Least-Squares Method for the Oseen Equation

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This article studies the least-squares finite element method for the linearized, stationary Navier–Stokes equation based on the stress-velocity-pressure formulation in d dimensions (d = 2 or 3). The least-squares functional is simply defined as the sum of the squares of the L^2 norm of the residuals. It is shown that the homogeneous least-squares functional is elliptic and continuous in the $H(\text{div};\Omega)^d \times H^1(\Omega)^d \times L^2(\Omega)$ norm. This immediately implies that the a priori error estimate of the conforming least-squares finite element approximation is optimal in the energy norm. The L^2 norm error estimate for the velocity is also established through a refined duality argument. Moreover, when the right-hand side $\mathbf{f}$ belongs only to $L^2(\Omega)^d$, we derive an a priori error bound in a weaker norm, that is, the $L^2(\Omega)^{d\times d} \times H^1(\Omega)^d \times L^2(\Omega)$ norm. © 2016 Wiley Periodicals, Inc. Numer Methods Partial Differential Eq 32: 1289–1303, 2016

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I. INTRODUCTION

Least-squares finite element methods for the numerical solution of second-order partial differential equations and systems have been intensively studied by many researchers (see, e.g., books [1, 2] and references therein). Numerical properties of the least-squares methods depend on the form of the first-order system and the choice of the least-squares norms. Basically, there are three types of the least-squares methods: the inverse approach (see, e.g., [2–4]), the div approach (see, e.g., [5–7]), and the div-curl approach (see, e.g., [8]). The inverse approach uses an inverse norm that is further replaced by either the weighted mesh-dependent norm (see [9]) or the discrete H^{-1} norm (see [10]) for computational feasibility. The corresponding homogeneous least-squares functionals for the div and the div-curl approaches are equivalent to the $H(\text{div})$ and the $H(\text{div}) \cap H(\mathbf{curl})$ norms for some variables, respectively. For the Stokes and Navier–Stokes equations, the least-squares methods are based on various first-order systems such as formulations of the vorticity-velocity-pressure, the stress-velocity, the stress-velocity-pressure, the velocity-gradient-pressure, and so forth.

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In [7], we developed and analyzed the div least-square methods for the stationary Stokes equation. The purpose of this article is to extend our study to the Oseen equations, that is, the linearized, stationary Navier–Stokes equation. Specifically, we introduce the div least-squares minimization problem based on the stress-velocity-pressure formulation, and show that the corresponding homogeneous least-squares functional is elliptic and continuous in the $H(\text{div};\Omega)^d$ norm for the stress, the $H^1(\Omega)^d$ norm for the velocity, and the $L^2(\Omega)$ norm for the pressure. Due to the convection term in the Oseen equation, it is difficult to prove the ellipticity of the corresponding homogeneous least-squares functional. This is also true for the scalar second-order elliptic partial differential equation (see, e.g., [11]). Our approach here is to first prove that the homogeneous functional plus the squares of the L^2 norm of the velocity is elliptic (see (3.1)) and then to remove this extra term by a compactness argument based on the well-posedness of the original problem.

The div least-squares finite element method is to solve the least-squares minimization problem in the conforming finite element subspace: the Raviart–Thomas (RT) element of the index $k \geq 0$ [12] for the stress, the continuous Lagrange element of degree $k+1$ for the velocity, and the piecewise discontinuous polynomials of degree k for the pressure. As the least-squares finite element method is stable, finite element spaces for those variables may be chosen independently. However, the above choice is the only combination leading to an optimal least-square finite element approximation with respect to the regularity, the degree of polynomial, and the number of degrees of freedom. Replacing the RT element by the BDM element [13], the approximation is still optimal on the regularity and on the degree of polynomial, but the BDM element has slightly more unknowns than that of the RT element.

The ellipticity of the least-square functional immediately implies the optimal a priori error estimate in the energy norm for the least-squares finite element method. Moreover, the method is not subject to the constraint that the mesh size is sufficiently small as other numerical methods. As the energy norm for the least-squares formulation uses the $H(\text{div};\Omega)^d$ norm for the tensor variable, it is natural that the error estimate in the energy norm is established under the assumption that the right-hand side $\mathbf{f}$ is smooth enough (see Theorem 5.1). This assumption is slightly stronger than that for the standard finite element method [14], and will be removed when we estimate the error in a weaker norm as in [6] (see Theorem 5.4). Finally, by using a refined duality argument presented in [5], we are able to obtain optimal L^2 norm error estimate for the velocity.

Least-squares finite element methods for the Oseen equation were studied in [15] and [16] based on the velocity-gradient-pressure and the velocity-vorticity-pressure formulations, respectively. The least-squares methods in [15] are the inverse and the div-curl approaches. Basically, the inverse approach is expensive and the div-curl approach requires extra regularity of the underlying problem. The least-squares method in [16] applies the simple L^2 norm least-squares approach to the velocity-vorticity-pressure formulation. The resulting least-squares functional is only stable in a norm weaker than that for the continuity. Hence, the resulting finite element approximation is not optimal with respect to the approximation space and the regularity of the underlying problem. For well balanced least-squares methods based on the velocity-vorticity-pressure formulation, see [17, 18].

An outline of the article is as follows. In Section II, we introduce the Oseen equation as well as its stress-velocity-pressure formulation. The well-posedness of the least-squares minimization is proved through establishing the ellipticity and continuity of the homogeneous least-squares functional in Section III. The least-squares finite element method and its a priori error estimates in various norms are presented in Sections IV and V respectively.

II. THE OSEEN EQUATION, LEAST-SQUARES FORMULATION AND SOME PRELIMINARIES

Let Ω be a bounded, open, connected subset of $\Re^d$ ($d = 2$ or 3) with a Lipschitz continuous boundary $\partial\Omega$. Denote the outward unit vector normal to the boundary by $\mathbf{n} = (n_1, \ldots, n_d)^t$. We partition the boundary of Ω into two open subsets Γ_D and Γ_N such that $\partial\Omega = \overline{\Gamma}_D \cup \overline{\Gamma}_N$ and $\Gamma_D \cap \Gamma_N = \emptyset$. For simplicity, we will assume that Γ_D is not empty (i.e., meas $(\Gamma_D) \neq 0$).

We use the standard notation and definitions for the Sobolev spaces $H^s(\Omega)^d$ and $H^s(\partial\Omega)^d$ for $s \geq 0$. The standard associated inner products are denoted by $(\cdot,\cdot)_{s,\Omega}$ and $(\cdot,\cdot)_{s,\partial\Omega}$, and their respective norms are denoted by $\|\cdot\|_{s,\Omega}$ and $\|\cdot\|_{s,\partial\Omega}$. (We suppress the superscript d because the dependence on dimension will be clear by context. We also omit the subscript Ω from the inner product and norm designation when there is no risk of confusion.) For $s = 0$, $H^s(\Omega)^d$ coincides with $L^2(\Omega)^d$. In this case, the inner product and norm will be denoted by $(\cdot,\cdot)$ and $|\cdot|$, respectively. Finally, set

$$H_D^1(\Omega) = \left\{q \in H^1(\Omega) \,:\, q = 0 \text{ on } \Gamma_D\right\}$$

and

$$H(\text{div};\Omega) = \left\{\mathbf{v} \in L^2(\Omega)^d \,:\, \nabla \cdot \mathbf{v} \in L^2(\Omega)\right\},$$

which is a Hilbert space under the norm

$$\|\mathrm{v}\|_{H(\text{div};\,\Omega)} = (||\mathbf{v}||^2 + ||\nabla \cdot \mathbf{v}||^2)^{\frac{1}{2}},$$

and define the subspace

$$H_N(\text{div};\Omega) = \{\mathbf{v} \in H(\text{div};\Omega) \,:\, \mathbf{n} \cdot \mathbf{v} = 0 \text{ on } \Gamma_N\}.$$

Let $\mathbf{f} = (f_1, \ldots, f_d)^t$ be a given external body force defined in Ω and $\mathbf{g} = (g_1, \ldots, g_d)^t$ be a given external surface traction applied on Γ_N. Let $\mathbf{u}(\mathbf{x}, t) = (u_1, \ldots, u_d)^t$ be the velocity vector field of a particle of fluid that is moving through $\mathbf{x}$ at time t, and let $\boldsymbol{\sigma} = (\sigma_{ij})_{d\times d}$ be the stress tensor field. Without loss of generality, we assume that the density is unit-valued. Then conservation of momentum implies both symmetry of the stress tensor and the local relation

$$\begin{cases} \frac{D\mathbf{u}}{Dt} - \nabla \cdot \boldsymbol{\sigma} = \mathbf{f} & \text{in} \quad \Omega, \\ \mathbf{n} \cdot \boldsymbol{\sigma} = \mathbf{g} & \text{on} \quad \Gamma_N, \end{cases} \tag{2.1}$$

where $\frac{D}{Dt}$ is the material derivative

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla = \frac{\partial}{\partial t} + \sum_{i=1}^{d} u_i \frac{\partial}{\partial x_i}.$$

Let ν be the viscosity constant, p the pressure, and

$$\epsilon(\mathbf{u}) = \frac{1}{2}(\nabla\,\mathbf{u} + (\nabla\,\mathbf{u})^t)$$

the deformation rate tensor, where $\nabla \mathbf{u}$ is the velocity gradient tensor with entries $(\nabla \mathbf{u})_{ij} = \partial u_i/\partial x_j$. Then, the constitutive law for incompressible Newtonian fluids is

$$\begin{cases} \boldsymbol{\sigma} = 2\,\nu\,\epsilon(\mathbf{u}) - p\,I & \text{in} \quad \Omega, \\ \nabla \cdot \mathbf{u} = 0 & \text{in} \quad \Omega. \end{cases} \tag{2.2}$$

A standard algorithmic treatment of (2.1) and (2.2) is to semidiscretize in time [19]. This leads to the Oseen equation:

$$\begin{cases} -\nabla \cdot \boldsymbol{\sigma} + \mathbf{b} \cdot \nabla \mathbf{u} + c\,\mathbf{u} = \mathbf{f} & \text{in} \quad \Omega, \\ \boldsymbol{\sigma} + p\,I - 2\,\nu\epsilon(\mathbf{u}) = \mathbf{0} & \text{in} \quad \Omega, \\ \nabla \cdot \mathbf{u} = 0 & \text{in} \quad \Omega \end{cases} \tag{2.3}$$

with the boundary conditions

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_D \quad \text{and} \quad \mathbf{n} \cdot \boldsymbol{\sigma} = \mathbf{0} \quad \text{on } \Gamma_N, \tag{2.4}$$

where $\mathbf{b} = (b_1, b_2, \cdots, b_d)^t \in L^\infty(\Omega)^d$ is the given vector-valued function and c is a positive constant. We assumed that $\mathbf{g} = \mathbf{0}$ for simplicity.

Let

$$L^2_N(\Omega) = \begin{cases} L^2(\Omega) & \text{if } \Gamma_N \neq \emptyset, \\ L^2_0(\Omega) = \left\{ q \in L^2(\Omega) \mid \int_\Omega q\,dx = 0 \right\} & \text{otherwise} \end{cases} \tag{2.5}$$

and

$$\mathbf{X}_N = \begin{cases} H_N(\text{div};\,\Omega)^d & \text{if } \Gamma_N \neq \emptyset, \\ \mathbf{X}_0 \equiv \left\{ \tau \in H(\text{div};\,\Omega)^d \mid \int_\Omega \text{tr}\boldsymbol{\tau}\,dx = 0 \right\} & \text{otherwise.} \end{cases} \tag{2.6}$$

Given $\mathbf{f} \in L^2(\Omega)^d$, we define the following least-squares functional:

$$G(\boldsymbol{\sigma}, \mathbf{u}, p; \mathbf{f}) = ||\mathbf{b} \cdot \nabla \mathbf{u} - \nabla \cdot \boldsymbol{\sigma} + c\mathbf{u} - \mathbf{f}||^2 + ||\boldsymbol{\sigma} + p\,I - 2\,\nu\epsilon(\mathbf{u})||^2 + ||\nabla \cdot \mathbf{u}||^2 \tag{2.7}$$

for all $(\sigma,\, \mathbf{u},\, p) \in \mathcal{V} \equiv \mathbf{X}_N \times H^1_D(\Omega)^d \times L^2_N(\Omega)$. Then, the least-squares minimization problem is to find $(\boldsymbol{\sigma},\, \mathbf{u},\, p) \in \mathcal{V}$ such that

$$G(\boldsymbol{\sigma},\, \mathbf{u},\, p\,;\, \mathbf{f}) = \inf_{(\boldsymbol{\tau},\, \mathbf{v},\, q) \in \mathcal{V}} G(\boldsymbol{\tau},\, \mathbf{v},\, q\,;\, \mathbf{f}). \tag{2.8}$$

The corresponding variational formulation is to find $(\boldsymbol{\sigma},\, \mathbf{u},\, p) \in \mathcal{V}$ such that

$$b(\boldsymbol{\sigma},\, \mathbf{u},\, p\,;\boldsymbol{\tau},\, \mathbf{v},\, q) = \mathcal{F}(\boldsymbol{\tau},\, \mathbf{v},\, q), \quad \forall\,(\boldsymbol{\tau},\, \mathbf{v},\, q) \in \mathcal{V}, \tag{2.9}$$

where the bilinear and linear forms are given by

$$\begin{aligned} b(\boldsymbol{\sigma}, \mathbf{u}, p\,;\, \boldsymbol{\tau}, \mathbf{v}, q) = {} & (\mathbf{b} \cdot \nabla \mathbf{u} - \nabla \cdot \boldsymbol{\sigma} + c\,\mathbf{u},\, \mathbf{b} \cdot \nabla \mathbf{v} - \nabla \cdot \boldsymbol{\tau} + c\,\mathbf{v}) \\ & + (\boldsymbol{\sigma} - 2\,\nu\epsilon(\mathbf{u}) + p\,I, \boldsymbol{\tau} - 2\,\nu\epsilon(\mathbf{v}) + q\,I) + (\nabla \cdot \mathbf{u},\, \nabla \cdot \mathbf{v}) \end{aligned} \tag{2.10}$$

$$\text{and} \quad \mathcal{F}(\boldsymbol{\tau}, \mathbf{v}, q) = (f, \mathbf{b} \cdot \nabla \mathbf{v} - \nabla \cdot \boldsymbol{\tau} + c\, \mathbf{v}), \tag{2.11}$$

respectively.

We will use the following notation, identity, and inequality. For the second order tensors $\boldsymbol{\sigma} = (\sigma_{ij})_{d\times d}$ and $\boldsymbol{\tau} = (\tau_{ij})_{d\times d}$, the inner product $(\boldsymbol{\sigma}, \boldsymbol{\tau})$ is defined by

$$(\boldsymbol{\sigma}, \boldsymbol{\tau}) = \int_\Omega \sum_{i,j=1}^{d} \sigma_{ij}\tau_{ij}\, \mathrm{d}\mathbf{x}.$$

If $\boldsymbol{\sigma}$ is symmetric and $\boldsymbol{\tau}$ is skew-symmetric, then

$$(\boldsymbol{\sigma}\, ,\, \boldsymbol{\tau}) = 0. \tag{2.12}$$

Let $\mathcal{A} : R^{d\times d} \to R^{d\times d}$ be a linear map defined by

$$\mathcal{A}\,\boldsymbol{\tau} = \boldsymbol{\tau} - \frac{1}{d}\,(\mathrm{tr}\,\boldsymbol{\tau})\, I, \quad \forall\, \boldsymbol{\tau} \in R^{d\times d},$$

then the following inequality (see [20]) holds:

$$\|\boldsymbol{\tau}\| \le C\, (\|\mathcal{A}\,\boldsymbol{\tau}\| + \|\nabla\cdot\boldsymbol{\tau}\|), \quad \forall\, \boldsymbol{\tau} \in \mathbf{X}_N, \tag{2.13}$$

where C is a positive constant, possibly depending on the domain Ω.

Finally, we provide the velocity-pressure form of the Oseen equation by eliminating the stress $\boldsymbol{\sigma}$ in (2.3):

$$\begin{cases} -\nabla\cdot(2\,\nu\epsilon(\mathbf{u}) - p\, I) + \mathbf{b}\cdot\nabla\mathbf{u} + c\mathbf{u} = \mathbf{f} & \text{in} \quad \Omega, \\ \nabla\cdot\mathbf{u} = 0 & \text{in} \quad \Omega \end{cases} \tag{2.14}$$

with the boundary conditions

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_D \quad \text{and} \quad \mathbf{n}\cdot(2\,\nu\epsilon(\mathbf{u}) - p\, I) = \mathbf{0} \quad \text{on } \Gamma_N. \tag{2.15}$$

Multiplying the equations in (2.14) by $\mathbf{v}$ and q, respectively, and integrating by parts, we obtain the variational form: find $(\mathbf{u}, p) \in H^1_D(\Omega)^d \times L^2_N(\Omega)$ such that

$$a(\mathbf{u},\ p; \mathbf{v},\ q) = f(\mathbf{v},\ q), \quad \forall(\mathbf{v}, q) \in H^1_D(\Omega)^d \times L^2_N(\Omega), \tag{2.16}$$

where the bilinear form $a(\cdot,\cdot)$ and the linear form $f(\cdot)$ are given by

$$a(\mathbf{u}, p; \mathbf{v}, q) = (2\nu\epsilon(\mathbf{u}) - p\, I, \epsilon(\mathbf{v})) + (\mathbf{b}\cdot\nabla\mathbf{u} + c\,\mathbf{u}, \mathbf{v}) + (\nabla\cdot\mathbf{u}, q), \quad f(\mathbf{v},\ q) = (\mathbf{f}, \mathbf{v}),$$

respectively.

III. WELL-POSEDNESS

In this section, we establish the well-posedness of problem (2.9). To this end, let

$$|||(\boldsymbol{\tau},\ \mathbf{v},\ q)||| = (||\mathbf{v}||_1^2 + ||q||^2 + ||\boldsymbol{\tau}||^2 + ||\nabla\cdot\boldsymbol{\tau}||^2)^{1/2}, \quad \forall\,(\boldsymbol{\tau},\ \mathbf{v},\ q) \in \mathcal{V}.$$

Lemma 3.1. *For all $(\tau, \mathbf{v}, q) \in \mathcal{V}$, there exists a positive constant C depending on ν, $\mathbf{b}$, c, and Ω such that*

$$|||(\boldsymbol{\tau},\ \mathbf{v},\ q)|||^2 \le C(G(\boldsymbol{\tau},\mathbf{v},q;\mathbf{0}) + ||\mathbf{v}||^2). \tag{3.1}$$

Proof. The proof of this lemma is similar to that of Theorem 3.2 in [7]. For the convenience of readers, we provide a brief proof here.

For any $(\boldsymbol{\tau}, \mathbf{v}, q) \in \mathcal{V}$, by the triangle, the Poincaré, and the Korn inequalities, we have

$$||\mathbf{b}\cdot\nabla\mathbf{v} + c\mathbf{v}|| \le C\,||\mathbf{v}||_1 \le C\,||\epsilon(\mathbf{v})||, \tag{3.2}$$

which, together with the triangle inequality, implies

$$||\nabla\cdot\boldsymbol{\tau}|| \le G^{1/2}(\boldsymbol{\tau},\mathbf{v},q;\mathbf{0}) + C\,||\epsilon(\mathbf{v})||. \tag{3.3}$$

As I and $\epsilon(\mathbf{v})$ are symmetric, the triangle inequality implies

$$\begin{aligned}\|\boldsymbol{\tau} - \boldsymbol{\tau}^t\| &= \|(\boldsymbol{\tau} + q\,I - 2\,\nu\epsilon(\mathbf{v})) - (\boldsymbol{\tau} + q\,I - 2\,\nu\epsilon(\mathbf{v}))^t\|\\ &\le 2\,\|\boldsymbol{\tau} + q\,I - 2\,\nu\epsilon(\mathbf{v})\| \le 2\,G^{1/2}(\boldsymbol{\tau},\mathbf{v},q;\mathbf{0}).\end{aligned}$$

By (2.12), integration by parts, the Cauchy-Schwarz inequality, and (3.2), we have

$$\begin{aligned}|(\boldsymbol{\tau},\ \nu\epsilon(\mathbf{v}))| &= \left|(\boldsymbol{\tau},\ \nu\nabla\mathbf{v}) - \left(\frac{\boldsymbol{\tau}-\boldsymbol{\tau}^t}{2},\ \nu\nabla\mathbf{v}\right)\right| = \left|(-\nabla\cdot\boldsymbol{\tau},\ \nu\mathbf{v}) - \left(\frac{\boldsymbol{\tau}-\boldsymbol{\tau}^t}{2},\ \nu\nabla\mathbf{v}\right)\right|\\ &= \left|(\mathbf{b}\cdot\nabla\mathbf{v} - \nabla\cdot\boldsymbol{\tau} + c\,\mathbf{v},\ \nu\mathbf{v}) - (\mathbf{b}\cdot\nabla\mathbf{v} + c\mathbf{v}, \nu\mathbf{v}) - \left(\frac{\boldsymbol{\tau}-\boldsymbol{\tau}^t}{2},\ \nu\nabla\mathbf{v}\right)\right|\\ &\le G^{1/2}(\boldsymbol{\tau},\mathbf{v},q;\mathbf{0})\,||\nu\mathbf{v}|| + C\,||\mathbf{v}||\,||\nu\mathbf{v}||_1 + G^{1/2}(\boldsymbol{\tau},\mathbf{v},q;\mathbf{0})\,||\nu\nabla\,\mathbf{v}||\\ &\le C\,(G(\boldsymbol{\tau},\ \mathbf{v},\ q\,;\mathbf{0}) + ||\mathbf{v}||^2)^{1/2}\,||\nu\epsilon(\mathbf{v})||,\end{aligned}$$

which, together with the fact that $(q\,I,\ \epsilon(\mathbf{v})) = (q,\ \nabla\cdot\mathbf{v})$ and the Cauchy–Schwarz inequality, gives

$$\begin{aligned}2\,\|\nu\epsilon(\mathbf{v})\|^2 &= (2\,\nu\epsilon(\mathbf{v}) - \boldsymbol{\tau} - q\,I,\ \nu\epsilon(\mathbf{v})) + (q,\ \nu\nabla\cdot\mathbf{v}) + (\boldsymbol{\tau},\ \nu\epsilon(\mathbf{v}))\\ &\le G^{1/2}(\boldsymbol{\tau},\mathbf{v},q;\mathbf{0})(\|\nu\epsilon(\mathbf{v})\| + \|\nu q\|) + C\,(G(\boldsymbol{\tau},\ \mathbf{v},\ q\,;\mathbf{0}) + \|\mathbf{v}\|^2)^{1/2}\,\|\nu\epsilon(\mathbf{v})\|.\end{aligned}$$

Hence, together with the ϵ-inequality, the above inequality implies

$$\|\epsilon(\mathbf{v})\|^2 \le C\,(G(\boldsymbol{\tau},\ \mathbf{v},\ q\,;\mathbf{0}) + \|\mathbf{v}\|^2 + G^{1/2}(\boldsymbol{\tau},\mathbf{v},q;\mathbf{0})\,\|q\|). \tag{3.4}$$

To bound $\|q\|$ in (3.4), by the triangle inequality we have

$$||q|| \leq \frac{1}{d}(||\mathrm{tr}(\boldsymbol{\tau} + q\, I - 2\,\nu\epsilon(\mathbf{v}))|| + \|\mathrm{tr}\,\boldsymbol{\tau}\| + 2\,\|\nu\nabla \cdot \mathbf{v}\|)$$
$$\leq C\,(G^{\frac{1}{2}}(\boldsymbol{\tau},\,\mathbf{v},\,q\,;\mathbf{0}) + \|\mathrm{tr}\,\boldsymbol{\tau}\|). \tag{3.5}$$

To bound $\|\mathrm{tr}\,\boldsymbol{\tau}\|$ in (3.5), the Cauchy-Schwarz inequality gives

$$\|\mathcal{A}\boldsymbol{\tau}\|^2 = (\boldsymbol{\tau},\,\mathcal{A}\boldsymbol{\tau}) = (\boldsymbol{\tau} + q\,I - 2\,\nu\epsilon(\mathbf{v}),\,\mathcal{A}\boldsymbol{\tau}) + (2\,\nu\epsilon(\mathbf{v}),\,\mathcal{A}\boldsymbol{\tau})$$
$$\leq (G^{\frac{1}{2}}(\boldsymbol{\tau},\,\mathbf{v},\,q\,;\mathbf{0}) + C\,\|\epsilon(\mathbf{v})\|)\|\mathcal{A}\boldsymbol{\tau}\|.$$

Hence,

$$\|\mathcal{A}\boldsymbol{\tau}\| \leq G^{\frac{1}{2}}(\boldsymbol{\tau},\,\mathbf{v},\,q\,;\mathbf{0}) + C\,\|\epsilon(\mathbf{v})\|,$$

which, together with (2.13) and (3.3), implies that

$$\|\mathrm{tr}\,\boldsymbol{\tau}\| \leq d\,\|\boldsymbol{\tau}\| \leq C\,(\|\mathcal{A}\boldsymbol{\tau}\| + \|\nabla\cdot\boldsymbol{\tau}\|) \leq C\,(G^{1/2}(\boldsymbol{\tau},\,\mathbf{v},\,q\,;\mathbf{0}) + \|\epsilon(\mathbf{v})\|). \tag{3.6}$$

Now, combining the upper bounds in (3.4)–(3.5), and (3.6) leads to

$$\|\epsilon(\mathbf{v})\|^2 \leq C\,(G(\boldsymbol{\tau},\,\mathbf{v},\,q\,;\mathbf{0}) + \|\mathbf{v}\|^2 + G^{1/2}(\boldsymbol{\tau},\,\mathbf{v},\,q\,;\mathbf{0})\,||\mathrm{tr}\,\boldsymbol{\tau}||)$$
$$\leq C\,(G(\boldsymbol{\tau},\,\mathbf{v},\,q\,;\mathbf{0}) + ||\mathbf{v}||^2 + G^{1/2}(\boldsymbol{\tau},\,\mathbf{v},\,q\,;\mathbf{0})\,||\epsilon(\mathbf{v})||).$$

Hence, by the ϵ-inequality, we have

$$\|\epsilon(\mathbf{v})\|^2 \leq C\,(G(\boldsymbol{\tau},\,\mathbf{v},\,q\,;\mathbf{0}) + \|\mathbf{v}\|^2),$$

which, together with (3.3, 3.6), and (3.5), implies that $\|\nabla\cdot\boldsymbol{\tau}\|$, $\|\boldsymbol{\tau}\|^2$, and $\|q\|^2$ are also bounded above by $G(\boldsymbol{\tau},\,\mathbf{v},\,q\,;\mathbf{0}) + \|\mathbf{v}\|^2$. This completes the proof of the lemma. ■

Theorem 3.2. *The homogeneous functional $G(\tau,\,\mathbf{v},\,q\,;\,\mathbf{0})$ is uniformly elliptic and continuous in $\mathcal{V}$; that is, there exist positive constants C_1 and C_2, depending on ν, $\mathbf{b}$, c, and Ω, such that*

$$C_1|||(\boldsymbol{\tau},\,\mathbf{v},\,q)|||^2 \leq G(\boldsymbol{\tau},\,\mathbf{v},\,q\,;\,\mathbf{0}) \leq C_2|||(\boldsymbol{\tau},\,\mathbf{v},\,q)|||^2, \quad \forall\,(\boldsymbol{\tau},\,\mathbf{v},\,q) \in \mathcal{V}. \tag{3.7}$$

Proof. The upper bound in (3.7) follows easily from the triangle inequality.

To show the validity of the lower bound in (3.7), we use the standard compactness argument. To this end, assume that the lower bound in (3.7) does not hold. Hence, there exists a sequence $(\boldsymbol{\tau}_n, \mathbf{v}_n, q_n) \in \mathcal{V}$ such that

$$|||(\boldsymbol{\tau}_n, \mathbf{v}_n, q_n)|||^2 = 1 \quad \text{and} \quad G(\boldsymbol{\tau}_n,\,\mathbf{v}_n,\,q_n\,;\mathbf{0}) < \frac{1}{n}. \tag{3.8}$$

As $H^1_D(\Omega)^d$ is compactly embedded in $L^2(\Omega)^d$, there exists a subsequence $\{\mathbf{v}_{n_i}\} \in H^1_D(\Omega)^d$, which converges in $L^2(\Omega)^d$. For any integers i and j and for any $(\boldsymbol{\tau}_{n_i}, \mathbf{v}_{n_i}, q_{n_i})$, $(\boldsymbol{\tau}_{n_j}, \mathbf{v}_{n_j}, q_{n_j}) \in \mathcal{V}$, it follows from Lemma 3.1 and the triangle inequality that

$$
\begin{aligned}
&|||(\boldsymbol{\tau}_{n_i} - \boldsymbol{\tau}_{n_j},\, \mathbf{v}_{n_i} - \mathbf{v}_{n_j},\, q_{n_i} - q_{n_j})|||^2 \\
&\quad \le C\, \left(G\left(\boldsymbol{\tau}_{n_i} - \boldsymbol{\tau}_{n_j},\, \mathbf{v}_{n_i} - \mathbf{v}_{n_j},\, q_{n_i} - q_{n_j}\,;\mathbf{0}\right) + ||\mathbf{v}_{n_i} - \mathbf{v}_{n_j}||^2\right) \\
&\quad \le C\, \left(\frac{1}{n_i} + \frac{1}{n_j} + \|\mathbf{v}_{n_i} - \mathbf{v}_{n_j}\|^2\right) \to 0,
\end{aligned}
$$

as $i, j \to \infty$. Therefore, $\{(\boldsymbol{\tau}_{n_i}, \mathbf{v}_{n_i}, q_{n_i})\}$ is a Cauchy sequence in the complete space $\mathcal{V}$. Hence, there exists $(\boldsymbol{\tau}_0, \mathbf{v}_0, q_0) \in \mathcal{V}$ such that

$$\lim_{i\to\infty} |||(\boldsymbol{\tau}_{n_i} - \boldsymbol{\tau}_0,\, \mathbf{v}_{n_i} - \mathbf{v}_0,\, q_{n_i} - q_0)||| = 0.$$

Next, we will show that

$$(\boldsymbol{\tau}_0,\, \mathbf{v}_0,\, q_0) = (\mathbf{0},\, \mathbf{0},\, 0), \tag{3.9}$$

which is contradictory with the first assumption in (3.8):

$$0 = |||(\boldsymbol{\tau}_0, \mathbf{v}_0, q_0)|||^2 = \lim_{i\to\infty} |||(\boldsymbol{\tau}_{n_i}, \mathbf{v}_{n_i}, q_{n_i})|||^2 = 1.$$

This in turn implies the validity of the lower bound in (3.7). To this end, for any $(\mathbf{w}, r) \in H_D^1(\Omega)^d \times L_N^2(\Omega)$, by the symmetry of I and $\epsilon(\mathbf{v}_{n_i})$, integration by parts, and the Cauchy–Schwarz inequality, we have

$$
\begin{aligned}
&|a(\mathbf{v}_{n_i}, q_{n_i}; \mathbf{w}, r)| \\
&\quad = |(2\,\nu\,\epsilon(\mathbf{v}_{n_i}) - q_{n_i} I, \nabla \mathbf{w}) + (\mathbf{b}\cdot\nabla\mathbf{v}_{n_i} + c\mathbf{v}_{n_i}, \mathbf{w}) + (\nabla\cdot\mathbf{v}_{n_i}, r)| \\
&\quad = |(2\,\nu\,\epsilon(\mathbf{v}_{n_i}) - q_{n_i} I - \boldsymbol{\tau}_{n_i}, \nabla \mathbf{w}) + (\mathbf{b}\cdot\nabla\mathbf{v}_{n_i} - \nabla\cdot\boldsymbol{\tau}_{n_i} + c\,\mathbf{v}_{n_i}, \mathbf{w}) + (\nabla\cdot\mathbf{v}_{n_i}, r)| \\
&\quad \le G^{1/2}(\boldsymbol{\tau}_{n_i}, \mathbf{v}_{n_i}, q_{n_i}; \mathbf{0})\,(||\mathbf{w}||_1^2 + ||r||^2)^{1/2} \le \frac{1}{\sqrt{n_i}}\,(||\mathbf{w}||_1^2 + ||r||^2)^{1/2}.
\end{aligned}
$$

Hence,

$$|a(\mathbf{v}_0, q_0; \mathbf{w}, r)| = \lim_{i\to\infty} |a(\mathbf{v}_{n_i}, q_{n_i}; \mathbf{w}, r)| \le 0,$$

which, together with the uniqueness of problem (2.16), implies

$$\mathbf{v}_0 = \mathbf{0} \quad \text{and} \quad q_0 = 0.$$

Now, $\boldsymbol{\tau}_0 = \mathbf{0}$ is a direct consequence of Lemma 3.1:

$$||\boldsymbol{\tau}_0||^2_{H(\mathrm{div};\Omega)} = \lim_{i\to\infty} ||\boldsymbol{\tau}_{n_i}||^2_{H(\mathrm{div};\Omega)} \le C \lim_{i\to\infty} (G(\boldsymbol{\tau}_{n_i}, \mathbf{v}_{n_i}, q_{n_i}; \mathbf{0}) + ||\mathbf{v}_{n_i}||^2) = 0.$$

This completes the proof of (3.9) and, hence, the theorem. ■

Proposition 3.3. *Problem (2.9) has a unique solution $(\boldsymbol{\sigma},\, \mathbf{u},\, p) \in \mathcal{V}$ satisfying the following a priori estimate:*

$$|||(\boldsymbol{\sigma},\, \mathbf{u},\, p)||| \le C\, ||f||.$$

Proof. It is easy to see that the linear form $\mathcal{F}(\boldsymbol{\tau}, \mathbf{v}, q)$ is bounded:

$$|\mathcal{F}(\boldsymbol{\tau}, \mathbf{v}, q)| \le C\,||f||\,|||(\boldsymbol{\tau}, \mathbf{v}, q)|||, \quad \forall\,(\boldsymbol{\tau}, \mathbf{v}, q) \in \mathcal{V}.$$

By Theorem 3.2 and the Lax–Milgram lemma, problem (2.9) has a unique solution $(\boldsymbol{\sigma}, \mathbf{u}, p) \in \mathcal{V}$. The a priori estimate is obtained as follows:

$$C\,|||(\boldsymbol{\sigma}, \mathbf{u}, p)|||^2 \le b(\boldsymbol{\sigma}, \mathbf{u}, p; \boldsymbol{\sigma}, \mathbf{u}, p) = \mathcal{F}(\boldsymbol{\sigma}, \mathbf{u}, p) \le C\,||f||\,|||(\boldsymbol{\sigma}, \mathbf{u}, p)|||.$$

This completes the proof of the proposition. ∎

IV. LEAST-SQUARES FINITE ELEMENT APPROXIMATION

For simplicity, consider the two-dimensional case ($d = 2$). Assuming that the domain Ω is polygonal, let $\mathcal{T}_h$ be a regular triangulation of Ω (see [21]) with triangular elements of size $\mathcal{O}(h)$. Let $P_k(K)$ be the space of polynomials of degree k on triangle K, and denote the local Raviart–Thomas space of index k on K by

$$RT_k(K) = P_k(K)^2 + \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} P_k(K).$$

Then, the standard $H(\text{div}; \Omega)$ conforming Raviart–Thomas space of index k [12], the standard (conforming) continuous piecewise polynomials of degree $k+1$, and the piecewise polynomials of degree k are defined, respectively, by

$$\Sigma_h^k = \left\{\boldsymbol{\tau} \in \mathbf{X}_N \;:\; \boldsymbol{\tau}|_K \in RT_k(K)^2,\ \forall\,K \in \mathcal{T}_h\right\} \subset \mathbf{X}_N, \tag{4.1}$$

$$V_h^{k+1} = \left\{\mathbf{v} \in C^0(\Omega)^2 \;:\; \mathbf{v}|_K \in P_{k+1}(K)^2,\ \forall\,K \in \mathcal{T}_h,\ \mathbf{v} = \mathbf{0} \text{ on } \Gamma_D\right\} \subset H_D^1(\Omega)^2, \tag{4.2}$$

$$M_h^k = \left\{p \in L_N^2(\Omega) \;:\; p|_K \in P_k(K),\ \forall\,K \in \mathcal{T}_h\right\} \subset L_N^2(\Omega). \tag{4.3}$$

These spaces have the following approximation properties: let $k \ge 0$ be an integer, and let $l \in (0, k+1]$:

$$\inf_{\boldsymbol{\tau} \in \Sigma_h^k} ||\boldsymbol{\sigma} - \boldsymbol{\tau}||_{H(\text{div};\Omega)} \le C\,h^l(||\boldsymbol{\sigma}||_l + ||\nabla\cdot\boldsymbol{\sigma}||_l) \tag{4.4}$$

for $\boldsymbol{\sigma} \in H^l(\Omega)^{2\times 2} \cap \mathbf{X}_N$ with $\nabla\cdot\boldsymbol{\sigma} \in H^l(\Omega)^2$ and

$$\inf_{\mathbf{v} \in V_h^{k+1}} ||\mathbf{u} - \mathbf{v}||_1 \le C\,h^l\,||\mathbf{u}||_{l+1} \tag{4.5}$$

for $\mathbf{u} \in H^{l+1}(\Omega)^2 \cap H_D^1(\Omega)^2$, and

$$\inf_{q \in M_h^k} ||p - q|| \le Ch^l||p||_l \tag{4.6}$$

for $p \in H^l(\Omega) \cap L_N^2(\Omega)$. Based on the smoothness of $\boldsymbol{\sigma}$, $\mathbf{u}$, and p, we will choose $k+1$ to be the smallest integer greater than or equal to l.

The least-squares finite element approximation to the Oseen equation based on the stress-velocity-pressure formulation is to find $(\boldsymbol{\sigma}^h, \mathbf{u}^h, p^h) \in \Sigma_h^k \times V_h^{k+1} \times M_h^k$ such that

$$G(\boldsymbol{\sigma}^h, \mathbf{u}^h, p^h; \mathbf{f}) = \min_{(\boldsymbol{\tau}, \mathbf{v}, q) \in \Sigma_h^k \times V_h^{k+1} \times M_h^k} G(\boldsymbol{\tau}, \mathbf{v}, q; \mathbf{f}). \tag{4.7}$$

Equivalently, it is to find $(\boldsymbol{\sigma}^h, \mathbf{u}^h, p^h) \in \Sigma_h^k \times V_h^{k+1} \times M_h^k$ such that

$$b(\boldsymbol{\sigma}^h, \mathbf{u}^h, p^h; \boldsymbol{\tau}, \mathbf{v}, q) = \mathcal{F}(\boldsymbol{\tau}, \mathbf{v}, q), \quad \forall (\boldsymbol{\tau}, \mathbf{v}, q) \in \Sigma_h^k \times V_h^{k+1} \times M_h^k. \tag{4.8}$$

By Theorem 3.2 and the fact that $\Sigma_h^k \times V_h^{k+1} \times M_h^k \subset \mathcal{V}$, problem (4.7) and equivalent problem (4.8) have a unique solution.

V. A PRIORI ERROR ESTIMATES

In this section, we establish a priori error estimates in both the energy norm and the L^2 norm. These estimates are obtained under the assumption that the right-hand side $\mathbf{f}$ is sufficiently smooth. When $\mathbf{f}$ is only in $L^2(\Omega)^d$, we are also able to derive an a priori error estimate in a norm which is weaker than the energy norm.

Let $(\boldsymbol{\sigma}, \mathbf{u}, p)$ and $(\boldsymbol{\sigma}^h, \mathbf{u}^h, p^h)$ be the solutions of (2.9) and (4.8), respectively. Denote by

$$E^h = \boldsymbol{\sigma} - \boldsymbol{\sigma}^h, \quad \mathbf{e}^h = \mathbf{u} - \mathbf{u}^h, \quad \text{and} \quad e^h = p - p^h. \tag{5.1}$$

Taking the difference between (2.9) and (4.8) gives the following orthogonality:

$$b(E^h, \mathbf{e}^h, e^h; \boldsymbol{\tau}, \mathbf{v}, q) = 0, \quad \forall (\boldsymbol{\tau}, \mathbf{v}, q) \in \Sigma_h^k \times V_h^{k+1} \times M_h^k. \tag{5.2}$$

A. Energy Norm Error Estimate

In this section, we obtain the quasi-optimal a priori error estimate in the energy norm.

Theorem 5.1. *Assume that* $\mathbf{f} \in H^l(\Omega)^2$ *and that the solution* $(\boldsymbol{\sigma}, \mathbf{u}, p)$ *of (2.9) is in* $H^l(\Omega)^{2\times 2} \times H^{l+1}(\Omega)^2 \times H^l(\Omega)$. *Let* $k+1$ *be the smallest integer greater than or equal to* l. *Then, with* $(\boldsymbol{\sigma}^h, \mathbf{u}^h, p^h) \in \Sigma_h^k \times V_h^{k+1} \times M_h^k$ *denoting the solution to (4.8), the following error estimate holds:*

$$|||(E^h, \mathbf{e}^h, e^h)||| \le C\, h^l(||\boldsymbol{\sigma}||_l + ||\mathbf{f}||_l + ||\mathbf{u}||_{l+1} + ||p||_l). \tag{5.3}$$

Proof. By the coercivity in (3.7), the orthogonality in (5.2), and the Cauchy–Schwarz inequality, it is straightforward to obtain the following Céa's lemma for the least-squares finite element approximation:

$$|||(E^h, \mathbf{e}^h, e^h)||| \le C \inf_{(\boldsymbol{\tau}, \mathbf{v}, q) \in \Sigma_h^k \times V_h^{k+1} \times M_h^k} |||(\boldsymbol{\sigma} - \boldsymbol{\tau}, \mathbf{u} - \mathbf{v}, p - q)|||.$$

Now, (5.3) follows from the approximation properties in (4.4)–(4.5), and (4.6) and the fact that

$$||\nabla \cdot \boldsymbol{\sigma}||_l \le ||\mathbf{f}||_l + ||\mathbf{b} \cdot \nabla u||_l + ||c\,\mathbf{u}||_l \le ||\mathbf{f}||_l + C\,||\mathbf{u}||_{l+1}.$$

This completes the proof of the theorem. ∎

B. L^2 Norm Error Estimate

As usual, we use a duality argument to establish the L^2 norm error estimate. Note that this argument for the div least-square finite element method is more complicated than that for the Galerkin finite element method.

To this end, for $\mathbf{f} \in L^2(\Omega)^2$, $g \in H^1(\Omega)$, and $\mathbf{g}_N \in H^{1/2}(\Gamma_N)$, consider the Oseen problem in (2.16) with a linear form defined by

$$f(\mathbf{v}, q) = (\mathbf{f}, \mathbf{v}) + (g, q) + \int_{\Gamma_N} \mathbf{g}_N \cdot \mathbf{v}\, ds.$$

Consider also the dual problem of (2.16): find $(\mathbf{z}, r) \in H^1_D(\Omega)^d \times L^2_N(\Omega)$ such that

$$a(\mathbf{v}, q; \mathbf{z}, r) = (\mathbf{f}, \mathbf{v}) + (g, q), \quad \forall\, (\mathbf{v}, q) \in H^1_D(\Omega)^d \times L^2_N(\Omega). \tag{5.4}$$

Assume that both problems have the full H^2 regularity:

$$||\mathbf{u}||_2 + ||p||_1 \le C \left(||\mathbf{f}|| + ||g||_1 + ||\mathbf{g}_N||_{\frac{1}{2},\Gamma_N} \right) \quad \text{and} \quad ||\mathbf{z}||_2 + ||r||_1 \le C\, (||\mathbf{f}|| + ||g||_1). \tag{5.5}$$

Lemma 5.2. *Assume that the regularity estimates in (5.5) hold. Then, there exists $(\boldsymbol{\gamma}, \mathbf{w}, q) \in \mathcal{V}$ such that*

$$||\mathbf{e}^h||^2 = b(E^h, \mathbf{e}^h, e^h; \boldsymbol{\gamma}, \mathbf{w}, q) \tag{5.6}$$

and that

$$||\boldsymbol{\gamma}||_1 + ||\nabla \cdot \boldsymbol{\gamma}||_1 + ||\mathbf{w}||_2 + ||q||_1 \le C\, ||\mathbf{e}^h||. \tag{5.7}$$

Proof. Let $(\mathbf{z}, r)$ be the solution of the dual problem in (5.4) with $\mathbf{f} = \mathbf{e}^h$ and $g = 0$. Then the regularity assumption in (5.5) gives

$$||\mathbf{z}||_2 + ||r||_1 \le C\, ||\mathbf{e}^h||. \tag{5.8}$$

Choose $(\mathbf{v}, q) = (\mathbf{e}^h, e^h)$ in (5.4) with $\mathbf{f} = \mathbf{e}^h$ and $g = 0$. The fact that $\epsilon(\mathbf{e}^h)$ and I are symmetric leads to

$$(2\nu\epsilon(\mathbf{e}^h) - e^h\, I,\, \epsilon(\mathbf{z})) = (2\nu\epsilon(\mathbf{e}^h) - e^h\, I,\, \nabla\mathbf{z}),$$

which, together with integrating by parts, gives

$$\begin{aligned} ||\mathbf{e}^h||^2 &= a(\mathbf{e}^h, e^h; \mathbf{z}, r) = (2\nu\epsilon(\mathbf{e}^h) - e^h\, I,\, \epsilon(\mathbf{z})) + (\mathbf{b}\cdot\nabla\mathbf{e}^h + c\,\mathbf{e}^h,\, \mathbf{z}) + (\nabla\cdot\mathbf{e}^h,\, r) \\ &= (2\nu\epsilon(\mathbf{e}^h) - e^h\, I,\, \nabla\mathbf{z}) + (\mathbf{b}\cdot\nabla\mathbf{e}^h + c\,\mathbf{e}^h,\, \mathbf{z}) + (\nabla\cdot\mathbf{e}^h,\, r) \\ &= (2\nu\epsilon(\mathbf{e}^h) - e^h\, I,\, \nabla\mathbf{z}) + (\nabla\cdot E^h,\, \mathbf{z}) + (\mathbf{b}\cdot\nabla\mathbf{e}^h + c\,\mathbf{e}^h - \nabla\cdot E^h,\, \mathbf{z}) + (\nabla\cdot\mathbf{e}^h,\, r) \\ &= (2\nu\epsilon(\mathbf{e}^h) - e^h\, I - E^h,\, \nabla\mathbf{z}) + (\mathbf{b}\cdot\nabla\mathbf{e}^h + c\,\mathbf{e}^h - \nabla\cdot E^h,\, \mathbf{z}) + (\nabla\cdot\mathbf{e}^h,\, r). \end{aligned} \tag{5.9}$$

Let $(\mathbf{w}, q)$ be the solution of the following Oseen problem:

$$\begin{cases} -\nabla\cdot(2\,\nu\epsilon(\mathbf{w}) - q\, I) + \mathbf{b}\cdot\nabla\mathbf{w} + c\,\mathbf{w} = \mathbf{z} - \Delta\mathbf{z} & \text{in} \quad \Omega, \\ \nabla\cdot\mathbf{w} = r & \text{in} \quad \Omega, \end{cases} \tag{5.10}$$

with boundary conditions

$$\mathbf{w} = \mathbf{0} \quad \text{on } \Gamma_D \quad \text{and} \quad \mathbf{n} \cdot (2\,\nu\epsilon(\mathbf{w}) - q\,I) = \mathbf{n} \cdot \nabla \mathbf{z} \quad \text{on } \Gamma_N, \tag{5.11}$$

and let

$$\boldsymbol{\gamma} = 2\,\nu\epsilon(\mathbf{w}) - q\,I - \nabla \mathbf{z},$$

then (5.6) follows from (5.9) and (5.10).

To prove the validity of (5.7), by the regularity assumption in (5.5), the triangle inequality, the trace theorem, and (5.8), we have

$$||\mathbf{w}||_2 + ||q||_1 \le C\,(||\mathbf{z} - \Delta \mathbf{z}|| + ||r||_1 + ||\nabla \mathbf{z} \cdot \mathbf{n}||_{\frac{1}{2},\Gamma_N}) \le C\,(||\mathbf{z}||_2 + ||r||_1) \le C\,||\mathbf{e}^h||.$$

It now follows from the triangle inequality and (5.8) that

$$||\boldsymbol{\gamma}||_1 = ||2\,\nu\epsilon(\mathbf{w}) - q\,I - \nabla \mathbf{z}||_1 \le C\,(||\mathbf{w}||_2 + ||q||_1 + ||\mathbf{z}||_2) \le C\,(||\mathbf{z}||_2 + ||r||_1) \le C\,||\mathbf{e}^h||$$

and that

$$||\nabla \cdot \boldsymbol{\gamma}||_1 = ||\mathbf{z} - \mathbf{b} \cdot \nabla \mathbf{w} - c\,\mathbf{w}||_1 \le C\,(||\mathbf{z}||_1 + ||\mathbf{w}||_2) \le C\,(||\mathbf{z}||_2 + ||r||_1) \le C\,||\mathbf{e}^h||.$$

This completes the proof of (5.7) and, hence, the lemma. ■

Theorem 5.3. *Under the assumptions of Theorem 5.1 and Lemma 5.2, the following L^2 norm error estimate holds:*

$$||\mathbf{e}^h|| \le C\,h\,|||(E^h, \mathbf{e}^h, e^h)||| \le C\,h^{l+1}\,(||\boldsymbol{\sigma}||_l + ||\mathbf{f}||_l + ||\mathbf{u}||_{l+1} + ||p||_l). \tag{5.12}$$

Proof. The second inequality in (5.12) is a direct consequence of the first inequality and Theorem 5.1. To prove the first inequality, take $(\boldsymbol{\gamma}, \mathbf{w}, q)$ as that in Lemma 5.2. For any $(\boldsymbol{\tau}^h, \mathbf{v}^h, q^h) \in \Sigma_h^k \times V_h^{k+1} \times M_h^k$, it follows from the orthogonality, the continuity, the approximation properties in (4.4)–(4.5), and (4.6), and (5.7) that

$$\begin{aligned}
||\mathbf{e}^h||^2 &= b(E^h, \mathbf{e}^h, e^h; \boldsymbol{\gamma}, \mathbf{w}, q) = b(E^h, \mathbf{e}^h, e^h; \boldsymbol{\gamma} - \boldsymbol{\gamma}^h, \mathbf{w} - \mathbf{w}^h, q - q^h) \\
&\le C\,|||(E^h, \mathbf{e}^h, e^h)||| \inf_{(\gamma^h, \mathbf{w}^h, q^h) \in \Sigma_h^k \times V_h^{k+1} \times M_h^k} |||(\boldsymbol{\gamma} - \boldsymbol{\gamma}^h, \mathbf{w} - \mathbf{w}^h, q - q^h)||| \\
&\le Ch\,|||(E^h, \mathbf{e}^h, e^h)|||(||\boldsymbol{\gamma}||_1 + ||\nabla \cdot \boldsymbol{\gamma}||_1 + ||\mathbf{w}||_2 + ||q||_1) \\
&\le Ch\,|||(E^h, \mathbf{e}^h, e^h)|||\,||\mathbf{e}^h||,
\end{aligned}$$

which implies the first inequality in (5.12). This completes the proof of the theorem. ■

C. Error Estimate With $f \in L^2(\Omega)^2$

The error estimate in the energy norm is obtained under the assumption that $\mathbf{f}$ is at least in $H^\alpha(\Omega)^2$ with $\alpha > 0$. Following the idea in [22, 5], we derive an error estimate in a weak norm when $\mathbf{f}$ is only in $L^2(\Omega)^2$.

To this end, let

$$RT_0 = \{\boldsymbol{\tau} \in H_N(\text{div};\, \Omega) \,:\, \boldsymbol{\tau}|_K \in RT_0(K), \quad \forall\, K \in \mathcal{T}_h\}$$
$$\text{and} \quad Q_h = \left\{q \in L^2(\Omega) : q|_K = constant \quad \forall\, K \in \mathcal{T}_h\right\}.$$

We will use the standard nodal interpolation operator $\pi_h : H_N(\text{div};\, \Omega) \to RT_0$ (see [23]). Define $\Pi_h : \mathbf{X}_N \to \Sigma_h^0$ by

$$\Pi_h \boldsymbol{\sigma} = (\pi_h \boldsymbol{\sigma}_1, \pi_h \boldsymbol{\sigma}_2),$$

then, Π_h satisfies the following properties:

$$||\boldsymbol{\sigma} - \Pi_h \boldsymbol{\sigma}|| \le Ch\, ||\boldsymbol{\sigma}||_1, \quad \forall\, \boldsymbol{\sigma} \in H^1(\Omega)^{2\times 2}, \tag{5.13}$$

$$(\nabla \cdot (\boldsymbol{\sigma} - \Pi_h \boldsymbol{\sigma}),\, \mathbf{q}) = 0, \quad \forall\, \mathbf{q} \in Q_h^2. \tag{5.14}$$

Theorem 5.4. *Let* $(\boldsymbol{\sigma},\, \mathbf{u},\, p)$ *be the solution of (2.9) and* $(\boldsymbol{\sigma}^h,\, \mathbf{u}^h,\, p^h)$ *the solution of (4.8) with* $k = 0$. *Assume that the regularity in (5.5) is valid. Then the following error estimate holds:*

$$||E^h|| + ||\mathbf{e}^h||_1 + ||e^h|| \le C\, h\, ||\mathbf{f}||. \tag{5.15}$$

Proof. Let $\mathbf{u}_I$ and p_I be interpolants of $\mathbf{u}$ and p that satisfy the approximation properties in (4.5) and (4.6), respectively. It follows from the triangle inequality and the approximation properties in (4.13, 4.5), and (4.6) that

$$\begin{aligned}
&||E^h|| + ||\mathbf{e}^h||_1 + ||e^h|| \\
&\quad \le ||\boldsymbol{\sigma} - \Pi_h \boldsymbol{\sigma}|| + ||\mathbf{u} - \mathbf{u}_I||_1 + ||p - p_I|| + ||\Pi_h \boldsymbol{\sigma} - \boldsymbol{\sigma}^h|| + ||\mathbf{u}_I - \mathbf{u}^h||_1 + ||p_I - p^h|| \\
&\quad \le Ch\, (||\boldsymbol{\sigma}||_1 + ||\mathbf{u}||_2 + ||p||_1) + C\, (||\Pi_h \boldsymbol{\sigma} - \boldsymbol{\sigma}^h|| + ||\mathbf{u}_I - \mathbf{u}^h||_1 + ||p_I - p^h||).
\end{aligned}$$

By (2.3) and the regularity estimate in (5.5), we have

$$||\boldsymbol{\sigma}||_1 + ||\mathbf{u}||_2 + ||p||_1 \le C\, (||\mathbf{u}||_2 + ||p||_1) \le C\, ||\mathbf{f}||.$$

Thus, to show the validity of (5.15), it suffices to prove that

$$||\Pi_h \boldsymbol{\sigma} - \boldsymbol{\sigma}^h|| + ||\mathbf{u}_I - \mathbf{u}^h||_1 + ||p_I - p^h|| \le Ch\, (||\boldsymbol{\sigma}||_1 + ||\mathbf{u}||_2 + ||p||_1). \tag{5.16}$$

The coercivity in (3.7) and the orthogonality in (5.2) lead to

$$\begin{aligned}
&C\, |||(\Pi_h \boldsymbol{\sigma} - \boldsymbol{\sigma}^h, \mathbf{u}_I - \mathbf{u}^h, p_I - p^h)|||^2 \\
&\quad \le b(\Pi_h \boldsymbol{\sigma} - \boldsymbol{\sigma}^h,\, \mathbf{u}_I - \mathbf{u}^h,\, p_I - p^h;\, \Pi_h \boldsymbol{\sigma} - \boldsymbol{\sigma}^h,\, \mathbf{u}_I - \mathbf{u}^h,\, p_I - p^h) \\
&\quad = b(\Pi_h \boldsymbol{\sigma} - \boldsymbol{\sigma}^h,\, \mathbf{u}_I - \mathbf{u}^h,\, p_I - p^h;\, \Pi_h \boldsymbol{\sigma} - \boldsymbol{\sigma},\, \mathbf{u}_I - \mathbf{u},\, p_I - p) = I_1 + I_2 + I_3,
\end{aligned} \tag{5.17}$$

where

$$I_1 = (-\nabla \cdot (\Pi_h \boldsymbol{\sigma} - \boldsymbol{\sigma}^h), -\nabla \cdot (\Pi_h \boldsymbol{\sigma} - \boldsymbol{\sigma})), \quad I_2 = (\mathbf{b} \cdot \nabla(\mathbf{u}_I - \mathbf{u}^h), -\nabla \cdot (\Pi_h \boldsymbol{\sigma} - \boldsymbol{\sigma})),$$

and

$$I_3 = (\Pi_h\boldsymbol{\sigma} - \boldsymbol{\sigma}^h - 2\nu\epsilon(\mathbf{u}_I - \mathbf{u}^h) + (p_I - p^h)\, I,\ \Pi_h\boldsymbol{\sigma} - \boldsymbol{\sigma} - 2\nu\epsilon(\mathbf{u}_I - \mathbf{u}) + (p_I - p)\, I)$$
$$+ (\mathbf{b}\cdot\nabla(\mathbf{u}_I - \mathbf{u}^h) - \nabla\cdot(\Pi_h\boldsymbol{\sigma} - \boldsymbol{\sigma}^h) + c(\mathbf{u}_I - u^h), \mathbf{b}\cdot\nabla(\mathbf{u}_I - \mathbf{u}) + c(\mathbf{u}_I - \mathbf{u}))$$
$$+ (\nabla\cdot(\mathbf{u}_I - \mathbf{u}^h), \nabla\cdot(\mathbf{u}_I - \mathbf{u})) + (c(\mathbf{u}_I - \mathbf{u}^h), -\nabla\cdot(\Pi_h\boldsymbol{\sigma} - \boldsymbol{\sigma})).$$

As $\nabla\cdot(\Pi_h\boldsymbol{\sigma} - \boldsymbol{\sigma}^h) \in Q_h^2$, (5.14) implies

$$I_1 = 0.$$

To bound I_2, let $\mathbf{b}_I$ be a piecewise constant function such that

$$||\mathbf{b} - \mathbf{b}_I||_{L^\infty} \le Ch\, ||\mathbf{b}||_{W_1^\infty} \le C\, h. \tag{5.18}$$

Notice that (5.14) implies

$$(\mathbf{b}_I\cdot\nabla(\mathbf{u}_I - \mathbf{u}^h), -\nabla\cdot(\Pi_h\boldsymbol{\sigma} - \boldsymbol{\sigma})) = 0,$$

which, together with the Cauchy–Schwarz inequality, (5.18) and the stability of the operator Π_h, implies

$$\begin{aligned} I_2 &= (\mathbf{b}\cdot\nabla(\mathbf{u}_I - \mathbf{u}^h),\ -\nabla\cdot(\Pi_h\boldsymbol{\sigma} - \boldsymbol{\sigma})) \\ &= ((\mathbf{b} - \mathbf{b}_I)\cdot\nabla(\mathbf{u}_I - \mathbf{u}^h),\ -\nabla\cdot(\Pi_h\boldsymbol{\sigma} - \boldsymbol{\sigma})) + (\mathbf{b}_I\cdot\nabla(\mathbf{u}_I - \mathbf{u}^h),\ -\nabla\cdot(\Pi_h\boldsymbol{\sigma} - \boldsymbol{\sigma})) \\ &= ((\mathbf{b} - \mathbf{b}_I)\cdot\nabla(\mathbf{u}_I - \mathbf{u}^h),\ -\nabla\cdot(\Pi_h\boldsymbol{\sigma} - \boldsymbol{\sigma})) \\ &\le C||\mathbf{b} - \mathbf{b}_I||_{L^\infty}||\mathbf{u}_I - \mathbf{u}^h||_1\, ||\nabla\cdot(\Pi_h\boldsymbol{\sigma} - \boldsymbol{\sigma})|| \le Ch\, ||\nabla\cdot\boldsymbol{\sigma}||\, ||\mathbf{u}_I - \mathbf{u}^h||_1. \end{aligned}$$

To bound I_3, it follows from the Cauchy–Schwarz inequality, (3.2), the triangle inequality, integration by parts, and the approximation properties in (4.13, 4.5), and (4.6) that

$$\begin{aligned} I_3 &\le ||\Pi_h\boldsymbol{\sigma} - \boldsymbol{\sigma}^h - 2\nu\epsilon(\mathbf{u}_I - \mathbf{u}^h) + (p_I - p^h)I||\, ||\Pi_h\boldsymbol{\sigma} - \boldsymbol{\sigma} - 2\nu\epsilon(\mathbf{u}_I - \mathbf{u}) + (p_I - p)\, I|| \\ &\quad + C\,(||\mathbf{u}_I - \mathbf{u}^h||_1 + ||\nabla\cdot(\Pi_h\boldsymbol{\sigma} - \boldsymbol{\sigma}^h)||)\,(||\mathbf{b}\cdot\nabla(\mathbf{u}_I - \mathbf{u})|| + ||c(\mathbf{u}_I - \mathbf{u}||) \\ &\quad + ||\nabla\cdot(\mathbf{u}_I - \mathbf{u}^h)||\, ||\nabla\cdot(\mathbf{u}_I - \mathbf{u})|| + (c\nabla(\mathbf{u}_I - \mathbf{u}^h),\ \Pi_h\boldsymbol{\sigma} - \boldsymbol{\sigma}) \\ &\le Ch\,(||\boldsymbol{\sigma}||_1 + ||\mathbf{u}||_2 + ||p||_1)\,(||\Pi_h\boldsymbol{\sigma} - \boldsymbol{\sigma}^h|| + ||\mathbf{u}_I - \mathbf{u}^h||_1 + ||p_I - p^h||) \\ &\quad + Ch\, ||\mathbf{u}||_2\,(||\mathbf{u}_I - \mathbf{u}^h||_1 + ||\nabla\cdot(\Pi_h\boldsymbol{\sigma} - \boldsymbol{\sigma}^h)||) + Ch\,(||\mathbf{u}||_2 + ||\boldsymbol{\sigma}||_1)\, ||\mathbf{u}_I - \mathbf{u}^h||_1 \\ &\le Ch\,(||\boldsymbol{\sigma}||_1 + ||\mathbf{u}||_2 + ||p||_1)|||(\Pi_h\boldsymbol{\sigma} - \boldsymbol{\sigma}^h,\ \mathbf{u}_I - \mathbf{u}^h,\ p_I - p^h)|||. \end{aligned}$$

Combining (5.17) with bounds for I_2 and I_3 gives

$$|||(\Pi_h\boldsymbol{\sigma} - \boldsymbol{\sigma}^h,\ \mathbf{u}_I - \mathbf{u}^h,\ p_I - p^h)||| \le Ch\,(||\boldsymbol{\sigma}||_1 + ||\nabla\cdot\boldsymbol{\sigma}|| + ||\mathbf{u}||_2 + ||p||_1).$$

which implies (5.16). This completes the proof of the theorem. ■

Remark 5.5. If, instead of the full $H^2(\Omega)$ regularity in (5.5), problem (2.9) admits only $H^{1+\alpha}(\Omega)$ regularity with $\alpha \in (0,\ 1)$, then Theorem 5.4 holds with the following estimate:

$$||E^h|| + ||\mathbf{e}^h||_1 + ||e^h|| \le C\, h^\alpha||\mathbf{f}||.$$

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