holds for all $w \in H^{1+\epsilon}(K)$ with $\Delta w \in L^2(K)$ and for all $v \in H^{1-\epsilon}(K)$ with $0 < \epsilon < 1/2$. Let $H^{-\epsilon}(K)$ be the dual of $H_0^\epsilon(K)$ which is the closure of $C_0^\infty(K)$ in the $H^\epsilon(K)$ norm. Since $H^\epsilon(K)$ is the same space as $H_0^\epsilon(K)$ for $\epsilon \in (0, 1/2)$ (see, e.g., Theorem 1.4.2.4 in [24]), ∇v is then in $H^{-\epsilon}(K)^2$. That is, the term $(\nabla w, \nabla v)_K$ in (2.7) can be viewed as a duality pairing between $H^\epsilon(K)^2$ and $H^{-\epsilon}(K)^2$. The validity of (2.7) follows from the standard density argument and the fact that (2.7) holds for $C^\infty(\bar{K})$ functions.

By the trace theorem [24], $v|_{\partial K}$ is in $H^{1/2-\epsilon}(\partial K)$. Hence, the formal boundary integral in the left-hand side of (2.7) may be regarded as the duality pairing between $H^{\epsilon-1/2}(\partial K)$ and $H^{1/2-\epsilon}(\partial K)$, which is defined by the right-hand side of (2.7). Since, for each edge $e \subset \partial K$, the trivial extension of functions in $H^{1/2-\epsilon}(e)$ by zero to all of ∂K belongs to $H^{1/2-\epsilon}(\partial K)$ (see, e.g., Theorem 1.5.2.3 in [24]), this interpretation enables us to define the duality pairing on each edge e of ∂K,

$$\int_e (\nabla w \cdot \mathbf{n})\, v\, ds := \langle \nabla w \cdot \mathbf{n}, v \rangle_e,$$

where $(\nabla w \cdot \mathbf{n})|_e \in H^{\epsilon-1/2}(e)$ and $v|_e \in H^{1/2-\epsilon}(e)$.

LEMMA 2.1. *Letting $K \in \mathcal{T}$, $e \in \partial K$, and $0 < \epsilon < 1/2$, for any $\phi \in H^{1+\epsilon}(K)$ with $\Delta\phi \in L^2(K)$, there exists a positive constant C independent of ϕ such that*

$$\|\nabla\phi \cdot \mathbf{n}\|_{\epsilon-1/2,e} \le C\left(\|\nabla\phi\|_{\epsilon,K} + h_K^{1-\epsilon}\|\Delta\phi\|_{0,K}\right). \tag{2.8}$$

Proof. Inequality (2.8) is contained in the proof of Corollary 3.3 on page 1384 of [10]. For the convenience of readers, we provide a proof here. For any $g \in H^{1/2-\epsilon}(e)$, there exists a lifting v_g of g such that $v_g \in H^{1-\epsilon}(K)$, $v_g|_e = g$, $v_g|_{\partial K \setminus e} = 0$, and

$$\|\nabla v_g\|_{-\epsilon,K} + h_K^{\epsilon-1}\|v_g\|_{0,K} \le c\|g\|_{1/2-\epsilon,e}.$$

It then follows from the Green's formula in (2.7), the Cauchy–Schwarz inequality, and the definition of the dual norm that

$$\begin{aligned}\langle \nabla\phi \cdot \mathbf{n}, g\rangle_e = \langle \nabla\phi \cdot \mathbf{n}, v_g\rangle_{\partial K} &= (\Delta\phi,\, v_g)_K + (\nabla\phi,\, \nabla v_g)_K \\ &\le \|\Delta\phi\|_{0,K}\|v_g\|_{0,K} + \|\nabla\phi\|_{\epsilon,K}\|\nabla v_g\|_{-\epsilon,K} \\ &\le C\left(\|\nabla\phi\|_{\epsilon,K} + h_K^{1-\epsilon}\|\Delta\phi\|_{0,K}\right)\|g\|_{1/2-\epsilon,e},\end{aligned}$$

which, combining with the definition of the dual norm

$$\|\nabla\phi\cdot\vec{n}\|_{\epsilon-\frac{1}{2},e}\ \cancel{\|\nabla\phi\|_{\epsilon-1/2,e}} = \sup_{g\in H^{1/2-\epsilon}(e)} \frac{\langle \nabla\phi \cdot \mathbf{n}, g\rangle_e}{\|g\|_{1/2-\epsilon,e}},$$

implies (2.8). This completes the proof of the lemma. □

Denote by $H^s(\mathcal{T})$ the broken Sobolev space of degree $s > 0$ with respect to $\mathcal{T}$,

$$H^s(\mathcal{T}) = \{v \in L^2(\Omega) : v|_K \in H^s(K) \quad \forall\, K \in \mathcal{T}\},$$

and denote its subspace by

$$V^s(\mathcal{T}) = \{v \in H^s(\mathcal{T}) \,:\, \nabla \cdot (k\nabla v) \in L^2(K) \quad \forall\, K \in \mathcal{T}\}. \tag{2.9}$$

Let u be the solution of problem (1.1)–(1.2); then it is well known from the regularity estimate [28, 24] that $u \in H^{1+\alpha}(\Omega)$ for some positive α which could be very small.

for any $K \in \mathcal{T}$. The so-called DG norm is defined as follows:

$$|\!|\!| v |\!|\!|_{DG} = \left(\|k^{\frac{1}{2}} \nabla_h v\|_{0,\Omega}^2 + \sum_{e \in \mathcal{E}_I \cup \mathcal{E}_D} h_e^{-1} W_e \|[\![v]\!]\|_{0,e}^2 \right)^{1/2}.$$

LEMMA 2.3 (uniqueness and coercivity).

(i) *The bilinear form $a_1(\cdot,\cdot)$ is coercive in $\mathcal{U}^{DG}$ with the coercivity constant $\min\{1,\gamma_1\}$, provided that $\gamma_1 > 0$, i.e,*

$$a_1(v,v) \ge \min\{1,\gamma_1\} |\!|\!| v |\!|\!|_{DG}^2 \quad \forall\, v \in \mathcal{U}^{DG}. \tag{2.16}$$

Thus the NIPG problem in (2.15) *has a unique solution, provided that $\gamma_1 > 0$.*

(ii) *Let w_+^e and w_-^e be weights satisfying* (2.1). *Then SIPG and IIPG problems* (2.15) *have a unique solution, provided that $\gamma_\theta > 2(1 \overset{-}{\cancel{+}} \theta)^2 \rho_{\mathcal{T}}$. Moreover, the symmetric/incomplete bilinear form $a_\theta(\cdot,\cdot)$ for $\theta = -1$ or 0 is coercive in $\mathcal{U}^{DG}$ with a coercivity constant $\alpha_0 \in (0,1)$ independent of the mesh size and the ratio $k_{\max}/k_{\min}$, i.e.,*

$$a_\theta(v,v) \ge \alpha_0 |\!|\!| v |\!|\!|_{DG}^2 \quad \forall\, v \in \mathcal{U}^{DG}, \tag{2.17}$$

for $\theta = -1$ and 0, provided that $\gamma_\theta > \frac{2(1 \overset{-}{\cancel{+}} \theta)^2 \rho_{\mathcal{T}}}{1-\alpha_0} + \alpha_0$.

Proof. Let δ be a positive constant to be determined. For any $v \in \mathcal{U}^{DG}$ and for $e \in \mathcal{E}_I \bigcup \mathcal{E}_D$, the Cauchy–Schwarz inequality and the inequality of arithmetic and geometric means give

$$2 \sum_{e \in \mathcal{E}_I \cup \mathcal{E}_D} \int_e \{k \nabla v \cdot \mathbf{n}_e\}_w [\![v]\!]\, ds \le \sum_{e \in \mathcal{E}_I \cup \mathcal{E}_D} \frac{\delta\, h_e}{W_e} \|\{k \nabla v \cdot \mathbf{n}_e\}_w\|_{0,e}^2 + \sum_{e \in \mathcal{E}_I \cup \mathcal{E}_D} \frac{W_e}{\delta\, h_e} \|[\![v]\!]\|_{0,e}^2. \tag{2.18}$$

For $e \in \mathcal{E}_I$, let $e = \partial K_+^e \cap \partial K_-^e$. Since $k \nabla v$ is constant on each element,

$$\begin{aligned} &\frac{h_e}{W_e} \|\{k \nabla v \cdot \mathbf{n}_e\}_w\|_{0,e}^2 \\ &\quad \le 2h_e^2 \left\{ \frac{(w_+^e)^2 k_+}{W_e} k_+ (\nabla v_+ \cdot \mathbf{n}_e)^2 + \frac{(w_-^e)^2 k_-}{W_e} k_- (\nabla v_- \cdot \mathbf{n}_e)^2 \right\} \\ &\quad \le 2h_e^2 \max\left\{ \frac{(w_+^e)^2 k_+^e}{W_e}, \frac{(w_-^e)^2 k_-^e}{W_e} \right\} \left(k_+ (\nabla v_+ \cdot \mathbf{n}_e)^2 + k_- (\nabla v_- \cdot \mathbf{n}_e)^2 \right). \end{aligned}$$

Similarly, for $e \in \mathcal{E}_D$ and $e \subset \partial K$, we have

$$\frac{h_e}{W_e} \|\{k \nabla v \cdot \mathbf{n}_e\}_w\|_{0,e}^2 = k_K^{-1} h_e^2 (k_K \nabla v_K \cdot \mathbf{n}_e)^2 = h_e^2 k_K (\nabla v_K \cdot \mathbf{n}_e)^2.$$

Summing up over all edges in $\mathcal{E}_I \cup \mathcal{E}_D$ and using (2.5) imply that

$$\sum_{e \in \mathcal{E}_I \cup \mathcal{E}_D} \frac{h_e}{W_e} \|\{k \nabla v \cdot \mathbf{n}\}_w\|_e^2 \le 2 \sum_{K \in \mathcal{T}} \sum_{e \in \mathcal{E}_K} h_e^2 k_K (\nabla v_K \cdot \mathbf{n}_e)^2. \tag{2.19}$$

It is proved in [3] that

$$\sum_{e \in \mathcal{E}_K} h_e^2 (\nabla v \cdot \mathbf{n}_K)^2 = 4 \vec{v}_K^T \mathbf{S}_K^2 \vec{v}_K,$$

where $\vec{v}_K$ is the vector of values of v at vertices of K. Since $k_K \vec{v}_K^T \mathbf{S}_K \vec{v}_K = (k\nabla v, \nabla v)_K$, thus

$$\sum_{e\in\mathcal{E}_K} h_e^2 k_K (\nabla v_K \cdot \mathbf{n}_e)^2 \le 4k_K \vec{v}_K \mathbf{S}_K^2 \vec{v}_K \le 4\rho(\mathbf{S}_K) k_K \vec{v}_K^T \mathbf{S}_K \vec{v}_K = 4\rho(\mathbf{S}_K)\,(k\nabla v, \nabla v)_K,$$

which, together with (2.19), leads to

$$\sum_{e\in\mathcal{E}_I\cup\mathcal{E}_D} \frac{h_e}{W_e} \|\{k\nabla v\cdot\mathbf{n}\}_w\|_e^2 \le 8\rho_{\mathcal{T}}(k\nabla_h v,\, \nabla_h v).$$

Using (2.18), we now have

$$\sum_{e\in\mathcal{E}_I\cup\mathcal{E}_D} \int_e \{k\nabla v\cdot\mathbf{n}_e\}_w [\![v]\!]\, ds \le 4\delta\rho_{\mathcal{T}}(k\nabla_h v,\, \nabla_h v) + \sum_{e\in\mathcal{E}_I\cup\mathcal{E}_D} \frac{W_e}{2\delta\, h_e} \|[\![v]\!]\|_{0,e}^2.$$

Hence,

$$a_\theta(v,\, v) \ge (1 - 4(1 \overset{-}{\cancel{+}} \theta)\delta\rho_{\mathcal{T}})(k\nabla_h v,\, \nabla_h v) + \sum_{e\in\mathcal{E}_I\cup\mathcal{E}_D} \left(\gamma_\theta - \frac{1 \overset{-}{\cancel{+}} \theta}{2\delta}\right) \frac{W_e}{h_e} \|[\![v]\!]\|_{0,e}^2.$$

γ_θ

For any constant $\alpha_0 \in [0, 1)$, assume that $\gamma_\theta > \frac{2(1 \overset{-}{\cancel{+}} \theta)^2 \rho_{\mathcal{T}}}{1-\alpha_0} + \alpha_0$; then there exists $\delta > 0$ such that $\frac{2(1-\alpha_0)}{1 \underset{-}{\cancel{+}} \theta} > \delta^{-1} > \frac{4(1 \overset{-}{\cancel{+}} \theta)\rho_{\mathcal{T}}}{1-\alpha_0}$, which is equivalent to

$$1 - 4(1 \overset{-}{\cancel{+}} \theta)\delta\rho_{\mathcal{T}} > \alpha_0 \quad \text{and} \quad \gamma_\theta - \frac{1 \overset{-}{\cancel{+}} \theta}{2\delta} > \alpha_0.$$

This implies the coercivity of $a_\theta(v, v)$ in (2.17) for any $\alpha_0 \in (0, 1)$. When $\alpha_0 = 0$, it yields that $a_\theta(v, v)$ is positive and definite in $\mathcal{U}^{DG}$ and, hence, problem (2.15) has a unique solution. This completes the proof of the lemma. $\square$

REMARK 2.4. *The constant γ_θ that appears in* [5] *is chosen to be greater than $(1+\theta)^2 \max_{K\in\mathcal{T}} k_K \rho(S_K)$, which depends on k for $\theta \ne -1$, and, hence, it is not optimal.*

3. A priori error estimate.

Let $e = u - u_{\mathcal{T}}$, where u and $u_{\mathcal{T}}$ are the solutions of (2.14) and (2.15), respectively. The difference of (2.14) and (2.15) yields the following error equation:

$$a_\theta(e,\, v) = 0 \quad \forall\ v \in \mathcal{U}^{DG}. \tag{3.1}$$

Let $\epsilon > 0$ be a very small constant, and define

$$\|v\|_{k,\epsilon,\Omega} = \|k^{1/2}\nabla v\|_{\epsilon,\Omega}.$$

Let $P_{\mathcal{T}} : H_{g,D}^{1+\epsilon}(\Omega) \to \mathcal{U}_g$ be the orthogonal projection operator from $H_{g,D}^{1+\epsilon}(\Omega)$ onto $\mathcal{U}_g$ with respect to the inner product associated with the norm $\|\cdot\|_{k,\epsilon,\Omega}$. Then the standard interpolation argument and an analysis similar to that for Proposition 2.4 in [11] give that for $\phi \in H_{g,D}^{1+\epsilon}(\Omega) \cap H^{1+s}(\mathcal{T})$ with $\epsilon \le s \le 1$,

$$\|k^{1/2}\nabla(\phi - P_{\mathcal{T}}\phi)\|_{\epsilon,\Omega} \le C \left(\sum_{K\in\mathcal{T}} h_K^{2(s-\epsilon)} \|k^{1/2}\nabla\phi\|_{s,K}^2\right)^{1/2}, \tag{3.2}$$

where C is a positive constant independent of the mesh size and the ratio $k_{\max}/k_{\min}$. For any $v \in H^{1+s}(\mathcal{T})$, $0 < s \le 1$, denote

$$B_s(h, v) = \left(\sum_{K \in \mathcal{T}} h_K^{2(s-\epsilon)} \|k^{1/2} \nabla v\|_{s,K}^2 \right)^{1/2} + \left(\sum_{K \in \mathcal{T}} h_K^2 k_K^{-1} \|f\|_{0,K}^2 \right)^{1/2}.$$

LEMMA 3.1. *Assume that the solution $u \in V^{1+\epsilon}(\mathcal{T})$ of problem (2.14) belongs to $H^{1+s}(\mathcal{T})$ with $0 < \epsilon \le s \le 1$. Then*

$$\sum_{e \in \mathcal{E}_I \cup \mathcal{E}_D} \int_e \{k \nabla (P_{\mathcal{T}} u - u) \cdot \mathbf{n}_e\}_w [\![P_{\mathcal{T}} u - u_{\mathcal{T}}]\!] \, ds \le C\, B_s(h, u) |\!|\!| P_{\mathcal{T}} u - u_{\mathcal{T}} |\!|\!|_{DG}, \tag{3.3}$$

where C is a positive constant independent of the mesh size and the ratio $k_{\max}/k_{\min}$.

Proof. Let $z = P_{\mathcal{T}} u - u$ and $z_{\mathcal{T}} = P_{\mathcal{T}} u - u_{\mathcal{T}}$. By using the definition of the dual norm, the triangle inequality, the inverse inequality, (2.5), Lemma 2.1, and (3.2), we have

$$\begin{aligned}
&\sum_{e \in \mathcal{E}_I \cup \mathcal{E}_D} \int_e \{k \nabla (P_{\mathcal{T}} u - u) \cdot \mathbf{n}_e\}_w [\![P_{\mathcal{T}} u - u_{\mathcal{T}}]\!] \, ds = \sum_{e \in \mathcal{E}_I \cup \mathcal{E}_D} \int_e \{k \nabla z \cdot \mathbf{n}_e\}_w [\![z_{\mathcal{T}}]\!] \, ds \\
&\le \sum_{e \in \mathcal{E}_I \cup \mathcal{E}_D} \|\{k \nabla z \cdot \mathbf{n}_e\}_w\|_{\epsilon - 1/2, e} \, \|[\![z_{\mathcal{T}}]\!]\|_{1/2 - \epsilon, e} \\
&\le \sum_{e \in \mathcal{E}_I \cup \mathcal{E}_D} \left(w_-^e \|k_- \nabla z|_{K_-} \cdot \mathbf{n}_e\|_{\epsilon - \frac{1}{2}, e} + w_+^e \|k_+ \nabla z|_{K_+} \cdot \mathbf{n}_e\|_{\epsilon - \frac{1}{2}, e} \right) h_e^{\epsilon - \frac{1}{2}} \|[\![z_{\mathcal{T}}]\!]\|_{0,e} \\
&\le \sum_{e \in \mathcal{E}_I \cup \mathcal{E}_D} \left(\|k_-^{1/2} \nabla z|_{K_-} \cdot \mathbf{n}_e\|_{\epsilon - \frac{1}{2}, e} + \|k_+^{1/2} \nabla z|_{K_+} \cdot \mathbf{n}_e\|_{\epsilon - \frac{1}{2}, e} \right) h_e^{\epsilon - \frac{1}{2}} W_e^{1/2} \|[\![z_{\mathcal{T}}]\!]\|_{0,e} \\
&\le C \sum_{e \in \mathcal{E}_I \cup \mathcal{E}_D} \left(\sum_{K \in \omega_e} h^{\epsilon} \|k^{\frac{1}{2}} \nabla z\|_{\epsilon, K} + h_K \|k^{\frac{1}{2}} \Delta z\|_{0,K} \right) W_e^{\frac{1}{2}} h_e^{-\frac{1}{2}} \|[\![z_{\mathcal{T}}]\!]\|_{0,e} \\
&\le C B_s(h, u) |\!|\!| P_{\mathcal{T}} u - u_{\mathcal{T}} |\!|\!|_{DG}.
\end{aligned}$$

This completes the proof of the lemma. $\square$

THEOREM 3.2. *Assume that the solution $u \in V^{1+\epsilon}(\mathcal{T})$ of problem (2.14) belongs to $H^{1+s}(\mathcal{T}) \cap H^{1+\epsilon}(\Omega)$ with $0 < \epsilon \le s \le 1$ and that $\gamma_\theta > \frac{2(1+\theta)^2 \rho_{\mathcal{T}}}{1-\alpha_0} + \alpha_0$ for $\theta = +1$, 0, and -1. Then we have the following a priori error bound:*

$$|\!|\!| u - u_{\mathcal{T}} |\!|\!|_{DG} \le C\, B_s(h, u), \tag{3.4}$$

where C is a positive constant independent of the mesh size and the ratio $k_{\max}/k_{\min}$.

Proof. The triangle inequality gives

$$|\!|\!| e |\!|\!|_{DG} \le |\!|\!| u - P_{\mathcal{T}} u |\!|\!|_{DG} + |\!|\!| P_{\mathcal{T}} u - u_{\mathcal{T}} |\!|\!|_{DG}.$$

Since $u - P_{\mathcal{T}} u$ is continuous and vanishes on Γ_D, thus

$$|\!|\!| u - P_{\mathcal{T}} u |\!|\!|_{DG} = \|k^{1/2} \nabla (u - P_{\mathcal{T}} u)\|_{0,\Omega} \le \|k^{1/2} \nabla (u - P_{\mathcal{T}} u)\|_{\epsilon,\Omega}.$$

Now, by (3.2) with $\phi = u$ it suffices to show that

$$|\!|\!| P_{\mathcal{T}} u - u_{\mathcal{T}} |\!|\!|_{DG} \le C \left(|\!|\!| u - P_{\mathcal{T}} u |\!|\!|_{DG} + B_s(h, u) \right). \tag{3.5}$$

To this end, using the coercivity of $a_\theta(\cdot,\cdot)$ in (2.17), the error equation in (3.1), the Cauchy–Schwarz inequality, and the fact that $[\![P_{\mathcal{T}}u - u]\!]_{\mathcal{E}_I\cup\mathcal{E}_D} = 0$, we have

$$\begin{aligned}
&\alpha_0 \|P_{\mathcal{T}}u - u_{\mathcal{T}}\|_{DG}^2 \le a_\theta(P_{\mathcal{T}}u - u_{\mathcal{T}},\, P_{\mathcal{T}}u - u_{\mathcal{T}}) = a_\theta(P_{\mathcal{T}}u - u,\, P_{\mathcal{T}}u - u_{\mathcal{T}})\\
&= (k\nabla_h(P_{\mathcal{T}}u - u),\, \nabla_h(P_{\mathcal{T}}u - u_{\mathcal{T}})) + \sum_{e\in\mathcal{E}_I\cup\mathcal{E}_D} \int_e \gamma_\theta h_e^{-1} W_e [\![P_{\mathcal{T}}u - u]\!][\![P_{\mathcal{T}}u - u_{\mathcal{T}}]\!]\, ds\\
&- \sum_{e\in\mathcal{E}_I\cup\mathcal{E}_D} \int_e (\{k\nabla(P_{\mathcal{T}}u - u)\cdot \mathbf{n}_e\}_w [\![P_{\mathcal{T}}u - u_{\mathcal{T}}]\!]\\
&\qquad - \theta\{k\nabla(P_{\mathcal{T}}u - u_{\mathcal{T}})\cdot \mathbf{n}_e\}_w [\![P_{\mathcal{T}}u - u]\!])\, ds\\
&\le C\, \|P_{\mathcal{T}}u - u\|_{DG} \|P_{\mathcal{T}}u - u_{\mathcal{T}}\|_{DG} + \left|\sum_{e\in\mathcal{E}_I\cup\mathcal{E}_D} \int_e \{k\nabla(P_{\mathcal{T}}u - u)\cdot \mathbf{n}_e\}_w [\![P_{\mathcal{T}}u - u_{\mathcal{T}}]\!]\, ds\right|,
\end{aligned}$$

which, together with Lemma 3.1, implies (3.5) and, hence, (3.4). This completes the proof of the theorem. $\square$

4. Oswald- and Clément-type interpolations. Denote by $\mathcal{N}$, $\mathcal{N}_D$, and $\mathcal{N}_K$ the sets of all vertices of the triangulation $\mathcal{T}$, on the Γ_D, and of element $K \in \mathcal{T}$, respectively. For any $z \in \mathcal{N}$, denote by ϕ_z the nodal basis function of $\mathcal{U}$, and let

$$\omega_z = \{K \in \mathcal{T} : K \subset \operatorname{supp}(\phi_z)\} \quad \text{and} \quad \hat{\omega}_z = \left\{K \in \omega_z : k_K = \max_{K'\in\omega_z} k_{K'}\right\}.$$

The number of elements in $\hat{\omega}_z$ is denoted by $cd(z)$. Also, denote by $\tilde{\mathcal{E}}_K$ the set of edges that share at least a vertex with K.

In this section and sections 5 and 7, assume that the distribution of the coefficients k_K for all $K \in \mathcal{T}$ is locally quasi-monotone [31], which is slightly weaker than Hypothesis 2.7 in [11]. For the convenience of the readers, we restate it here.

DEFINITION 4.1. *Given a vertex $z \in \mathcal{N}$, the distribution of coefficients k_K, $K \in \omega_z$, is said to be* quasi-monotone *with respect to the vertex z if there exists a subset $\tilde{\omega}_{K,z,qm}$ of ω_z such that the union of elements in $\tilde{\omega}_{K,z,qm}$ is a Lipschitz domain and that the following hold:*

- *if $z \in \mathcal{N}\backslash\mathcal{N}_D$, then $\{K\} \cup \hat{\omega}_z \subset \tilde{\omega}_{K,z,qm}$ and $k_K \le \max_{K'\in\omega_z} k_{K'}$;*
- *if $z \in \mathcal{N}_D$, then $K \in \tilde{\omega}_{K,z,qm}$, $\partial\tilde{\omega}_{K,z,qm} \cap \Gamma_D \ne \emptyset$, and $k_K \le \max_{K'\in\omega_z} k_{K'}$.*

The distribution of coefficients k_K, $K \in \mathcal{T}$, is said to be locally quasi-monotone *if it is quasi-monotone with respect to every vertex $z \in \mathcal{N}$.*

For a given function $v \in \mathcal{U}^{DG}$, define the Oswald interpolation operator $\mathcal{I} : \mathcal{U}^{DG} \to \mathcal{U}_g$ by

$$\mathcal{I}v = \sum_{z\in\mathcal{N}} \mathcal{I}v(z)\phi_z(x),$$

where the nodal value of the interpolant $\mathcal{I}v$ at z is defined by

$$\mathcal{I}v(z) = \begin{cases} g_D(z) & \text{if } z \in \mathcal{N}_D,\\[2mm] \dfrac{1}{cd(z)} \displaystyle\sum_{K\in\hat{\omega}_z} v_K(z) & \text{if } z \in \mathcal{N}\backslash\mathcal{N}_D \end{cases}$$