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FIRST-ORDER SYSTEM LEAST SQUARES FOR THE SIGNORINI CONTACT PROBLEM IN LINEAR ELASTICITY*

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Abstract. A first-order system least squares formulation for the Signorini problem modeling frictionless contact in linear elasticity is studied. In addition to the displacement field, the stress tensor is used as an independent process variable. A contact boundary term is added to the usual least squares functional in order to achieve coercivity and continuity in appropriate norms. The discrete functional is shown to constitute an a posteriori error estimator on which an adaptive refinement strategy may be based. As finite element spaces, standard conforming piecewise polynomials for the displacement approximation are combined with Raviart–Thomas elements for the rows in the stress tensor. Computational results for a test problem of Hertzian contact illustrate the effectiveness of our least squares approach.

Key words. first-order system least squares, Signorini contact problem, boundary functional, a posteriori error estimator

AMS subject classifications. 65N30, 65N50, 74M15

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1. Introduction.

In this paper, a first-order system least squares formulation for the Signorini contact problem is studied. This approach constitutes a generalization of the least squares method for linear elasticity investigated in [7, 6]. The least squares functional is shown to be coercive to the energy norm of the error with respect to both independent variables, namely, the displacement field and the stress tensor. The least squares functional may also be bounded from above by this energy norm plus some extra boundary term. For the specific choices of finite element spaces used in this paper, this extra term is shown to converge at the same order as the energy norm. This allows us to evaluate the least squares functional locally and use it as an a posteriori error estimator on which an adaptive refinement strategy may be based. As an alternative, a least squares formulation related to the error of the strains is proposed as an extension of the approach in [8]. That formulation has the advantage of retaining its validity in the limit of an incompressible material.

The fact that the least squares functional constitutes an a posteriori error estimator for adaptive refinement is one of the main reasons for the popularity of this class of methods. During the last two decades, various least squares formulations were proposed for many types of boundary value problems in different application areas; see [1] for a survey of approaches that have been published. Other recent least squares methods concerned with nonlinear models in solid mechanics include those in [13, 19].

Contact problems arise in many engineering applications including metal forming or reliability studies of vehicle parts. From a mathematical point of view, contact problems involve the optimization of a nonlinear energy functional under constraints. For the Signorini contact problem investigated in the present paper, the energy

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function is quadratic and the (equality and inequality) constraints are linear. In this case, the nonlinearity of the problem is solely due to the inequality constraints. For approaches to a posteriori error estimation for contact problems, see, e.g., [9, 2]. Adaptive methods for rather general contact models are also treated in [15]; see also [21] for general background on computational contact mechanics. The least squares formulation presented here is not restricted to the Signorini contact problem for linear elasticity. Our treatment of the contact boundary condition can be combined with least squares formulation for other deformation models, for example, those for plastic behavior in [19, 17]. An extension of our least squares approach to contact with friction is currently being developed and will be presented elsewhere.

The contact conditions are treated by augmenting the least squares functional with an additional boundary functional in our formulation. Boundary functional approaches for the treatment of boundary conditions in the context of first-order systems have also been the subject of [18]. The finite element realization of the least squares contact formulation is done by standard conforming elements for the displacement and Raviart–Thomas elements for the stress components. Optimal order of approximation is achieved with respect to suitably scaled H^1 and $H(\mathrm{div})$ norms for displacement and stress, respectively.

In the next section the first-order system least squares formulation for the Signorini contact problem is derived which leads to a minimization problem for a quadratic functional subject to affine constraints. Section 3 presents the proof that the least squares functional is equivalent to an appropriate norm of the error. This implies that the least squares functional constitutes an a posteriori error estimator which may be used for adaptive refinement. An alternative least squares formulation which leads to approximations with respect to a strain-based norm of the error is investigated in section 4. Specific choices of finite element spaces for the approximation of the displacement and stress variables are proposed in section 5 along with details of the implementation. Finally, section 6 presents computational results which illustrate the effectiveness of our least squares approach.

2. First-order system least squares formulation of Signorini's problem.

We start from the equations of linear elasticity in the form

$$\begin{aligned} \mathrm{div}\,\boldsymbol{\sigma} &= \mathbf{0}\,, \\ \boldsymbol{\sigma} - \mathcal{C}\boldsymbol{\varepsilon}(\mathbf{u}) &= \mathbf{0}\,, \end{aligned} \tag{2.1}$$

where $\boldsymbol{\varepsilon}(\mathbf{u}) = (\nabla\mathbf{u} + \nabla\mathbf{u}^T)/2$ denotes the linear strain tensor and $\mathcal{C}$ describes the linear material law given by

$$\mathcal{C}\boldsymbol{\varepsilon}(\mathbf{u}) = \frac{E}{1+\nu}\left(\boldsymbol{\varepsilon}(\mathbf{u}) + \frac{\nu}{1-2\nu}(\mathrm{tr}\,\boldsymbol{\varepsilon}(\mathbf{u}))\,\mathbf{I}\right),$$

with the elasticity modulus E and Poisson ratio ν. The boundary of $\Omega \subset \mathbb{R}^d$ is divided into three parts as $\partial\Omega = \Gamma_D \cup \Gamma_N \cup \Gamma_C$. We consider boundary conditions of the form $\mathbf{u} = \mathbf{u}^D$ on Γ_D, $\boldsymbol{\sigma}\cdot\mathbf{n} = \mathbf{0}$ on Γ_N, and contact conditions

$$\begin{aligned} \mathbf{n}\cdot\mathbf{u} - g &\le 0\,, \\ \mathbf{n}\cdot(\boldsymbol{\sigma}\cdot\mathbf{n}) &\le 0\,, \\ \mathbf{t}\cdot(\boldsymbol{\sigma}\cdot\mathbf{n}) &= 0 \end{aligned} \tag{2.2}$$

on Γ_C (for contact without friction). Here, $\mathbf{n}$ and $\mathbf{t}$ denote the normal and tangential directions, respectively, on $\partial\Omega$, and g represents the gap, i.e., the distance of a boundary point in the reference configuration $\partial\Omega$ to the obstacle. The first condition in

(2.2) stands for the nonpenetration condition, the second for the direction of surface pressure, and the third for the absence of friction. In addition, a complementarity condition is valid which states that on Γ_C either $\mathbf{n}\cdot\mathbf{u}-g$ or $\mathbf{n}\cdot(\boldsymbol{\sigma}\cdot\mathbf{n})$ vanishes. The mechanical interpretation is that, at each point on the contact boundary not in contact with the obstacle, the traction force is zero in the normal direction.

The displacement conditions on Γ_D and traction conditions on Γ_N are built explicitly into the approximation spaces in our approach. To this end, we define

$$\begin{aligned} H^1_D(\Omega) &= \{q \in H^1(\Omega) : q = 0 \text{ on } \Gamma_D\},\\ H_N(\mathrm{div},\Omega) &= \{\mathbf{w} \in H(\mathrm{div},\Omega) : \mathbf{w}\cdot\mathbf{n} = 0 \text{ on } \Gamma_N\}\,. \end{aligned}$$

Throughout this paper, the norm on $L^2(\Omega)$, $L^2(\Omega)^d$, or $L^2(\Omega)^{d\times d}$ will be abbreviated by $\|\cdot\|$ and the corresponding inner product by $(\cdot\,,\cdot)$. We assume that Γ_D is a subset of $\partial\Omega$ with positive measure (length if $d=2$, area in the three-dimensional case) such that Korn's inequality is valid in the form

$$\|\mathbf{v}\|^2 + \|\nabla\mathbf{v}\|^2 \le C_K\|\boldsymbol{\varepsilon}(\mathbf{v})\|^2, \tag{2.3}$$

with a constant $C_K > 0$ (cf. [3, sect. VI.3]).

Motivated by the results in [7, 6] for linear elasticity computations, one may consider minimizing the quadratic least squares functional

$$\mathcal{F}(\mathbf{u},\boldsymbol{\sigma}) = \|\mathrm{div}\,\boldsymbol{\sigma}\|^2 + \|\mathcal{C}^{-1/2}\boldsymbol{\sigma} - \mathcal{C}^{1/2}\boldsymbol{\varepsilon}(\mathbf{u})\|^2 \tag{2.4}$$

among all $\mathbf{u} = \mathbf{u}^D + \hat{\mathbf{u}}$ with $\hat{\mathbf{u}} \in H^1_D(\Omega)^d$ and $\boldsymbol{\sigma} \in H_N(\mathrm{div},\Omega)^d$ which satisfy the linear constraints (2.2) and the complementarity condition

$$\langle \mathbf{n}\cdot(\boldsymbol{\sigma}\cdot\mathbf{n}), \mathbf{n}\cdot\mathbf{u} - g\rangle_{\Gamma_C} = 0\,. \tag{2.5}$$

For the precise definition of the inequality constraints in the trace spaces $H^{1/2}(\Gamma_C)$ (for $\mathbf{u}$) and $H^{-1/2}(\Gamma_C)$ (for $\boldsymbol{\sigma}\cdot\mathbf{n}$), see [11, sects. 5.3 and 5.5]. By $\langle\cdot\,,\cdot\rangle_{\Gamma_C}$ the duality pairing of these trace spaces is meant (see also [11, sect. 5.5]).

The only drawback of this approach is that the complementarity condition (2.5) is not linear. Without this complementarity condition the minimization problem would be for a quadratic functional under affine constraints which would simplify its solution considerably. Simply ignoring the complementarity condition is not possible since this would destroy the well posedness of the problem. Therefore, we consider the augmented least squares functional

$$\mathcal{F}_C(\mathbf{u},\boldsymbol{\sigma}) = \|\mathrm{div}\,\boldsymbol{\sigma}\|^2 + \|\mathcal{C}^{-1/2}\boldsymbol{\sigma} - \mathcal{C}^{1/2}\varepsilon(\mathbf{u})\|^2 + \langle \mathbf{n}\cdot\mathbf{u} - g, \mathbf{n}\cdot(\boldsymbol{\sigma}\cdot\mathbf{n})\rangle_{\Gamma_C}\,. \tag{2.6}$$

This quadratic functional is minimized for $\mathbf{u} = \mathbf{u}^D + \hat{\mathbf{u}}$ with $\hat{\mathbf{u}} \in H^1_D(\Omega)^d$ and $\boldsymbol{\sigma} \in H_N(\mathrm{div},\Omega)^d$ subject to the affine constraints (2.2). The usual trace theorems imply $\mathbf{u} \in H^{1/2}(\Gamma_C)$ and $\boldsymbol{\sigma}\cdot\mathbf{n} \in H^{-1/2}(\Gamma_C)$. If the geometry of the contact problem is such that $g \in H^{1/2}(\Gamma_C)$, then $\mathbf{n}\cdot\mathbf{u}-g \in H^{1/2}(\Gamma_C)$. Combined with $\mathbf{n}\cdot(\boldsymbol{\sigma}\cdot\mathbf{n}) \in H^{-1/2}(\Gamma_C)$, this ensures that the boundary functional term in (2.6) is well-defined. Let us denote by $\Gamma_{C,d}$ that part of the contact boundary where $\mathbf{n}\cdot\mathbf{u} - g = 0$ holds. This is the contact zone where the first constraint in (2.2) is active. Due to the complementarity condition (2.5) and the fact that both terms $\mathbf{n}\cdot\mathbf{u} - g$ and $\mathbf{n}\cdot(\boldsymbol{\sigma}\cdot\mathbf{n})$ are not allowed to be positive, we have $\mathbf{n}\cdot(\boldsymbol{\sigma}\cdot\mathbf{n}) = 0$ on $\Gamma_{C,s} = \Gamma_C\backslash\Gamma_{C,d}$. If this decomposition of the contact boundary as $\Gamma_C = \Gamma_{C,d}\cup\Gamma_{C,s}$ were known beforehand, then the contact problem would simply be a standard boundary value problem with (2.2) and (2.5) replaced by

$$\begin{aligned} \mathbf{n} \cdot \mathbf{u} - g &= 0 \text{ on } \Gamma_{C,d} , \\ \mathbf{n} \cdot (\boldsymbol{\sigma} \cdot \mathbf{n}) &= 0 \text{ on } \Gamma_{C,s} , \\ \mathbf{t} \cdot (\boldsymbol{\sigma} \cdot \mathbf{n}) &= 0 \text{ on } \Gamma_C . \end{aligned}$$

Regularity results for linear elasticity (see, e.g., [16]) can therefore be applied to contact problems. Under certain assumptions on the smoothness of the contact boundary, we have, for some $\beta > 0$, $\mathbf{u} \in H^{1+\beta}(\Omega)^d$, $\boldsymbol{\sigma} \in H^{\beta}(\Omega)^{d\times d}$, and $\operatorname{div} \boldsymbol{\sigma} \in H^{\beta}(\Omega)^d$. This implies that, for some $\alpha > 0$, $\mathbf{n} \cdot \mathbf{u} - g \in H^{1/2+\alpha}(\Gamma_C)$ and $\mathbf{n} \cdot (\boldsymbol{\sigma} \cdot \mathbf{n}) \in H^{-1/2+\alpha}(\Gamma_C)$ hold. Note, however, that α is restricted by the fact that the type of boundary condition changes at the interface between $\Gamma_{C,d}$ and $\Gamma_{C,s}$ even if Γ_C is smooth. Our finite element approximation results in the following sections will depend on such regularity assumptions.

The basic idea of the least squares finite element method lies in performing the minimization of (2.6) subject to the constraints (2.2) with respect to finite element spaces $\mathbf{U}_h \subset H^1_D(\Omega)^d$ and $\boldsymbol{\Sigma}_h \subset H_N(\operatorname{div}, \Omega)^d$. The specific choice of these finite element spaces will be discussed further below. The resulting problem consists of minimizing the quadratic functional (2.6) among all $\mathbf{u}_h = \mathbf{u}^D + \hat{\mathbf{u}}_h$ with $\hat{\mathbf{u}}_h \in \mathbf{U}_h$ and $\boldsymbol{\sigma}_h \in \boldsymbol{\Sigma}_h$ which satisfy the affine constraints (2.2).

3. The least squares functional as an a posteriori estimator. Our aim is to show that $\mathcal{F}_C(\mathbf{u}_h, \boldsymbol{\sigma}_h)$ constitutes an a posteriori error estimator, i.e., that its square root is bounded from below and from above by the approximation error scaled by suitable positive constants. More precisely, with the norm on $H^1(\Omega)^d \times H(\operatorname{div}, \Omega)^d$ given by

$$|||(\mathbf{u}, \boldsymbol{\sigma})||| = \left(\|\mathcal{C}^{1/2} \boldsymbol{\varepsilon}(\mathbf{u})\|^2 + \|\operatorname{div} \boldsymbol{\sigma}\|^2 + \|\mathcal{C}^{-1/2} \boldsymbol{\sigma}\|^2 \right)^{1/2} , \tag{3.1}$$

we will show that there exists a positive constant C_R such that

$$\mathcal{F}_C(\mathbf{u}_h, \boldsymbol{\sigma}_h) \geq C_R |||(\mathbf{u} - \mathbf{u}_h, \boldsymbol{\sigma} - \boldsymbol{\sigma}_h)|||^2 \tag{3.2}$$

and

$$\mathcal{F}_C(\mathbf{u}_h, \boldsymbol{\sigma}_h) \leq 2 |||(\mathbf{u} - \mathbf{u}_h, \boldsymbol{\sigma} - \boldsymbol{\sigma}_h)|||^2 + \langle \mathbf{n} \cdot \mathbf{u}_h - g, \mathbf{n} \cdot (\boldsymbol{\sigma}_h \cdot \mathbf{n}) \rangle_{\Gamma_C} \tag{3.3}$$

hold for all $(\mathbf{u}_h, \boldsymbol{\sigma}_h) \in (\mathbf{u}^D + \mathbf{U}_h) \times \boldsymbol{\Sigma}_h$ which satisfy conditions (2.2) on the contact boundary. The inequality in (3.2) establishes reliability of the error estimator, while the property (3.3) stands for its efficiency. The right-hand side in (3.3) contains an additional error term which will be handled separately in the finite element error estimates. Depending on the regularity of the underlying problem and on the selected finite element spaces, this additional error term may converge at the same (or higher) order as $|||(\mathbf{u} - \mathbf{u}_h, \boldsymbol{\sigma} - \boldsymbol{\sigma}_h)|||^2$. This will be discussed in more detail below in connection with the specific choice of finite element spaces.

LEMMA 3.1. *Let* $(\mathbf{u}, \boldsymbol{\sigma}) \in H^1(\Omega)^d \times H(\operatorname{div}, \Omega)^d$ *be the exact solution of the contact problem satisfying* (2.1) *under the constraints* (2.2) *and* (2.5). *Moreover, let* $(\mathbf{u}_h, \boldsymbol{\sigma}_h) \in (\mathbf{u}^D + \mathbf{U}_h) \times \boldsymbol{\Sigma}_h$ *be such that the constraints* (2.2) *are satisfied. Then,*

$$\langle \mathbf{n} \cdot \mathbf{u}_h - g, \mathbf{n} \cdot (\boldsymbol{\sigma}_h \cdot \mathbf{n}) \rangle_{\Gamma_C} \geq \langle \mathbf{n} \cdot (\mathbf{u}_h - \mathbf{u}), \mathbf{n} \cdot ((\boldsymbol{\sigma}_h - \boldsymbol{\sigma}) \cdot \mathbf{n}) \rangle_{\Gamma_C} . \tag{3.4}$$

Proof. Due to the complementarity condition, the contact boundary may be divided as $\Gamma_C = \Gamma_{C,d} \cup \Gamma_{C,s}$. On $\Gamma_{C,d}$, $\mathbf{n} \cdot \mathbf{u} - g = 0$; i.e., this is the boundary part which is in contact with the obstacle. The remainder of the contact boundary $\Gamma_{C,s}$ is in equilibrium which means that $\mathbf{n} \cdot (\boldsymbol{\sigma} \cdot \mathbf{n}) = 0$ holds there. The inequality constraints (2.2) imply

$$\langle \mathbf{n} \cdot \mathbf{u}_h - g, \mathbf{n} \cdot (\boldsymbol{\sigma} \cdot \mathbf{n}) \rangle_{\Gamma_{C,d}} \geq 0 \text{ and } \langle \mathbf{n} \cdot \mathbf{u} - g, \mathbf{n} \cdot (\boldsymbol{\sigma}_h \cdot \mathbf{n}) \rangle_{\Gamma_{C,s}} \geq 0 \,.$$

This leads to

$$\begin{aligned}
&\langle \mathbf{n} \cdot \mathbf{u}_h - g, \mathbf{n} \cdot (\boldsymbol{\sigma}_h \cdot \mathbf{n}) \rangle_{\Gamma_C} \\
&\quad \geq \langle \mathbf{n} \cdot \mathbf{u}_h - g, \mathbf{n} \cdot ((\boldsymbol{\sigma}_h - \boldsymbol{\sigma}) \cdot \mathbf{n}) \rangle_{\Gamma_{C,d}} + \langle \mathbf{n} \cdot (\mathbf{u}_h - \mathbf{u}), \mathbf{n} \cdot (\boldsymbol{\sigma}_h \cdot \mathbf{n}) \rangle_{\Gamma_{C,s}} \\
&\quad = \langle \mathbf{n} \cdot (\mathbf{u}_h - \mathbf{u}), \mathbf{n} \cdot ((\boldsymbol{\sigma}_h - \boldsymbol{\sigma}) \cdot \mathbf{n}) \rangle_{\Gamma_{C,d}} + \langle \mathbf{n} \cdot (\mathbf{u}_h - \mathbf{u}), \mathbf{n} \cdot ((\boldsymbol{\sigma}_h - \boldsymbol{\sigma}) \cdot \mathbf{n}) \rangle_{\Gamma_{C,s}} \\
&\quad = \langle \mathbf{n} \cdot (\mathbf{u}_h - \mathbf{u}), \mathbf{n} \cdot ((\boldsymbol{\sigma}_h - \boldsymbol{\sigma}) \cdot \mathbf{n}) \rangle_{\Gamma_C} . \qquad \square
\end{aligned}$$

THEOREM 3.2. *Let* $(\mathbf{u}, \boldsymbol{\sigma}) \in H^1(\Omega)^d \times H(\mathrm{div}, \Omega)^d$ *be the exact solution of the contact problem satisfying* (2.1) *under the constraints* (2.2) *and* (2.5). *Moreover, let* $(\mathbf{u}_h, \boldsymbol{\sigma}_h) \in (\mathbf{u}^D + \mathbf{U}_h) \times \boldsymbol{\Sigma}_h$ *be such that the constraints* (2.2) *are satisfied. Then,* (3.2) *holds with a constant* $C_R > 0$.

Proof. Using the fact that the exact solution of the contact problem satisfies (2.1) one obtains

$$\begin{aligned}
\mathcal{F}_C(\mathbf{u}_h, \boldsymbol{\sigma}_h) &= \|\mathrm{div}\, (\boldsymbol{\sigma}_h - \boldsymbol{\sigma})\|^2 + \|\mathcal{C}^{-1/2}(\boldsymbol{\sigma}_h - \boldsymbol{\sigma}) - \mathcal{C}^{1/2}\boldsymbol{\varepsilon}(\mathbf{u}_h - \mathbf{u})\|^2 \\
&\quad + \langle \mathbf{n} \cdot \mathbf{u}_h - g, \mathbf{n} \cdot (\boldsymbol{\sigma}_h \cdot \mathbf{n}) \rangle_{\Gamma_C} \\
&\geq \|\mathrm{div}\, (\boldsymbol{\sigma}_h - \boldsymbol{\sigma})\|^2 + \|\mathcal{C}^{-1/2}(\boldsymbol{\sigma}_h - \boldsymbol{\sigma}) - \mathcal{C}^{1/2}\boldsymbol{\varepsilon}(\mathbf{u}_h - \mathbf{u})\|^2 \\
&\quad + \langle \mathbf{n} \cdot (\mathbf{u}_h - \mathbf{u}), \mathbf{n} \cdot ((\boldsymbol{\sigma}_h - \boldsymbol{\sigma}) \cdot \mathbf{n}) \rangle_{\Gamma_C} \,,
\end{aligned} \tag{3.5}$$

where the last inequality follows from Lemma 3.1.

The remainder of the proof follows along similar lines as for Theorem 2.1 in [6]. We use the decomposition of an arbitrary matrix-valued function $\boldsymbol{\tau} \in L^2(\Omega)^{d \times d}$ into its symmetric and antisymmetric part,

$$\boldsymbol{\tau} = \mathrm{sy}\, \boldsymbol{\tau} + \mathrm{as}\, \boldsymbol{\tau} \text{ with } \mathrm{sy}\, \boldsymbol{\tau} = \frac{\boldsymbol{\tau} + \boldsymbol{\tau}^T}{2} \,,\ \mathrm{as}\, \boldsymbol{\tau} = \frac{\boldsymbol{\tau} - \boldsymbol{\tau}^T}{2} \,.$$

Obviously $(\mathrm{sy}\, \boldsymbol{\tau}, \mathrm{as}\, \boldsymbol{\tau}) = 0$ which implies

$$\|\boldsymbol{\tau}\|^2 = \|\mathrm{sy}\, \boldsymbol{\tau}\|^2 + \|\mathrm{as}\, \boldsymbol{\tau}\|^2 \geq \|\mathrm{as}\, \boldsymbol{\tau}\|^2 \,.$$

If this estimate is applied with $\boldsymbol{\tau} = \mathcal{C}^{-1/2}(\boldsymbol{\sigma}_h - \boldsymbol{\sigma}) - \mathcal{C}^{1/2}(\boldsymbol{\varepsilon}(\mathbf{u}_h) - \boldsymbol{\varepsilon}(\mathbf{u}))$, we obtain

$$\begin{aligned}
\|\mathcal{C}^{-1/2}(\boldsymbol{\sigma}_h - \boldsymbol{\sigma}) - \mathcal{C}^{1/2}(\boldsymbol{\varepsilon}(\mathbf{u}_h) - \boldsymbol{\varepsilon}(\mathbf{u}))\|^2 &\geq \|\mathrm{as}(\mathcal{C}^{-1/2}(\boldsymbol{\sigma}_h - \boldsymbol{\sigma}))\|^2 \\
&= \frac{1+\nu}{E} \|\mathrm{as}(\boldsymbol{\sigma}_h - \boldsymbol{\sigma})\|^2
\end{aligned} \tag{3.6}$$

(note that $\mathrm{as}(\mathcal{C}^{1/2}(\boldsymbol{\varepsilon}(\mathbf{u}_h) - \boldsymbol{\varepsilon}(\mathbf{u}))) = \mathbf{0}$). With this, (3.5) turns into

$$
\begin{aligned}
\mathcal{F}_C(\mathbf{u}_h, \boldsymbol{\sigma}_h) \geq{}& \|\mathrm{div}\,(\boldsymbol{\sigma}_h - \boldsymbol{\sigma})\|^2 + \frac{1}{2}\|\mathcal{C}^{-1/2}(\boldsymbol{\sigma}_h - \boldsymbol{\sigma}) - \mathcal{C}^{1/2}\boldsymbol{\varepsilon}(\mathbf{u}_h - \mathbf{u})\|^2 \\
&+ \frac{1+\nu}{2E}\|\mathrm{as}(\boldsymbol{\sigma}_h - \boldsymbol{\sigma})\|^2 + \langle \mathbf{n}\cdot(\mathbf{u}_h - \mathbf{u}), \mathbf{n}\cdot((\boldsymbol{\sigma}_h - \boldsymbol{\sigma})\cdot\mathbf{n})\rangle_{\Gamma_C} \\
\geq{}& \gamma(1-\gamma)\left(\frac{1}{\gamma}\|\mathrm{div}\,(\boldsymbol{\sigma}_h - \boldsymbol{\sigma})\|^2\right. \\
&+ \frac{1}{2(1-\gamma)}\|\mathcal{C}^{-1/2}(\boldsymbol{\sigma}_h - \boldsymbol{\sigma}) - \mathcal{C}^{1/2}\boldsymbol{\varepsilon}(\mathbf{u}_h - \mathbf{u})\|^2 \\
&\left.+\frac{1+\nu}{2\gamma E}\|\mathrm{as}(\boldsymbol{\sigma}_h - \boldsymbol{\sigma})\|^2 + \langle \mathbf{n}\cdot(\mathbf{u}_h - \mathbf{u}), \mathbf{n}\cdot((\boldsymbol{\sigma}_h - \boldsymbol{\sigma})\cdot\mathbf{n})\rangle_{\Gamma_C}\right) \\
=:{}& \gamma(1-\gamma)\hat{\mathcal{F}}_C(\mathbf{u}_h, \boldsymbol{\sigma}_h),
\end{aligned}
\tag{3.7}
$$

where $\gamma \in (0,1)$ will be chosen appropriately further below. Using the orthogonal decomposition into symmetric and antisymmetric parts for an arbitrary $\boldsymbol{\tau} \in H_N(\mathrm{div},\Omega)^d$ also gives us

$$
\begin{aligned}
(\boldsymbol{\tau}, \boldsymbol{\varepsilon}(\mathbf{v})) &= (\mathrm{sy}\,\boldsymbol{\tau}, \boldsymbol{\varepsilon}(\mathbf{v})) + (\mathrm{as}\,\boldsymbol{\tau}, \boldsymbol{\varepsilon}(\mathbf{v})) = (\mathrm{sy}\,\boldsymbol{\tau}, \boldsymbol{\varepsilon}(\mathbf{v})) \\
&= (\mathrm{sy}\,\boldsymbol{\tau}, \nabla\,\mathbf{v}) = (\boldsymbol{\tau}, \nabla\,\mathbf{v}) - (\mathrm{as}\,\boldsymbol{\tau}, \nabla\,\mathbf{v}) \\
&= \langle \boldsymbol{\tau}\cdot\mathbf{n}, \mathbf{v}\rangle_{\Gamma_C} - (\mathrm{div}\,\boldsymbol{\tau}, \mathbf{v}) - (\mathrm{as}\,\boldsymbol{\tau}, \nabla\,\mathbf{v})
\end{aligned}
$$

for any $\mathbf{v} \in H^1_D(\Omega)^d$. Inserting $\boldsymbol{\tau} = \boldsymbol{\sigma}_h - \boldsymbol{\sigma}$ and $\mathbf{v} = \mathbf{u}_h - \mathbf{u}$ leads to

$$
\begin{aligned}
(\boldsymbol{\sigma}_h - \boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\mathbf{u}_h - \mathbf{u})) &= \langle (\boldsymbol{\sigma}_h - \boldsymbol{\sigma})\cdot\mathbf{n}, (\mathbf{u}_h - \mathbf{u})\rangle_{\Gamma_C} \\
&\quad - (\mathrm{div}\,(\boldsymbol{\sigma}_h - \boldsymbol{\sigma}), \mathbf{u}_h - \mathbf{u}) - (\mathrm{as}\,(\boldsymbol{\sigma}_h - \boldsymbol{\sigma}), \nabla\,(\mathbf{u}_h - \mathbf{u})) \\
&= \langle \mathbf{n}\cdot((\boldsymbol{\sigma}_h - \boldsymbol{\sigma})\cdot\mathbf{n}), \mathbf{n}\cdot(\mathbf{u}_h - \mathbf{u})\rangle_{\Gamma_C} \\
&\quad - (\mathrm{div}\,(\boldsymbol{\sigma}_h - \boldsymbol{\sigma}), \mathbf{u}_h - \mathbf{u}) - (\mathrm{as}\,(\boldsymbol{\sigma}_h - \boldsymbol{\sigma}), \nabla\,(\mathbf{u}_h - \mathbf{u}))\,.
\end{aligned}
\tag{3.8}
$$

We may therefore rewrite (3.7) as

$$
\begin{aligned}
\hat{\mathcal{F}}_C(\mathbf{u}_h\,, \boldsymbol{\sigma}_h) ={}& \frac{1}{\gamma}\|\mathrm{div}\,(\boldsymbol{\sigma}_h - \boldsymbol{\sigma})\|^2 + \frac{1}{2(1-\gamma)}\|\mathcal{C}^{-1/2}(\boldsymbol{\sigma}_h - \boldsymbol{\sigma})\|^2 \\
&+ \frac{1}{2(1-\gamma)}\|\mathcal{C}^{1/2}\boldsymbol{\varepsilon}(\mathbf{u}_h - \mathbf{u})\|^2 - \frac{1}{1-\gamma}(\boldsymbol{\sigma}_h - \boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\mathbf{u}_h - \mathbf{u})) \\
&+ \frac{1+\nu}{2\gamma E}\|\mathrm{as}(\boldsymbol{\sigma}_h - \boldsymbol{\sigma})\|^2 + \langle \mathbf{n}\cdot(\mathbf{u}_h - \mathbf{u}), \mathbf{n}\cdot((\boldsymbol{\sigma}_h - \boldsymbol{\sigma})\cdot\mathbf{n})\rangle_{\Gamma_C} \\
={}&\frac{1}{\gamma}\|\mathrm{div}\,(\boldsymbol{\sigma}_h - \boldsymbol{\sigma})\|^2 + \frac{1}{2(1-\gamma)}\|\mathcal{C}^{-1/2}(\boldsymbol{\sigma}_h - \boldsymbol{\sigma})\|^2 + \frac{1}{2(1-\gamma)}\|\mathcal{C}^{1/2}\boldsymbol{\varepsilon}(\mathbf{u}_h - \mathbf{u})\|^2 \\
&- \frac{\gamma}{1-\gamma}(\boldsymbol{\sigma}_h - \boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\mathbf{u}_h - \mathbf{u})) - \langle \mathbf{n}\cdot((\boldsymbol{\sigma}_h - \boldsymbol{\sigma})\cdot\mathbf{n}), \mathbf{n}\cdot(\mathbf{u}_h - \mathbf{u})\rangle_{\Gamma_C} \\
&+ (\mathrm{div}\,(\boldsymbol{\sigma}_h - \boldsymbol{\sigma}), \mathbf{u}_h - \mathbf{u}) + (\mathrm{as}\,(\boldsymbol{\sigma}_h - \boldsymbol{\sigma}), \nabla\,(\mathbf{u}_h - \mathbf{u})) \\
&+ \frac{1+\nu}{2\gamma E}\|\mathrm{as}(\boldsymbol{\sigma}_h - \boldsymbol{\sigma})\|^2 + \langle \mathbf{n}\cdot(\mathbf{u}_h - \mathbf{u}), \mathbf{n}\cdot((\boldsymbol{\sigma}_h - \boldsymbol{\sigma})\cdot\mathbf{n})\rangle_{\Gamma_C}
\end{aligned}
$$

which leads to

$$
\begin{aligned}
\hat{\mathcal{F}}_C(\mathbf{u}_h,\boldsymbol{\sigma}_h) &= \frac{1}{\gamma}\|\mathrm{div}\,(\boldsymbol{\sigma}_h-\boldsymbol{\sigma})\|^2 + \frac{1}{2(1-\gamma)}\|\mathcal{C}^{-1/2}(\boldsymbol{\sigma}_h-\boldsymbol{\sigma})\|^2\\
&+ \frac{1}{2(1-\gamma)}\|\mathcal{C}^{1/2}\boldsymbol{\varepsilon}(\mathbf{u}_h-\mathbf{u})\|^2 - \frac{\gamma}{1-\gamma}(\boldsymbol{\sigma}_h-\boldsymbol{\sigma},\boldsymbol{\varepsilon}(\mathbf{u}_h-\mathbf{u}))\\
&+ (\mathrm{div}\,(\boldsymbol{\sigma}_h-\boldsymbol{\sigma}),\mathbf{u}_h-\mathbf{u}) + (\mathrm{as}\,(\boldsymbol{\sigma}_h-\boldsymbol{\sigma}),\nabla\,(\mathbf{u}_h-\mathbf{u})) + \frac{1+\nu}{2\gamma E}\|\mathrm{as}(\boldsymbol{\sigma}_h-\boldsymbol{\sigma})\|^2\\
&= \frac{1}{\gamma}\|\mathrm{div}\,(\boldsymbol{\sigma}_h-\boldsymbol{\sigma})\|^2 + \frac{1}{2}\|\mathcal{C}^{-1/2}(\boldsymbol{\sigma}_h-\boldsymbol{\sigma})\|^2 + \frac{1}{2}\|\mathcal{C}^{1/2}\boldsymbol{\varepsilon}(\mathbf{u}_h-\mathbf{u})\|^2\\
&+ \frac{\gamma}{2(1-\gamma)}\|\mathcal{C}^{-1/2}(\boldsymbol{\sigma}_h-\boldsymbol{\sigma}) - \mathcal{C}^{1/2}\boldsymbol{\varepsilon}(\mathbf{u}_h-\mathbf{u})\|^2\\
&+ (\mathrm{div}\,(\boldsymbol{\sigma}_h-\boldsymbol{\sigma}),\mathbf{u}_h-\mathbf{u}) + (\mathrm{as}\,(\boldsymbol{\sigma}_h-\boldsymbol{\sigma}),\nabla\,(\mathbf{u}_h-\mathbf{u})) + \frac{1+\nu}{2\gamma E}\|\mathrm{as}(\boldsymbol{\sigma}_h-\boldsymbol{\sigma})\|^2\\
&\geq \frac{1}{\gamma}\|\mathrm{div}\,(\boldsymbol{\sigma}_h-\boldsymbol{\sigma})\|^2 + \frac{1}{2}\|\mathcal{C}^{-1/2}(\boldsymbol{\sigma}_h-\boldsymbol{\sigma})\|^2 + \frac{1}{2}\|\mathcal{C}^{1/2}\boldsymbol{\varepsilon}(\mathbf{u}_h-\mathbf{u})\|^2\\
&- \frac{1}{2\gamma}\|\mathrm{div}\,(\boldsymbol{\sigma}_h-\boldsymbol{\sigma})\|^2 - \frac{\gamma}{2}\|\mathbf{u}_h-\mathbf{u}\|^2 - \frac{\gamma E}{2(1+\nu)}\|\nabla\,(\mathbf{u}_h-\mathbf{u})\|^2\,.
\end{aligned}
\tag{3.9}
$$

Korn's inequality (2.3) implies

$$
\begin{aligned}
\|\mathcal{C}^{1/2}\boldsymbol{\varepsilon}(\mathbf{u}_h-\mathbf{u})\|^2 &= (\mathcal{C}\boldsymbol{\varepsilon}(\mathbf{u}_h-\mathbf{u}),\boldsymbol{\varepsilon}(\mathbf{u}_h-\mathbf{u}))\\
&= \frac{E}{1+\nu}\left(\boldsymbol{\varepsilon}(\mathbf{u}_h-\mathbf{u}) + \frac{\nu}{1-2\nu}(\mathrm{tr}\,\boldsymbol{\varepsilon}(\mathbf{u}_h-\mathbf{u}))\mathbf{I},\boldsymbol{\varepsilon}(\mathbf{u}_h-\mathbf{u})\right)\\
&= \frac{E}{1+\nu}\left(\|\boldsymbol{\varepsilon}(\mathbf{u}_h-\mathbf{u})\|^2 + \frac{\nu}{1-2\nu}\|\mathrm{tr}\,\boldsymbol{\varepsilon}(\mathbf{u}_h-\mathbf{u})\|^2\right)\\
&\geq \frac{E}{1+\nu}\|\boldsymbol{\varepsilon}(\mathbf{u}_h-\mathbf{u})\|^2 \geq \frac{E}{1+\nu}\,\frac{1}{C_K}\left(\|\nabla\,(\mathbf{u}_h-\mathbf{u})\|^2 + \|\mathbf{u}_h-\mathbf{u}\|^2\right)\,.
\end{aligned}
$$

This leads to

$$
\|\mathbf{u}_h-\mathbf{u}\|^2 + \frac{E}{1+\nu}\|\nabla\,(\mathbf{u}_h-\mathbf{u})\|^2 \leq C_0\|\mathcal{C}^{1/2}\boldsymbol{\varepsilon}(\mathbf{u}_h-\mathbf{u})\|^2,
\tag{3.10}
$$

with $C_0 = C_K\ \max\{1,(1+\nu)/E\}$. If we combine (3.10) with (3.9), we obtain

$$
\begin{aligned}
\hat{\mathcal{F}}_C(\mathbf{u}_h,\boldsymbol{\sigma}_h) \geq{}& \frac{1}{2\gamma}\|\mathrm{div}\,(\boldsymbol{\sigma}_h-\boldsymbol{\sigma})\|^2 + \frac{1}{2}\|\mathcal{C}^{-1/2}(\boldsymbol{\sigma}_h-\boldsymbol{\sigma})\|^2\\
&+ \frac{1}{2}(1-\gamma C_0)\|\mathcal{C}^{1/2}\boldsymbol{\varepsilon}(\mathbf{u}_h-\mathbf{u})\|^2\,.
\end{aligned}
\tag{3.11}
$$

Finally, choosing $\gamma = \min\{1/(2C_0),1/2\} \in (0,1)$ gives us

$$
\hat{\mathcal{F}}_C(\mathbf{u}_h,\boldsymbol{\sigma}_h) \geq \|\mathrm{div}\,(\boldsymbol{\sigma}_h-\boldsymbol{\sigma})\|^2 + \frac{1}{2}\|\mathcal{C}^{-1/2}(\boldsymbol{\sigma}_h-\boldsymbol{\sigma})\|^2 + \frac{1}{4}\|\mathcal{C}^{1/2}\boldsymbol{\varepsilon}(\mathbf{u}_h-\mathbf{u})\|^2
$$

which means that (3.2) holds with $C_R = \gamma(1-\gamma)/4$. ◻

THEOREM 3.3. *Let* $(\mathbf{u},\boldsymbol{\sigma}) \in H^1(\Omega)^d \times H(\mathrm{div},\Omega)^d$ *be the exact solution of the contact problem satisfying* (2.1) *under the constraints* (2.2) *and* (2.5). *If* $(\mathbf{u}_h,\boldsymbol{\sigma}_h) \in (\mathbf{u}^D+\mathbf{U}_h)\times\boldsymbol{\Sigma}_h$ *satisfies the constraints* (2.2), *then* (3.3) *holds.*

Proof. We start again from (3.5) and obtain

$$\begin{aligned} \mathcal{F}_C(\mathbf{u}_h,\boldsymbol{\sigma}_h) &= \|\mathrm{div}\,(\boldsymbol{\sigma}_h-\boldsymbol{\sigma})\|^2 + \|\mathcal{C}^{-1/2}(\boldsymbol{\sigma}_h-\boldsymbol{\sigma}) - \mathcal{C}^{1/2}\boldsymbol{\varepsilon}(\mathbf{u}_h-\mathbf{u})\|^2 \\ &\quad + \langle \mathbf{n}\cdot\mathbf{u}_h - g, \mathbf{n}\cdot(\boldsymbol{\sigma}_h\cdot\mathbf{n})\rangle_{\Gamma_C} \\ &\le \|\mathrm{div}\,(\boldsymbol{\sigma}_h-\boldsymbol{\sigma})\|^2 + 2\left(\|\mathcal{C}^{-1/2}(\boldsymbol{\sigma}_h-\boldsymbol{\sigma})\|^2 + \|\mathcal{C}^{1/2}\boldsymbol{\varepsilon}(\mathbf{u}_h-\mathbf{u})\|^2\right) \\ &\quad + \langle \mathbf{n}\cdot\mathbf{u}_h - g, \mathbf{n}\cdot(\boldsymbol{\sigma}_h\cdot\mathbf{n})\rangle_{\Gamma_C} \\ &\le 2|||(\mathbf{u}-\mathbf{u}_h,\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)|||^2 + \langle \mathbf{n}\cdot\mathbf{u}_h - g, \mathbf{n}\cdot(\boldsymbol{\sigma}_h\cdot\mathbf{n})\rangle_{\Gamma_C} \end{aligned} \tag{3.12}$$

which proves (3.3). ∎

The constant C_R in (3.2) and (3.3) depends on the material parameters, in particular, on Young's modulus E. In order to avoid this dependence on E, it is suggested to scale the problem in such a way that E is of order one. This is always possible by changing the scale in which the stress is measured. Alternatively, the different terms in the least squares functional (2.6) may be scaled accordingly in dependence on E. Our reliability and efficiency properties are uniform in the incompressible limit as $\nu \to 0.5$.

4. Another least squares formulation. The results in the previous section are concerned with a least squares functional which measures the error in energy norms with respect to both variables, displacements and stresses. Alternatively, one may use a least squares formulation that is valid for incompressible materials, which generalizes the approach investigated in [8]. It will turn out that this least squares functional measures the error in a norm related to the strain variable.

The constitutive equation in (2.1) may be rewritten as

$$\mathcal{C}^{-1}\boldsymbol{\sigma} - \boldsymbol{\varepsilon}(\mathbf{u}) = \mathbf{0}\,. \tag{4.1}$$

This leads to the differently weighted augmented least squares functional

$$\mathcal{G}_C(\mathbf{u},\boldsymbol{\sigma}) = \|\mathrm{div}\,\boldsymbol{\sigma}\|^2 + \|\mathcal{C}^{-1}\boldsymbol{\sigma} - \boldsymbol{\varepsilon}(\mathbf{u})\|^2 + \langle \mathbf{n}\cdot\mathbf{u} - g, \mathbf{n}\cdot(\boldsymbol{\sigma}\cdot\mathbf{n})\rangle_{\Gamma_C} \tag{4.2}$$

defined for $\mathbf{u} = \mathbf{u}^D + \hat{\mathbf{u}}$ with $\hat{\mathbf{u}} \in H^1_D(\Omega)^d$ and $\boldsymbol{\sigma} \in H_N(\mathrm{div},\Omega)^d$ subject to the affine constraints in (2.2).

LEMMA 4.1 (see [8]). *The following inequalities hold:*

$$\|\boldsymbol{\tau}\| \le C\,\left(\sqrt{(\mathcal{C}^{-1}\boldsymbol{\tau},\,\boldsymbol{\tau})} + \|\mathrm{div}\,\boldsymbol{\tau}\|_{-1,D}\right) \quad \forall\,\boldsymbol{\tau} \in H_N(\mathrm{div},\Omega)^d, \tag{4.3}$$

$$\|\boldsymbol{\tau} - \boldsymbol{\tau}^t\| \le C\,\|\mathcal{C}^{-1}\boldsymbol{\tau} - \boldsymbol{\varepsilon}(\mathbf{v})\| \quad \forall\,\boldsymbol{\tau} \in L^2(\Omega)^{d\times d}\ \forall\,\mathbf{v} \in H^1(\Omega)^d, \tag{4.4}$$

$$\|\mathcal{C}^{-1}\boldsymbol{\tau}\| \le \frac{1+\nu}{E}\,\|\boldsymbol{\tau}\| \quad \forall\,\boldsymbol{\tau} \in L^2(\Omega)^{d\times d}\,, \tag{4.5}$$

where $\|\cdot\|_{-1,D}$ *denotes the standard norm of the dual space of* $H^1_D(\Omega)$.

THEOREM 4.2. *Let* $(\mathbf{u},\boldsymbol{\sigma}) \in H^1(\Omega)^d \times H(\mathrm{div},\Omega)^d$ *be the exact solution of the contact problem satisfying* (2.1) *under the constraints* (2.2) *and* (2.5). *Moreover, let* $(\mathbf{u}_h,\boldsymbol{\sigma}_h) \in (\mathbf{u}^D + \mathbf{U}_h) \times \boldsymbol{\Sigma}_h$ *be such that the constraints* (2.2) *are satisfied. Then,*

$$\mathcal{G}_C(\mathbf{u}_h,\boldsymbol{\sigma}_h) \ge C_R'\left(\|\mathbf{u}-\mathbf{u}_h\|^2_{1,\Omega} + \|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|^2 + \|\mathrm{div}\,(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)\|^2\right), \tag{4.6}$$

with a constant $C_R' > 0$.

Proof. Using the fact that $(\mathbf{u},\boldsymbol{\sigma})$ satisfies (2.1), one obtains

$$\begin{aligned}\mathcal{G}_C(\mathbf{u}_h,\boldsymbol{\sigma}_h) = &\|\mathrm{div}\,(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)\|^2 + \|\mathcal{C}^{-1}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)-\varepsilon(\mathbf{u}-\mathbf{u}_h)\|^2\\ &+\langle \mathbf{n}\cdot\mathbf{u}_h - g, \mathbf{n}\cdot(\boldsymbol{\sigma}_h\cdot\mathbf{n})\rangle_{\Gamma_C}\,.\end{aligned} \tag{4.7}$$

The fact that $\mathbf{t}\cdot((\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)\cdot\mathbf{n}) = 0$ on Γ_C implies $(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)\cdot\mathbf{n} = (\mathbf{n}\cdot(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)\cdot\mathbf{n})\,\mathbf{n}$ on Γ_C. By integration by parts; Lemma 3.1; the Cauchy–Schwarz, the Korn, and the triangle inequalities; and (4.7), (4.4), and (4.5), we have

$$\begin{aligned}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h,\varepsilon(\mathbf{u}-\mathbf{u}_h)) &= \langle(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)\cdot\mathbf{n},\mathbf{u}-\mathbf{u}_h\rangle_{\Gamma_C} - (\mathrm{div}\,(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h),\mathbf{u}-\mathbf{u}_h)\\ &\quad -(\mathrm{as}\,(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h),\nabla(\mathbf{u}-\mathbf{u}_h))\\ &= \langle\mathbf{n}\cdot((\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)\cdot\mathbf{n}),\mathbf{n}\cdot(\mathbf{u}-\mathbf{u}_h)\rangle_{\Gamma_C} - (\mathrm{div}\,(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h),\mathbf{u}-\mathbf{u}_h)\\ &\quad -(\mathrm{as}\,(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h),\nabla(\mathbf{u}-\mathbf{u}_h))\\ &\le \langle\mathbf{n}\cdot\mathbf{u}_h-g,\,\mathbf{n}\cdot(\boldsymbol{\sigma}_h\cdot\mathbf{n})\rangle_{\Gamma_C}\\ &\quad + C\,\left(\|\mathrm{div}\,(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)\| + \|\mathrm{as}\,(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)\|\right)\|\varepsilon(\mathbf{u}-\mathbf{u}_h)\|\\ &\le \mathcal{G}_C(\mathbf{u}_h,\boldsymbol{\sigma}_h)\\ &\quad + C\,\left(\|\mathrm{div}\,(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)\| + \|\mathcal{C}^{-1}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)-\varepsilon(\mathbf{u}-\mathbf{u}_h)\|\right)\\ &\qquad\quad \left(\|\varepsilon(\mathbf{u}-\mathbf{u}_h)-\mathcal{C}^{-1}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)\| + \|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|\right)\\ &\le \mathcal{G}_C(\mathbf{u}_h,\boldsymbol{\sigma}_h) + C\,\sqrt{\mathcal{G}_C(\mathbf{u}_h,\boldsymbol{\sigma}_h)}\,\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|\,.\end{aligned}$$

It then follows from the Cauchy–Schwarz inequality and (4.3) that

$$\begin{aligned}(\mathcal{C}^{-1}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h),\boldsymbol{\sigma}-\boldsymbol{\sigma}_h) &= (\mathcal{C}^{-1}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)-\varepsilon(\mathbf{u}-\mathbf{u}_h),\boldsymbol{\sigma}-\boldsymbol{\sigma}_h) + (\varepsilon(\mathbf{u}-\mathbf{u}_h),\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)\\ &\le \|\mathcal{C}^{-1}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)-\varepsilon(\mathbf{u}-\mathbf{u}_h)\|\,\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|\\ &\quad + \mathcal{G}_C(\mathbf{u}_h,\boldsymbol{\sigma}_h) + C\,\sqrt{\mathcal{G}_C(\mathbf{u}_h,\boldsymbol{\sigma}_h)}\,\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|\\ &\le C\,\mathcal{G}_C(\mathbf{u}_h,\boldsymbol{\sigma}_h)\\ &\quad + C\,\sqrt{\mathcal{G}_C(\mathbf{u}_h,\boldsymbol{\sigma}_h)}\,\left(\sqrt{(\mathcal{C}^{-1}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h),\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)} + \|\mathrm{div}\,(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)\|_{-1,D}\right)\,.\end{aligned}$$

Hence,

$$(\mathcal{C}^{-1}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h),\boldsymbol{\sigma}-\boldsymbol{\sigma}_h) \le C\,\mathcal{G}_C(\mathbf{u}_h,\boldsymbol{\sigma}_h)\,,$$

which, together with (4.3), implies

$$\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|^2 \le C\,\mathcal{G}_C(\mathbf{u}_h,\boldsymbol{\sigma}_h). \tag{4.8}$$

Using the Korn and the triangle inequalities, (4.5), and (4.8) lead to

$$\begin{aligned}\|\mathbf{u}-\mathbf{u}_h\|^2_{1,\Omega} &\le C\,\|\varepsilon(\mathbf{u}-\mathbf{u}_h)\|^2\\ &\le C\,\left(\|\varepsilon(\mathbf{u}-\mathbf{u}_h)-\mathcal{C}^{-1}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)\|^2 + \|\mathcal{C}^{-1}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)\|^2\right)\\ &\le C\,\mathcal{G}_C(\mathbf{u}_h,\boldsymbol{\sigma}_h)\,.\end{aligned} \tag{4.9}$$

Now, (4.6) is a direct consequence of (4.8), (4.9), and the fact that the inequality $\|\mathrm{div}\,(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)\|^2 \le \mathcal{G}_C(\mathbf{u}_h,\boldsymbol{\sigma}_h)$ holds. This completes the proof of the theorem. ∎

THEOREM 4.3. *Under the assumptions of Theorem* 4.2, *we have*

$$\begin{aligned}\mathcal{G}_C(\mathbf{u}_h,\boldsymbol{\sigma}_h) \le 2\,\|\varepsilon(\mathbf{u}-\mathbf{u}_h)\|^2_{1,\Omega} + \frac{2(1+\nu)}{E}\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|^2 + \|\mathrm{div}\,(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)\|^2\\ + \langle\mathbf{n}\cdot\mathbf{u}_h-g,\mathbf{n}\cdot(\boldsymbol{\sigma}_h\cdot\mathbf{n})\rangle_{\Gamma_C}\,.\end{aligned} \tag{4.10}$$

Proof. (4.10) follows easily from (4.7), the triangle inequality, and (4.5). ∎

5. Finite element approximation. For the finite element approximation, we need to specify suitable spaces $\mathbf{U}_h \subset H^1_D(\Omega)^d$ and $\mathbf{\Sigma}_h \subset H_N(\mathrm{div},\Omega)^d$ to represent the displacement field and stress tensor, respectively. A natural choice for $\mathbf{U}_h$ would be, for each displacement component, to use continuous functions which consist of piecewise polynomials of given degree $k \geq 1$ on a given triangulation $\mathcal{T}_h$. For the strain-based formulation of section 4, the resulting finite element approximation is uniform in the incompressible limit as $\nu \to 1/2$. If one is interested in results which are uniform in the incompressible limit also for the energy-based formulation based on the functional $\mathcal{F}_C(\,\cdot\,)$, then either $k \geq 4$ has to be chosen or nonconforming elements, e.g., those of degree 2 by Fortin and Soulie [10], need to be used. We will not investigate the incompressible limit in this work and choose conforming elements of degree $k = 1$ or $k = 2$ instead in our computations. Raviart–Thomas spaces for each row of the stress tensor are an appropriate choice for $\mathbf{\Sigma}_h$. These spaces consist of piecewise polynomials of degree k in each component with the restriction that the normal component is of degree $k-1$ along each $(d-1)$-dimensional hyperplane in each element. Moreover, the normal component is continuous at the interface between elements (edges for $d = 2$, faces in the three-dimensional case).

The standard finite element interpolation estimates (cf. [3, sect. II.6] and [4, sect. 14.3]) lead to

$$\|\boldsymbol{\varepsilon}(\mathbf{u} - \mathcal{I}_h \mathbf{u})\|^2 \leq C_I h^{2k} |\mathbf{u}|^2_{k+1,\Omega}, \tag{5.1}$$

with a constant C_I, where $\mathcal{I}_h : H^1_D(\Omega)^d \to \mathbf{U}_h$ denotes the corresponding projection. On the other hand, using estimates for the interpolation operator $\mathcal{R}_h : H_N(\mathrm{div},\Omega)^d \to \mathbf{\Sigma}_h$ associated with Raviart–Thomas elements (cf. [5, sect. III.3]), we obtain

$$\|\mathrm{div}\,(\boldsymbol{\sigma} - \mathcal{R}_h \boldsymbol{\sigma})\|^2 + \|\boldsymbol{\sigma} - \mathcal{R}_h \boldsymbol{\sigma}\|^2 \leq C_I h^{2k} \left(|\mathrm{div}\,\boldsymbol{\sigma}|^2_{k,\Omega} + |\boldsymbol{\sigma}|^2_{k,\Omega} \right) . \tag{5.2}$$

The construction of $\mathcal{R}_h$ is such that, restricted to an edge (or face) E, it coincides with the local $L^2(E)$ orthogonal projection Q_h onto the polynomials of degree $k-1$ (cf. [5, sect. III.3]). This property will be used in the following lemma in order to obtain interpolation bounds for the boundary functional term.

LEMMA 5.1. *For each edge E of the triangulation $\mathcal{T}_h$, let $\mathcal{Q}_h$ denote the $L^2(E)$ orthogonal projection onto the polynomials of degree $k-1$. Then, for the interpolation operator $\mathcal{R}_h$ associated with the Raviart–Thomas elements,*

$$\mathbf{n} \cdot (\mathcal{R}_h \boldsymbol{\sigma} \cdot \mathbf{n}) = \mathcal{Q}_h(\mathbf{n} \cdot (\boldsymbol{\sigma} \cdot \mathbf{n})) \tag{5.3}$$

holds.

Proof. From the remarks before Lemma 5.1 we have, by construction,

$$\mathcal{R}_h \boldsymbol{\sigma} \cdot \mathbf{n} = \mathcal{Q}_h(\boldsymbol{\sigma} \cdot \mathbf{n})\,,$$

where the $L^2(E)$ projection $\mathcal{Q}_h$ is meant componentwise for $\boldsymbol{\sigma} \cdot \mathbf{n}$ here. The definition of $\mathcal{Q}_h$ implies

$$\langle \mathcal{Q}_h(\boldsymbol{\sigma} \cdot \mathbf{n}), \mathbf{w} \rangle_E = \langle \boldsymbol{\sigma} \cdot \mathbf{n}, \mathbf{w} \rangle_E$$

for all $\mathbf{w}$ (componentwise) polynomials of degree $k-1$. In particular, by choosing $\mathbf{w} = w\,\mathbf{n}$, where w is a scalar polynomial of degree $k-1$, one obtains

$$\langle \mathcal{Q}_h(\boldsymbol{\sigma} \cdot \mathbf{n}), w\,\mathbf{n} \rangle_E = \langle \boldsymbol{\sigma} \cdot \mathbf{n}, w\,\mathbf{n} \rangle_E$$

and therefore

$$\langle \mathbf{n}\cdot(\mathcal{Q}_h(\boldsymbol{\sigma}\cdot\mathbf{n})), w\rangle_E = \langle \mathbf{n}\cdot(\boldsymbol{\sigma}\cdot\mathbf{n}), w\rangle_E \,.$$

This means that $\mathbf{n}\cdot(\mathcal{Q}_h(\boldsymbol{\sigma}\cdot\mathbf{n}))$ coincides with the (scalar) $L^2(E)$ orthogonal projection $\mathcal{Q}_h$ of $\mathbf{n}\cdot(\boldsymbol{\sigma}\cdot\mathbf{n})$. ∎

The additional boundary functional term in (3.3) may therefore be estimated directly as

$$\begin{aligned}\langle \mathbf{n}\cdot\mathcal{I}_h\mathbf{u}-g, \mathbf{n}\cdot(\mathcal{R}_h\boldsymbol{\sigma}\cdot\mathbf{n})\rangle_{\Gamma_C} &= \langle \mathcal{I}_h(\mathbf{n}\cdot\mathbf{u})-g, \mathcal{Q}_h(\mathbf{n}\cdot(\boldsymbol{\sigma}\cdot\mathbf{n}))\rangle_{\Gamma_C}\\ &= \langle \mathcal{I}_h(\mathbf{n}\cdot\mathbf{u}-g), \mathcal{Q}_h(\mathbf{n}\cdot(\boldsymbol{\sigma}\cdot\mathbf{n}))\rangle_{\Gamma_C} + \langle \mathcal{I}_h g - g, \mathcal{Q}_h(\mathbf{n}\cdot(\boldsymbol{\sigma}\cdot\mathbf{n}))\rangle_{\Gamma_C}\,.\end{aligned} \tag{5.4}$$

For simplicity of notation, the same symbols $\mathcal{I}_h$, $\mathcal{Q}_h$, and $\mathcal{R}_h$ are used here to denote the corresponding projections for scalar or vector-valued functions. The first term on the right-hand side of (5.4) may be estimated if we assume that there are only a finite number of such edges where $\Gamma_{C,d}$ and $\Gamma_{C,s}$ meet. On edges completely contained in $\Gamma_{C,d}$ or $\Gamma_{C,s}$, either $\mathcal{I}_h(\mathbf{n}\cdot\mathbf{u}-g)$ or $\mathcal{Q}_h(\mathbf{n}\cdot(\boldsymbol{\sigma}\cdot\mathbf{n}))$ vanishes. Our analysis of the boundary term is restricted to the two-dimensional case and to low-order elements $k=1$ or $k=2$. Note that this is the situation that we face in our numerical computations in the next section.

LEMMA 5.2. *Assume that the exact solution of* (2.1), (2.2), *and* (2.5) *is such that there are only a finite number of edges where $\Gamma_{C,d}$ and $\Gamma_{C,s}$ meet. Then, for $k=1,2$,*

$$\begin{aligned}&\langle \mathbf{n}\cdot\mathcal{I}_h\mathbf{u}-g, \mathbf{n}\cdot(\mathcal{R}_h\boldsymbol{\sigma}\cdot\mathbf{n})\rangle_{\Gamma_C}\\ &\qquad\le C_I h^{2k}\left(\|\mathbf{n}\cdot\mathbf{u}-g\|_{1/2+k,\Gamma_C} + \|g\|_{2k,\Gamma_C}\right)\|\mathbf{n}\cdot(\boldsymbol{\sigma}\cdot\mathbf{n})\|_{-1/2+k,\Gamma_C}\,.\end{aligned} \tag{5.5}$$

Proof. The first term on the right-hand side in (5.4) satisfies

$$\langle \mathcal{I}_h(\mathbf{n}\cdot\mathbf{u}-g), \mathcal{Q}_h(\mathbf{n}\cdot(\boldsymbol{\sigma}\cdot\mathbf{n}))\rangle_{\Gamma_C} = \sum_{E\cap\Gamma_{C,d}\neq\emptyset, E\cap\Gamma_{C,s}\neq\emptyset} \langle \mathcal{I}_h(\mathbf{n}\cdot\mathbf{u}-g), \mathcal{Q}_h(\mathbf{n}\cdot(\boldsymbol{\sigma}\cdot\mathbf{n}))\rangle_E\,,$$

where, as explained above, only a finite number of edges appear in the sum. With the abbreviations $p=\mathbf{n}\cdot\mathbf{u}-g$ and $q=\mathbf{n}\cdot(\boldsymbol{\sigma}\cdot\mathbf{n})$, we may treat each edge term $\langle \mathcal{I}_h p, \mathcal{Q}_h q\rangle_E$ separately. Let ξ_0 denote the edge midpoint and ξ_d and ξ_s the end points of E contained in $\Gamma_{C,d}$ and $\Gamma_{C,s}$, respectively. Then, the fact that the integrand constitutes a polynomial of degree 3 may be used to obtain

$$\begin{aligned}\langle \mathcal{I}_h p, \mathcal{Q}_h q\rangle_E &= \int_E (\mathcal{I}_h p)(\mathcal{Q}_h q)\,ds\\ &= h_E\left(\frac{1}{6}(\mathcal{I}_h p)(\xi_s)(\mathcal{Q}_h q)(\xi_s) + \frac{2}{3}(\mathcal{I}_h p)(\xi_0)(\mathcal{Q}_h q)(\xi_0)\right)\,,\end{aligned}$$

where we used the fact that $(\mathcal{I}_h p)(\xi_d)=0$. In the case $k=1$ ($\mathcal{Q}_h q$ constant),

$$(\mathcal{Q}_h q)(\xi_0) = (\mathcal{Q}_h q)(\xi_s) = \frac{1}{h_E}\int_E q(\xi)\,d\xi\,,$$

and this leads to

$$\langle \mathcal{I}_h p, \mathcal{Q}_h q\rangle_E = \left(\frac{1}{6}(\mathcal{I}_h p)(\xi_s) + \frac{2}{3}(\mathcal{I}_h p)(\xi_0)\right)\int_E q(\xi)\,d\xi = \frac{1}{2}p(\xi_s)\int_E q(\xi)\,d\xi\,.$$

For $k = 2$ ($\mathcal{Q}_h q$ linear) we obtain

$$(\mathcal{Q}_h q)(\xi_0) = \frac{1}{h_E} \int_E q(\xi)\, d\xi \ , \ \frac{1}{6}\left((\mathcal{Q}_h q)(\xi_s) - (\mathcal{Q}_h q)(\xi_0)\right) = \frac{1}{h_E^2} \int_E q(\xi)(\xi - \xi_0)\, d\xi \ ,$$

and this implies

$$\langle \mathcal{I}_h p, \mathcal{Q}_h q \rangle_E = \left(\frac{1}{6} p(\xi_s) + \frac{2}{3} p(\xi_0) \right) \int_E q(\xi)\, d\xi + p(\xi_s) \frac{1}{h_E} \int_E q(\xi)(\xi - \xi_0)\, d\xi \ .$$

In any case,

$$\langle \mathcal{I}_h p, \mathcal{Q}_h q \rangle_E \leq 2 \sup_E |p(\xi)| \int_E |q(\xi)|\, d\xi \ . \tag{5.6}$$

The fact that p vanishes in a neighborhood of ξ_d may be used to get

$$|p(\xi)| = \left| \int_{\xi_d}^{\xi} p'(\eta)\, d\eta \right| \leq \int_{\xi_d}^{\xi} |p'(\eta)|\, d\eta \leq h_E^{1/2} \|p'\|_{0,E}$$

for all $\xi \in E$ if $p \in H^1(E)$. Similarly, if $p \in H^2(E)$,

$$|p'(\xi)| \leq h_E^{1/2} \|p''\|_{0,E}$$

and therefore

$$|p(\xi)| \leq h_E \left(\int_E \|p''\|_{0,E}^2 \right)^{1/2} = h_E^{3/2} \|p''\|_{0,E} \ .$$

More generally, if $p \in H^{1+l}(E)$,

$$|p(\xi)| \leq h_E^{1/2+l} \|p^{(1+l)}\|_{0,E} \ .$$

Similarly, the fact that q vanishes in a neighborhood of ξ_s implies

$$\int_E |q(\xi)|\, d\xi = \int_E \left| \int_{\xi_s}^{\xi} q'(\eta)\, d\eta \right| d\xi \leq h_E^{3/2} \|q'\|_{0,E}$$

if $q \in H^1(E)$, and, more generally, if $q \in H^l(E)$,

$$\int_E |q(\xi)|\, d\xi \leq h_E^{1/2+l} \|q^{(l)}\|_{0,E} \ .$$

The desired estimate

$$\sup_E |p(\xi)| \int_E |q(\xi)|\, d\xi \leq C h_E^{2k} \|p\|_{1/2+k,E} \|q\|_{-1/2+k,E} \tag{5.7}$$

for $k = 1, 2$ follows from interpolation between these Sobolev spaces. Combining (5.7) with (5.6), we end up with

$$\langle \mathcal{I}_h(\mathbf{n} \cdot \mathbf{u} - g), \mathcal{Q}_h(\mathbf{n} \cdot (\boldsymbol{\sigma} \cdot \mathbf{n})) \rangle_{\Gamma_C} \leq C_I h_E^{2k} \|\mathbf{n} \cdot \mathbf{u} - g\|_{1/2+k,\Gamma_C} \|\mathbf{n} \cdot (\boldsymbol{\sigma} \cdot \mathbf{n})\|_{-1/2+k,\Gamma_C}$$

for the first term in (5.4). For the second term in (5.4), we may write

$$\langle \mathcal{I}_h g - g, \mathcal{Q}_h q\rangle_E = \int_E K_1(\xi)\,(g\mathcal{Q}_h q)''(\xi)\,d\xi$$

or

$$\langle \mathcal{I}_h g - g, \mathcal{Q}_h q\rangle_E = \int_E K_3(\xi)\,(g\mathcal{Q}_h q)^{(4)}(\xi)\,d\xi$$

for $k = 1, 2$, respectively. Here, K_1 and K_3 denote the Peano kernel for the trapezoidial and Simpson quadrature rules, respectively (cf. [20, sect. 3.2]). Note that $(\mathcal{I}_h g)(\mathcal{Q}_h q)$ is a polynomial of degree 1 and 3, respectively, in the cases $k = 1$ and $k = 2$. For $k = 1$ we immediately get

$$\begin{aligned}\langle \mathcal{I}_h g - g, \mathcal{Q}_h q\rangle_E &\le \int_E |K_1(\xi)|\,|(g\mathcal{Q}_h q)''(\xi)|\,d\xi\\ &\le \frac{1}{8}h_E^2\int_E |g''(\xi)|\,|\mathcal{Q}_h q(\xi)|\,d\xi\\ &\le \frac{1}{8}h_E^2\|g''\|_{0,E}\|\mathcal{Q}_h q\|_{0,E} \le \frac{1}{8}h_E^2\|g\|_{2,E}\|q\|_{0,E}\,,\end{aligned}$$

where we used the fact that $\mathcal{Q}_h q$ is constant on E. For $k = 2$, using the fact that $\mathcal{Q}_h q$ is linear on E, we obtain

$$\begin{aligned}\langle \mathcal{I}_h g - g, \mathcal{Q}_h q\rangle_E &\le \int_E |K_3(\xi)|\,|(g\mathcal{Q}_h q)^{(4)}(\xi)|\,d\xi\\ &\le \frac{1}{1152}h_E^4\int_E \Big(|g^{(4)}(\xi)|\,|\mathcal{Q}_h q(\xi)| + 4|g'''(\xi)|\,|(\mathcal{Q}_h q)'(\xi)|\Big)\,d\xi\\ &\le \frac{1}{288}h_E^4\Big(\|g^{(4)}\|_{0,E}\,\|\mathcal{Q}_h q\|_{0,E} + \|g'''\|_{0,E}\,\|(\mathcal{Q}_h q)'\|_{0,E}\Big)\\ &\le \frac{C_1}{288}h_E^4\|g\|_{4,E}\,\|q\|_{1,E}\,.\end{aligned}$$

The last of the above inequalities follows from the H^1-stability of the local L^2-projection $\mathcal{Q}_h$, i.e., from the existence of a constant C_1 such that

$$|\mathcal{Q}_h q|_{1,E} \le C_1 |q|_{1,E} \text{ for all } q \in H^1(E). \qquad \square$$

THEOREM 5.3. *Let* $(\mathbf{u}, \boldsymbol{\sigma}) \in H^1(\Omega)^d \times H(\mathrm{div}, \Omega)^d$ *be the exact solution of the contact problem satisfying* (2.1) *under the constraints* (2.2) *and* (2.5). *Moreover, let* $(\mathbf{u}_h, \boldsymbol{\sigma}_h) \in (\mathbf{u}^D + \mathbf{U}_h) \times \boldsymbol{\Sigma}_h$ *be the least squares finite element approximation minimizing* (2.6) *or* (4.2) *under the constraints* (2.2) *and using the finite element spaces of degree* k *used earlier in this section. Then, if the problem is sufficiently regular such that the right-hand side exists,*

$$\begin{aligned}&\|\boldsymbol{\varepsilon}\,(\mathbf{u} - \mathbf{u}_h)\| + \|\mathrm{div}\,(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\| + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|\\ &\quad \le C\,h^k\,\big(|\mathbf{u}|_{k+1,\Omega} + |\mathrm{div}\,\boldsymbol{\sigma}|_{k,\Omega} + |\boldsymbol{\sigma}|_{k,\Omega}\\ &\qquad + \|\mathbf{n}\cdot\mathbf{u} - g\|_{1/2+k,\Gamma_C} + \|g\|_{2k,\Gamma_C} + \|\mathbf{n}\cdot(\boldsymbol{\sigma}\cdot\mathbf{n})\|_{-1/2+k,\Gamma_C}\big)\end{aligned} \tag{5.8}$$

holds. In the case of the least squares formulation based on (4.2), *the same constant* C *may be chosen for all values of* ν.

Proof. Theorems 3.2 and 3.3 imply

$$\begin{aligned}\|\boldsymbol{\varepsilon}\,(\mathbf{u}-\mathbf{u}_h)\|^2 + \|\mathrm{div}\,(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)\|^2 + \|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|^2 &\le C_1\mathcal{F}_C(\mathbf{u}_h,\boldsymbol{\sigma}_h) \le C_1\mathcal{F}_C(\mathcal{I}_h\mathbf{u},\mathcal{R}_h\boldsymbol{\sigma})\\ &\le C_2\left(\|\boldsymbol{\varepsilon}\,(\mathbf{u}-\mathcal{I}_h\mathbf{u})\|^2 + \|\mathrm{div}\,(\boldsymbol{\sigma}-\mathcal{R}_h\boldsymbol{\sigma})\|^2 + \|\boldsymbol{\sigma}-\mathcal{R}_h\boldsymbol{\sigma}\|^2\right.\\ &\quad\left.+\langle \mathbf{n}\cdot\mathcal{I}_h\mathbf{u}-g, \mathbf{n}\cdot(\mathcal{R}_h\boldsymbol{\sigma}\cdot\mathbf{n})\rangle_{\Gamma_C}\right)\ .\end{aligned}$$

Similarly, Theorems 4.2 and 4.3 imply

$$\begin{aligned}\|\boldsymbol{\varepsilon}\,(\mathbf{u}-\mathbf{u}_h)\|^2 + \|\mathrm{div}\,(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)\|^2 + \|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|^2 &\le C_1'\mathcal{F}_C(\mathbf{u}_h,\boldsymbol{\sigma}_h) \le C_1'\mathcal{F}_C(\mathcal{I}_h\mathbf{u},\mathcal{R}_h\boldsymbol{\sigma})\\ &\le C_2'\left(\|\boldsymbol{\varepsilon}\,(\mathbf{u}-\mathcal{I}_h\mathbf{u})\|^2 + \|\mathrm{div}\,(\boldsymbol{\sigma}-\mathcal{R}_h\boldsymbol{\sigma})\|^2 + \|\boldsymbol{\sigma}-\mathcal{R}_h\boldsymbol{\sigma}\|^2\right.\\ &\quad\left.+\langle \mathbf{n}\cdot\mathcal{I}_h\mathbf{u}-g, \mathbf{n}\cdot(\mathcal{R}_h\boldsymbol{\sigma}\cdot\mathbf{n})\rangle_{\Gamma_C}\right)\ .\end{aligned}$$

Combining this with (5.1), (5.2), and (5.5) leads to (5.8). ∎

One more remark on the validity of our least squares approach in the incompressible limit seems appropriate at this point. The result of Theorem 5.3 is stated only with a constant uniformly for all values of ν in the case of the functional $\mathcal{G}_C(\cdot,\cdot)$ investigated in section 4. For the functional $\mathcal{F}_C(\cdot,\cdot)$ introduced earlier in section 2, uniform approximation properties could also be shown by using the energy-related norm $|||(\,\cdot\,,\,\cdot\,)|||$ defined in (3.1) and employing robust finite element spaces for the displacement approximation with respect to the energy norm, like, e.g., the nonconforming Fortin–Soulie elements [10] (see [6] for details on their performance in the least squares context).

In the finite element implementation, the constraints (2.2) are replaced by pointwise constraints at the nodes which belong to the edges contained in the contact boundary. For example, if $k=1$, the constraints

$$\begin{aligned}(\mathbf{n}\cdot\mathbf{u}_h-g)|_{x_i} &\le 0 \text{ for all } x_i\in\mathcal{V}_h\cap\Gamma_C\ ,\\ \mathbf{n}\cdot(\boldsymbol{\sigma}_h\cdot\mathbf{n})|_{x_i} &\le 0 \text{ for all } x_i\in\mathcal{M}_h\cap\Gamma_C\ ,\\ \mathbf{t}\cdot(\boldsymbol{\sigma}_h\cdot\mathbf{n})|_{x_i} &= 0 \text{ for all } x_i\in\mathcal{M}_h\cap\Gamma_C\end{aligned}\tag{5.9}$$

are incorporated, where $\mathcal{V}_h$ and $\mathcal{M}_h$ denote the set of all vertices and edge midpoints, respectively, associated with the triangulation $\mathcal{T}_h$. In this case, it is actually easy to see that (5.9) and (2.2) are actually equivalent since only piecewise linear functions on Γ_C are involved. In the case $k=2$, the constraints

$$\begin{aligned}(\mathbf{n}\cdot\mathbf{u}_h-g)|_{x_i} &\le 0 \text{ for all } x_i\in(\mathcal{V}_h\cup\mathcal{M}_h)\cap\Gamma_C\ ,\\ \mathbf{n}\cdot(\boldsymbol{\sigma}_h\cdot\mathbf{n})|_{x_i} &\le 0 \text{ for all } x_i\in(\mathcal{E}_h^-\cup\mathcal{E}_h^+)\cap\Gamma_C\ ,\\ \mathbf{t}\cdot(\boldsymbol{\sigma}_h\cdot\mathbf{n})|_{x_i} &= 0 \text{ for all } x_i\in(\mathcal{E}_h^-\cup\mathcal{E}_h^+)\cap\Gamma_C\ ,\end{aligned}\tag{5.10}$$

where $\mathcal{E}_h^-$ and $\mathcal{E}_h^+$ denote the starting point and end point, respectively, of each edge in the triangulation $\mathcal{T}_h$, are no longer equivalent to (2.2). This is due to the fact that enforcing $\mathbf{n}\cdot\mathbf{u}_h-g\le 0$ at both ends and at the midpoint of an edge does not mean that the quadratic function $\mathbf{n}\cdot\mathbf{u}_h-g$ is nonnegative at the other points on the edge.

The minimization of the quadratic functional (2.6) under affine constraints like (5.9) or (5.10) is carried out by an active set method (cf. [14, sect. 16]). Such a method identifies sets of active constraints $I_A(\mathbf{u}_h)$ and $I_A(\boldsymbol{\sigma}_h)$ such that the minimization problem with respect to equality constraints where

$$\begin{aligned}(\mathbf{n}\cdot\mathbf{u}_h-g)|_{x_i} &= 0 \text{ for all } x_i\in I_A(\mathbf{u}_h)\ ,\\ \mathbf{n}\cdot(\boldsymbol{\sigma}_h\cdot\mathbf{n})|_{x_i} &= 0 \text{ for all } x_i\in I_A(\boldsymbol{\sigma}_h)\end{aligned}\tag{5.11}$$

replace the inequality constraints in (5.9) or (5.10) is equivalent to the inequality constrained problem. This is done iteratively by solving a sequence of minimization problems under affine equality constraints.

6. Computational results. The numerical tests for our least squares finite element method are performed for a Hertzian contact problem under plane strain conditions. The test example is taken from [12] and consists of a half-circle with center at $(0, 0.4)$ and radius 0.4 which is in contact with the plane $x_2 = 0$. On top of the half-circle constant displacement boundary conditions of $\mathbf{u}^D = (0, -0.005)$ are prescribed. The material parameters are $E = 270269\text{N/mm}^2$ and $\nu = 0.248$. For the numerical treatment, the problem is rescaled in such a way that the Lamé parameter $\mu = E/(2(1+\nu))$ equals one. This is necessary in order to make sure that the constants in the results of section 3 remain of moderate size. Note that this means that this rescaling alters only the size of the stresses. The correct values can easily be computed afterwards. Our first combination of trial spaces consists of conforming linear finite elements (P_1) for the displacements and lowest order Raviart–Thomas elements (RT_0) for the stresses.

The results in Table 6.1 indicate an asymptotic behavior of the least squares functional like

$$\mathcal{F}_C(\mathbf{u}_h, \boldsymbol{\sigma}_h) \sim N_h^{-1} ,$$

where N_h denotes the number of degrees of freedom. This is the optimal approximation order that can be achieved with linear finite element spaces. Note that the number of active constraints is only a tiny fraction of the size of the full system throughout the computations.

For the quadratic case, i.e., conforming P_2 elements combined with next-to-lowest order RT_1 Raviart–Thomas elements, the results in Table 6.2 indicate that a much better approximation is achieved on all levels in relation to the degrees of freedom. The optimal convergence behavior achievable would imply

$$\mathcal{F}_C(\mathbf{u}_h, \boldsymbol{\sigma}_h) \sim N_h^{-2} ,$$

which is obviously not quite reached. Interestingly, the error reduction slows down considerably during the fifth refinement step which may be due to a suboptimal refinement rule not sufficiently appreciating the low regularity of the problem. It may also be due to the pointwise treatment of the contact conditions in (5.10). The triangulation on the fifth refinement level in Table 6.2 is plotted in Figure 6.1.

Figure 6.2 shows the stress distribution $|\boldsymbol{\sigma}_h|$ computed on this triangulation.

Table 6.1
Reduction of least squares functional on adaptively refined triangulations P_1/RT_0.

	dim $\mathbf{U}_h$	dim $\boldsymbol{\Sigma}_h$	$\mathcal{F}_C(\mathbf{u}_h, \boldsymbol{\sigma}_h)$	$\#I_A(\mathbf{u}_h)$	$\#I_A(\boldsymbol{\sigma}_h)$
$l=0$	102	306	2.23 e-5	1	14
$l=1$	178	526	4.88 e-6	1	18
$l=2$	304	896	2.03 e-6	3	22
$l=3$	532	1578	8.95 e-7	7	25
$l=4$	932	2746	4.93 e-7	15	39
$l=5$	1650	4898	3.02 e-7	24	43
$l=6$	2952	8772	1.76 e-7	29	66
$l=7$	5286	15750	1.08 e-7	47	80
$l=8$	9436	28136	6.66 e-8	59	115

Table 6.2
Reduction of least squares functional on adaptively refined triangulations P_2/RT_1.

	dim $\mathbf{U}_h$	dim $\mathbf{\Sigma}_h$	$\mathcal{F}_C(\mathbf{u}_h, \boldsymbol{\sigma}_h)$	$\#I_A(\mathbf{u}_h)$	$\#I_A(\boldsymbol{\sigma}_h)$
$l = 0$	392	988	9.24 e-6	1	30
$l = 1$	766	1908	4.17 e-7	5	38
$l = 2$	1484	3700	9.75 e-8	7	52
$l = 3$	2832	7076	3.11 e-8	15	68
$l = 4$	5336	13296	1.31 e-8	29	90
$l = 5$	9704	24160	1.15 e-8	57	116

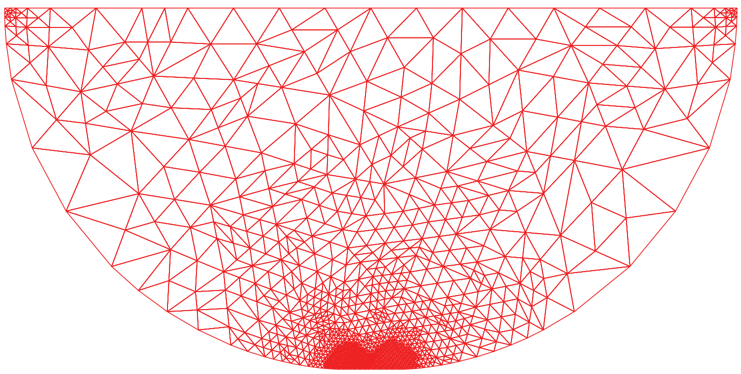

Fig. 6.1. *Triangulation of deformed region after five adaptive refinement steps.*

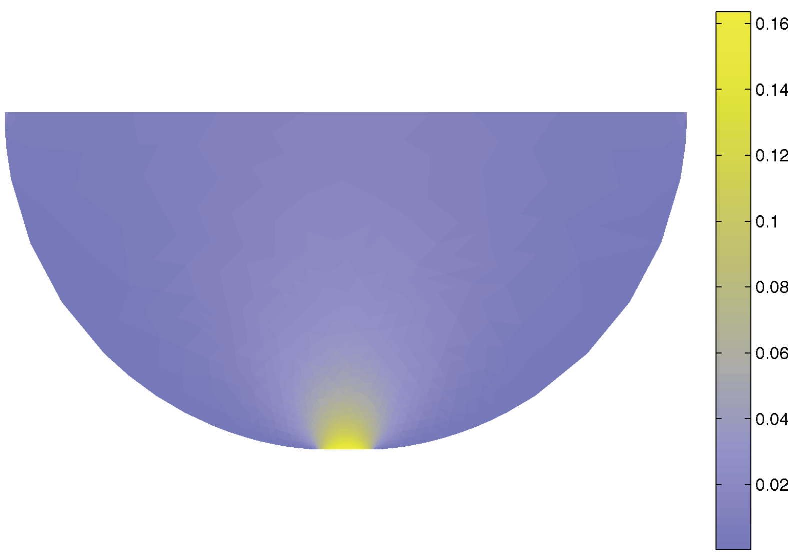

Fig. 6.2. *Stress distribution after five adaptive refinement steps.*

Finally, some comments on the nature of our test problem and on the performance of the adaptive implementation of our least squares approach are appropriate. The strong localization of the refinement zone indicates that this problem is far from H^2-regular. It can already be anticipated from Figure 6.1 and becomes apparent on closer inspection at higher resolution that the adaptive refinement is concentrated at the end points of the active contact boundary. At these points, the boundary conditions change from clamped (displacement) to traction-free (normal stress) causing severe singularities. The theoretical results in [16] indicate that $\mathbf{u} \in H^{1+\beta}(\Omega)^2$ with $\beta \in (1/4, 1/2)$. Overall, our least squares formulation appears to perform very well in an adaptive setting and leads to optimal convergence behavior for linear finite elements. Improved convergence is achieved with quadratic elements, but the asymptotic behavior expected from the experience with standard linear elasticity computations is not quite reached.

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