v \in \mathscr{V}$, by inequality (3.5), we have

$$\begin{aligned} a^s(P_0 v, P_0 v) &= a(P_0 v, P_0 v) - b(P_0 v, P_0 v) \\ &= a(v, P_0 v) - b(P_0 v, P_0 v) \\ &= a^s(v, P_0 v) + b((I - P_0) v, P_0 v) \\ &\leqslant a^s(P_0 v, v) + c\|(I - P_0) v\|\, |||\, P_0 v\, |||. \end{aligned}$$

It follows from Lemma B that

$$\begin{aligned} a^s(P_0 v, P_0 v) &\leqslant a^s(P_0 v, v) + cH^\alpha\, |||\, v\, |||\; |||\, P_0 v\, ||| \\ &\leqslant a^s(P_0 v, v) + \frac{1}{4} |||\, P_0 v\, |||^2 + (cH^\alpha\, |||\, v\, |||)^2, \end{aligned}$$

which certainly implies (4.9). For any $j \in \{1, 2, \dots, J\}$ and any $v \in \mathscr{V}$, since support of $P_j v$ is Ω_j whose diameter is H_j, an elementary estimate shows that $\|P_j v\| \leqslant cH_j ||| P_j v |||$. Hence, by (3.3), (4.1), and (3.5) we have

$$\begin{aligned} a^s(P_j v, P_j v) &= a(v, P_j v) - b(P_j v, P_j v) \\ &= a^s(v, P_j v) + b\big((I - P_j)v, P_j v\big) \\ &\leqslant a^s(P_j v, v) + c\,|||(I - P_j)v|||_j \|P_j v\| \\ &\leqslant a^s(P_j v, v) + cH\,|||v|||_j\,|||P_j v||| \\ &\leqslant a^s(P_j v, v) + \frac{1}{4}|||P_j v|||^2 + \big(cH\,|||v|||_j\big)^2. \end{aligned}$$

(4.10) follows. An analogous argument leads to (4.11). ■

Lemma 4.1 and the definitions of the subspaces $\mathscr{V}_j$ guarantee Assumption 2.3 with an appropriate constant C and $\tau' = \tau'' = cH^2$. As a straightforward consequence of Remark 2.3, we have

PROPOSITION 4.1. *There exists a constant $H_0 > 0$ such that, if $H \leqslant H_0$, then Algorithm 4.1 and its alternative are uniformly convergent, i.e.,*

$$|||E_1||| \leqslant \gamma_1 \tag{4.12}$$

and

$$|||E_2||| \leqslant \gamma_2 \tag{4.13}$$

where $\gamma_1 < 1$ and $\gamma_2 < 1$ are constants that do not depend on the number of subdomains J, the maximum size of the subdomains H, or the mesh size of the fine triangulation h.

Fix $j \in \{0, 1, \ldots, J\}$ and define the L^2 projection Q_j from $\mathscr{V}$ onto $\mathscr{V}_j$ and the linear operator $A_j: \mathscr{V}_j \to \mathscr{V}_j$ by

$$(Q_j v, \varphi) = (v, \varphi), \quad \forall v \in \mathscr{V}, \quad \forall \varphi \in \mathscr{V}_j, \tag{4.14}$$

and

$$(A_j v, \varphi) = a(v, \varphi) \quad \forall v, \varphi \in \mathscr{V}_j, \tag{4.15}$$

respectively. Fix $j \in \{1, 2, \ldots, J\}$ and define the linear operators $A_j^s: \mathscr{V}_j \to \mathscr{V}_j$ and $A: \mathscr{V} \to \mathscr{V}$ by

$$(A_j^s v, \varphi) = a^s(v, \varphi) \quad \forall v \in \mathscr{V}_j, \quad \forall \varphi \in \mathscr{V}_j \tag{4.16}$$

and

$$(Av, \varphi) = a(v, \varphi) \quad \forall v, \varphi \in \mathscr{V}, \tag{4.17}$$

respectively. It is easy to see that, for any $j \in \{0, 1, \ldots, J\}$,

$$P_j = A_j^{-1} Q_j A \tag{4.18}$$

and

$$P_j^s = (A_j^s)^{-1} Q_j A. \tag{4.19}$$

Thus, to avoid solving either nonsymmetric indefinite or symmetric positive definite subproblems in Algorithm 4.1 and its alternative, we introduce a scaled preconditioner R_j for either A_j or A_j^s that satisfies the property that, for any $v \in \mathscr{V}_j$, either

$$a^s(R_j A_j v, R_j A_j v) \leqslant a^s(R_j A_j v, v) + C_{\tilde{R}} H_j^{\beta} a^s(v, v) (\beta > 1)$$

$$C_R a^s(v, v) \leqslant a^s(v, R_j A_j v) \tag{4.20}$$

or

$$a^s(R_j A_j^s v, R_j A_j^s v) \leqslant a^s(R_j A_j^s v, v)$$

$$C_R a^s(v, v) \leqslant a^s(v, R_j A_j^s v), \qquad (4.21)$$

$j = 1, 2, \ldots, J$. The development of preconditioners has been the subject of intensive research (cf. [1–3, 11, 12]). Let $T_0 = P_0$ and $T_j = R_j Q_j A$ and consider the following Schwarz alternating procedure applied to (3.8) using inexact subproblem solvers.

ALGORITHM 4.2. *Given an approximation $u^l \in \mathscr{V}$ to the solution u of* (3.8), *define the next approximation $u^{l+1} \in \mathscr{V}$ as follows*:

1. *Set* $W_{-1} = u^l$.
2. *For* $j = 0, 1, \ldots, J$ *in turn, define* W_j *by*

$$W_j = W_{j-1} + T_j(u - W_{j-1}).$$

3. *Set* $u^{l+1} = W_J$.

The following lemma establishes Assumption 2.3 for an appropriate constant C and either $\tau' = \tau'' = cH^\beta$ or $\tau' = \tau'' = cH^2$.

LEMMA 4.2. *For any $v \in \mathscr{V}$ for all $j \in \{1, 2, \ldots, J\}$, if* (4.20) *holds, then*

$$a^s(T_j v, T_j v) \leqslant \frac{4}{3} a^s(T_j v, v) + cH^\beta a_j^s(v, v), \qquad (4.22)$$

and if (4.21) *holds, then*

$$a^s(T_j v, T_j v) \leqslant \frac{4}{3} a^s(T_j v, v) + cH^2 a_j^s(v, v). \qquad (4.23)$$

PROOF. Fix $j \in \{1, 2, \ldots, J\}$ and $v \in \mathscr{V}_j$. Then by (4.18), (4.20), the definitions of the linear operators T_j, Q_j, and A, and (3.5) we have

$$\begin{aligned}
a^s(T_jv, T_jv) &= a^s(R_jA_jP_jv, R_jA_jP_jv) \\
&\leqslant a^s(R_jA_jP_jv, P_jv) + C_{\tilde{R}}H_j^{\beta}a^s(P_jv, P_jv) \\
&= a^s(T_jv, v) + b(T_jv, (I - P_j)v) + C_{\tilde{R}}H_j^{\beta}a^s(P_jv, P_jv) \\
&\leqslant a^s(T_jv, v) + c\|T_jv\| \, |||(I - P_j)v|||_j + C_{\tilde{R}}H_j^{\beta}a^s(P_jv, P_jv) \\
&\leqslant a^s(T_jv, v) + \frac{1}{4}a^s(T_jv, T_jv) + cH_j^2a_j^s(v, v) \\
&\quad + C_{\tilde{R}}H_j^{\beta}a^s(P_jv, P_jv),
\end{aligned}$$

which implies (4.22). (4.23) follows from the above proof by replacing A_j by A_j^s and setting $C_{\tilde{R}} = 0$. ∎

Assumption 2.2 follows from

LEMMA 4.3. *If either* (4.20) *or* (4.21) *holds, then there exists a constant* $H_0 > 0$ *such that, if* $H \leqslant H_0$, *then*

$$c\,|||v|||^2 \leqslant \sum_{j=0}^{J} a^s(T_jv, T_jv) \tag{4.24}$$

for any $v \in \mathscr{V}$.

PROOF. Without loss of generality, we only prove inequality (4.24) by assuming (4.20). Fix $j \in \{1, 2, \ldots, J\}$ and $v \in \mathscr{V}$. Then by (4.20) we have

$$\begin{aligned}
a^s(P_jv, P_jv) &\leqslant \frac{1}{C_R}a^s(P_jv, T_jv) \\
&\leqslant \frac{1}{C_R}\,|||P_jv|||\,|||T_jv|||.
\end{aligned}$$

Hence,

$$||| P_j v ||| \leqslant \frac{1}{C_R} ||| T_j v ||| .$$

By the inequalities (3.5) and Lemma B, for any $v_0 \in \mathscr{V}_0$ we have

$$\begin{aligned} a(P_0 v, v_0) - b(v, v_0) &= a^s(P_0 v, v_0) + b((P_0 - I)v, v_0) \\ &\leqslant ||| P_0 v ||| \, ||| v_0 ||| + c\|(P_0 - I)v\| \, ||| v_0 ||| \\ &\leqslant ||| P_0 v ||| \, ||| v_0 ||| + cH^\alpha ||| v ||| \, ||| v_0 ||| . \end{aligned}$$

It follows from the above inequalities and Lemma A that

$$\begin{aligned} a^s(v, v) &= \sum_{j=0}^{J} a^s(v, v_j) \\ &= \sum_{j=0}^{J} \left(a(P_j v, v_j) - b(v, v_j) \right) \\ &\leqslant c \sum_{j=0}^{J} ||| P_j v ||| \, ||| v_j ||| + c\left(H^\alpha ||| v_0 ||| + \sum_{j=1}^{J} H ||| v_j ||| \right) ||| v ||| \\ &\leqslant \frac{c}{C_R} \sum_{j=0}^{J} ||| T_j v ||| \, ||| v_j ||| + c(H^{2\alpha} + JH^2)^{1/2} ||| v ||| \left(\sum_{j=0}^{J} ||| v_j |||^2 \right)^{1/} \\ &\leqslant \frac{cC_0^{1/2}}{C_R} \left(\sum_{j=0}^{J} ||| T_j v |||^2 \right)^{1/2} ||| v ||| + cC_0^{1/2}(H^{2\alpha} + JH^2)^{1/2} ||| v |||^2 . \end{aligned}$$

The proof is completed by using H_0 smaller enough so that $cC_0^{1/2}(H^{2\alpha} + JH^2)$ is bounded above by a constant less than one. ∎

Letting

$$E_3 = (I - T_J) \cdots (I - T_0),$$

then as a direct result of Remark 2.3 we have

PROPOSITION 4.2. *Under the assumptions of Lemma* 4.3, *there exists a constant* $H_0 > 0$ *such that, if* $H \leqslant H_0$, *then Algorithm* 4.2 *is uniformly convergent, i.e.,*

$$|||E_3||| \leqslant \gamma_3 \tag{4.25}$$

where $\gamma_3 < 1$ *is a constant that does not depend on the number of subdomains* J, *the maximum size of the subdomains* H, *and the mesh size of the fine triangulation* h.

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