# Generalized Prager-Synge Identity and Robust Equilibrated Error Estimators for Discontinuous Elements

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**Abstract.** The well-known Prager-Synge identity is valid in $H^1(\Omega)$ and serves as a foundation for developing equilibrated a posteriori error estimators for continuous elements. In this paper, we introduce a new identity, that may be regarded as a generalization of the Prager-Synge identity, to be valid for *piecewise* $H^1(\Omega)$ functions for diffusion problems.

For nonconforming finite element approximation of arbitrary odd order, we improve the current methods by proposing a *fully explicit* approach that recovers an equilibrated flux in $H(\mathrm{div};\Omega)$ through a local element-wise scheme. The local efficiency for the recovered flux is robust with respect to the diffusion coefficient jump regardless of its distribution.

For discontinuous elements, we note that the typical approach of recovering a $H^1$ function for the nonconforming error can be proved robust only under some restrictive assumptions. To promote the unconditional robustness of the error estimator with respect to the diffusion coefficient jump, we propose to recover a gradient in $H(\mathrm{curl};\Omega)$ space through a simple *explicit* averaging technique over facets. Our resulting error estimator is proved to be globally reliable and locally efficient regardless of the coefficient distribution. Nevertheless, the reliability constant is no longer to be 1.

## 1 Introduction

Equilibrated a posteriori error estimators have attracted much interest recently due to the guaranteed reliability bound with the reliability constant being one. This property implies that they are perfect for discretization error control on both coarse and fine meshes. Error control on coarse meshes is important but difficult for computationally challenging problems.

For the conforming finite element approximation, a mathematical foundation of equilibrated estimators is the Prager-Synge identity [35] that is valid in $H^1(\Omega)$ (see Section 3). Based on this identity, various equilibrated estimators have been studied recently by many researchers (see, e.g., [32, 24, 34, 22, 23, 7, 3, 37, 11, 13, 14, 38, 19, 15, 26]). The key ingredient of the equilibrated estimators for the continuous elements is local recovery of an equilibrated (locally conservative) flux in the $H(\mathrm{div};\Omega)$ space through the numerical flux. By using a partition of unity, Ladevèze and Leguillon [32] initiated a local procedure to reduce the construction of an equilibrated flux to vertex patch based local calculations. For the continuous linear finite element

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approximation to the Poisson equation in two dimensions, an equilibrated flux in the lowest order Raviart-Thomas space was explicitly constructed in [11, 13]. This explicit approach does not lead to robust equilibrated estimator with respect to the coefficient jump without introducing a constraint minimization (see [19]). The constraint minimization on each vertex patch may be efficiently solved by first computing an equilibrated flux and then calculating a divergence free correction. For recent developments, see [15] and references therein.

Recovery of equilibrated fluxes for discontinuous elements has also been studied by many researchers. For discontinuous Garlerkin (DG) methods, equilibrated fluxes in Raviart-Thomas (RT) spaces were explicitly reconstructed in [2] for linear elements and in [25] for higher order elements. For nonconforming finite element methods, existing *explicit* equilibrated flux recoveries in RT spaces seem to be limited to the linear Crouzeix-Raviart (CR) and the quadratic Fortin-Soulie elements by Marini [33] (see [1] in the context of estimator) and Kim [30], respectively. For higher order nonconforming elements, existing recovery techniques for conforming elements (see, e.g., [13, 14, 25]) may be directly applied, but all these recoveries need to solve vertex-patch minimization problems. By solving element-wise minimization problems, a local reconstruction procedure was proposed by Ainsworth and Rankin in [4]. Their recovered flux is not in the $H(div)$ conforming spaces. Nevertheless, the resulting estimator provides a guaranteed upper bound. Another implicit approach recovering fluxes in the RT spaces is proposed by Becker and collaborators in [9] in which properties of the nonconforming solutions are explored.

One purpose of this paper is to establish the Prager-Synge identity for piecewise $H^1(\Omega)$ functions in both two and three dimensions. This is proceeded by first establishing an Prager-Synge inequality (see Lemma 3.1) and then showing the validity of the identity through a Helmholtz decomposition. For Poisson equation with pure Dirichlet boundary conditions, a non-optimal inequality was obtained earlier by Braess, Fraunholz, and Hoppe in [12]; and a slightly more general inequality than that of Lemma 3.1 was proved in [26] by introducing the elliptic projection of the discontinuous finite element approximation as done by Kim in [31].

Based on the generalized Prager-Synge identity and an equivalent form (see Corollary 3.5), the construction of an equilibrated a posteriori error estimator for discontinuous finite element solutions is reduced to recover an equilibrated flux in $H(\text{div};\Omega)$ and to recover either a potential function in $H^1(\Omega)$ or a curl free vector-valued function in $H(\text{curl};\Omega)$. The energy norm of the difference between the recovered flux (gradient or potential) and the corresponding numerical one is then used as the conforming (nonconforming) error estimator.

Another contribution of this paper is to introduce a fully explicit post-processing procedure for recovering an equilibrated flux in the RT space of index $k-1$ for the nonconforming elements of any odd order of $k \geq 1$. Currently, we are not able to extend our recovery technique to even orders. This is because structures of the nonconforming finite element spaces of even and odd orders are fundamentally different. In theory, our recovered flux appears to be the same as in [9]. However, the explicit formula is only provided for the first order Crouzeix-Raviart element in [9] and due to the nature of their approach local patch problems need to be solved for higher order elements. Based on our recovery, the resulting conforming error estimator can be proved locally efficient regardless of the coefficient jump. To our knowledge, this is the only existing flux recovery for higher order nonconforming elements that has such property. For other methods, e.g., see [4], the robust efficiency requires that the distribution of the diffusion coefficient is quasi-monotone (see [**?**])

Recovery of a potential function in $H^1(\Omega)$ for discontinuous elements was studied by many researchers (see, e.g., [4, 2, 12, 26]). The resulting a posteriori error estimator based on $H^1$

recovery can be locally efficient. Nevertheless, to show independence of the efficiency constant on the jump, it also has to assume quasi-monotone distribution on the diffusion coefficient. As an alternative to $H^1$ recovery, one can also recover a gradient in the curl free space. Local approaches for recovering equilibrated flux in [11, 13, 19, 14, 15] may be directly applied (at least in two dimensions) to obtain a gradient in the curl-free space. As mentioned previously, this approach again requires solutions of local constraint minimization problems over vertex patches. The resulting a posteriori error estimator will again suffer from the conditional robustness for the efficiency constant.

In this paper, to promote the unconditional robustness for both the conforming and nonconforming errors, we will employ a simple averaging technique over facets to recover a gradient in $H(\text{curl};\Omega)$. Due to the fact that the recovered gradient is not necessarily curl free, the reliability constant of the resulting estimator is no longer one. However, it turns out that the curl free constraint is not essential and, theoretically we are able to prove that the resulting estimator has the robust local reliability as well as the robust local efficiency without the quasi-monotone assumption. This is compatible with our recent result in [17] on the residual error estimator for discontinuous elements.

This paper is organized as follows. The diffusion problem and the finite element mesh are introduced in Section 2. The generalized Prager-Synge identity for piecewise $H^1(\Omega)$ functions are established in Section 3. In Section 4, we briefly introduce the nonconforming finite element approximation and the explicit recoveries of the equilibrated flux and the gradient. The resulting a posteriori error estimator is also described in Section 4. Global reliability and local efficiency of the estimator are proved in Section 5. Finally, numerical results are presented in Section 6.

## 2 Model problem

Let $\Omega$ be a bounded polygonal domain in $\mathbb{R}^d, d = 2, 3$, with Lipschitz boundary $\partial\Omega = \overline{\Gamma}_D \cup \overline{\Gamma}_N$, where $\overline{\Gamma}_D \cap \overline{\Gamma}_N = \emptyset$. For simplicity, assume that $\text{meas}_{d-1}(\Gamma_D) \neq 0$. Considering the diffusion problem:

$$-\nabla \cdot (A\nabla u) = f \quad \text{in} \quad \Omega, \tag{2.1}$$

with boundary conditions

$$u = 0 \;\text{ on } \Gamma_D \quad \text{and} \quad -A\nabla u \cdot \mathbf{n} = g \;\text{ on } \Gamma_N,$$

where $\nabla\cdot$ and $\nabla$ are the respective divergence and gradient operators; $\mathbf{n}$ is the outward unit vector normal to the boundary; $f \in L^2(\Omega)$ and $g \in H^{-1/2}(\Gamma_N)$ are given scalar-valued functions; and the diffusion coefficient $A(x)$ is symmetric, positive definite, and piecewise constant full tensor with respect to the domain $\overline{\Omega} = \cup_{i=1}^n \overline{\Omega}_i$. Here we assume that the subdomain, $\Omega_i$ for $i = 1, \cdots, n$, is open and polygonal.

We use the standard notations and definitions for the Sobolev spaces. Let

$$H^1_D(\Omega) = \left\{ v \in H^1(\Omega) \,:\, v = 0 \text{ on } \Gamma_D \right\}.$$

Then the corresponding variational problem of (2.1) is to find $u \in H^1_D(\Omega)$ such that

$$a(u,\, v) := (A\nabla u, \nabla v) = (f, v) - \langle g, v\rangle_{\Gamma_N}, \quad \forall\, v \in H^1_D(\Omega), \tag{2.2}$$

where $(\cdot,\cdot)_\omega$ is the $L^2$ inner product on the domain $\omega$. The subscript $\omega$ is omitted when $\omega = \Omega$.

### 2.1 Triangulation

Let $\mathcal{T} = \{K\}$ be a finite element partition of $\Omega$ that is regular, and denote by $h_K$ the diameter of the element $K$. Furthermore, assume that the interfaces,

$$\Gamma = \{\partial\Omega_i \cap \partial\Omega_j : i \neq j \text{ and } i, j = 1, \cdots, n\},$$

do not cut through any element $K \in \mathcal{T}$. Denote the set of all facets of the triangulation $\mathcal{T}$ by

$$\mathcal{E} := \mathcal{E}_I \cup \mathcal{E}_D \cup \mathcal{E}_N,$$

where $\mathcal{E}_I$ is the set of interior element facets, and $\mathcal{E}_D$ and $\mathcal{E}_N$ are the sets of boundary facets belonging to the respective $\Gamma_D$ and $\Gamma_N$. In this paper, we use the notion facet to represent the $d-1$ topological structure of the mesh with elements in the $d$ dimensions. Note that for meshes of two (three) dimensional domains, facets are corresponding to edges (faces). For each $F \in \mathcal{E}$, denote by $h_F$ the length of $F$ and by $\mathbf{n}_F$ a unit vector normal to $F$. Let $K_F^+$ and $K_F^-$ be the two elements sharing the common facet $F \in \mathcal{E}_I$ such that the unit outward normal of $K_F^-$ coincides with $\mathbf{n}_F$. When $F \in \mathcal{E}_D \cup \mathcal{E}_N$, $\mathbf{n}_F$ is the unit outward normal to $\partial\Omega$ and denote by $K_F^-$ the element having the facet $F$. Note here that the term facet refers to the $d-1$ dimensional entity of the mesh. In 2D, a facet is equivalent to an edge and in 3D, it is equivalent to a face.

## 3 Generalized Prager-Synge inequality

For the conforming finite element approximation, the foundation of the equilibrated a posteriori error estimator is the Prager-Synge identity [35]. That is, let $u \in H_D^1(\Omega)$ be the solution of (2.1), then

$$\|A^{1/2}\nabla (u-w)\|^2 + \|A^{-1/2}\boldsymbol{\tau} + A^{1/2}\nabla u\|^2 = \|A^{-1/2}\boldsymbol{\tau} + A^{1/2}\nabla w\|^2$$

for all $w \in H_D^1(\Omega)$ and for all $\boldsymbol{\tau} \in \Sigma_f(\Omega)$, where $\Sigma_f(\Omega)$ is the so-called equilibrated flux space defined by

$$\Sigma_f(\Omega) = \Big\{\boldsymbol{\tau} \in H(\mathrm{div};\Omega) : \nabla\cdot\boldsymbol{\tau} = f \text{ in } \Omega \;\text{ and }\; \boldsymbol{\tau}\cdot\mathbf{n} = g \text{ on } \Gamma_N\Big\}.$$

Here, $H(\mathrm{div};\Omega) \subset L^2(\Omega)^d$ denotes the space of all vector-valued functions whose divergence are in $L^2(\Omega)$. The Prager-Synge identity immediately leads to

$$\|A^{1/2}\nabla (u-w)\|^2 \le \inf_{\boldsymbol{\tau}\in\Sigma_f(\Omega)} \|A^{-1/2}\boldsymbol{\tau} + A^{1/2}\nabla w\|^2. \tag{3.1}$$

Choosing $w \in H_D^1(\Omega)$ to be the conforming finite element approximation, then (3.1) implies that

$$\eta_\tau := \|A^{-1/2}\boldsymbol{\tau} + A^{1/2}\nabla w\|, \quad \forall\, \boldsymbol{\tau} \in \Sigma_f(\Omega) \tag{3.2}$$

is a reliable estimator with the reliability constant being one.

We now proceed to establish a generalization of (3.1) for piecewise $H^1(\Omega)$ functions with applications to nonconforming and discontinuous Galerkin finite element approximations. To this end, denote the broken $H^1(\Omega)$ space with respect to $\mathcal{T}$ by

$$H^1(\mathcal{T}) = \Big\{v \in L^2(\Omega) \,:\, v|_K \in H^1(K), \quad \forall\, K \in \mathcal{T}\Big\}.$$

Define $\nabla_h$ be the discrete gradient operator on $H^1(\mathcal{T})$ such that for any $v \in H^1(\mathcal{T})$

$$(\nabla_h v)|_K = \nabla(v|_K), \quad \forall K \in \mathcal{T}.$$

**Lemma 3.1.** *Let $u \in H^1_D(\Omega)$ be the solution of* (2.1)*. In both two and three dimensions, for all $w \in H^1(\mathcal{T})$, we have*

$$\|A^{1/2}\nabla_h(u-w)\|^2 \le \inf_{\boldsymbol{\tau}\in\Sigma_f(\Omega)} \|A^{-1/2}\boldsymbol{\tau} + A^{1/2}\nabla_h w\|^2 + \inf_{v\in H^1_D(\Omega)} \|A^{1/2}\nabla_h(v-w)\|^2. \tag{3.3}$$

*Proof.* Firstly, it is easy to see that

$$\|A^{1/2}\nabla_h(u-w)\|^2 = \|A^{1/2}\nabla_h w + A^{-1/2}\boldsymbol{\tau}\|^2 - \|A^{1/2}\nabla u + A^{-1/2}\boldsymbol{\tau}\|^2 - 2(\nabla_h(u-w), A\nabla u + \boldsymbol{\tau}). \tag{3.4}$$

For all $\boldsymbol{\tau} \in \Sigma_f(\Omega)$ and for all $v \in H^1_D(\Omega)$, it follows from integration by parts and the Cauchy-Schwarz and Young's inequalities that

$$\begin{aligned} 2\,(\nabla_h(u-w), A\nabla u + \boldsymbol{\tau}) &= 2\,(\nabla(u-v), A\nabla u + \boldsymbol{\tau}) + 2\,(\nabla_h(v-w), A\nabla u + \boldsymbol{\tau}) \\ &= 2\,(\nabla_h(v-w), A\nabla u + \boldsymbol{\tau}) \\ &\le \|A^{1/2}\nabla_h(v-w)\|^2 + \|A^{1/2}\nabla u + A^{-1/2}\boldsymbol{\tau}\|^2. \end{aligned}$$

which, together with (3.4), implies

$$\|A^{1/2}\nabla_h(u-w)\|^2 \le \|A^{1/2}\nabla_h w + A^{-1/2}\boldsymbol{\tau}\|^2 + \|A^{1/2}\nabla_h(v-w)\|^2. \tag{3.5}$$

Since the above inequality is valid for all $\boldsymbol{\tau} \in \Sigma_f(\Omega)$ and all $v \in H^1_D(\Omega)$, this implies the validity of (3.3) and, hence, the lemma. □

**Remark 3.2.** *For Poisson equation with pure Dirichlet boundary conditions, a suboptimal result is also proved earlier in [12] by Braess, Fraunholz, and Hoppe:*

$$\|\nabla_h(u-w)\| \le \inf_{\boldsymbol{\tau}\in\Sigma_f(\Omega)} \|\nabla w + \boldsymbol{\tau}\| + 2 \inf_{v\in H^1_0(\Omega)} \|\nabla_h(v-w)\|;$$

*recently, a slightly more general inequality than that of Lemma 3.1 was proved in [26] by introducing the elliptic projection of the discontinuous finite element approximation as done by Kim in [31].*

For each $F \in \mathcal{E}$, in two dimensions, assume that $\mathbf{n}_F = (n_{1,F}, n_{2,F})$, then denote by $\mathbf{t}_F = (-n_{2,F}, n_{1,F})$ the unit vector tangent to $F$ and by $\mathbf{s}_F$ and $\mathbf{e}_F$ the start and end points of $F$, respectively, such that $\mathbf{e}_F - \mathbf{s}_F = h_F \mathbf{t}_F$.

Let

$$\mathcal{H} = \begin{cases} \left\{ v \in H^1(\Omega) : \int_\Omega v\,dx = 0 \text{ and } \dfrac{\partial v}{\partial \mathbf{t}} = 0 \text{ on } \Gamma_N \right\} & \text{in } 2D, \\ \left\{ \boldsymbol{\tau} \in H^1(\Omega)^3 : \nabla\cdot\boldsymbol{\tau} = 0 \text{ and } (\nabla\times\boldsymbol{\tau})\cdot\mathbf{n} = 0 \text{ on } \Gamma_N \right\} & \text{in } 3D, \end{cases}$$

where $\nabla\times$ is the classical curl operator in three dimentions.

For a scalar-valued function $v \in H^1(\Omega)$, we define the formal adjoint operator of the curl in two dimensions by

$$\nabla^\perp v = \left( \frac{\partial v}{\partial y}, -\frac{\partial v}{\partial x} \right).$$

For any $w \in H^1(\mathcal{T})$, let $\phi \in H^1_D(\Omega)$ be the solution of

$$(A\nabla\phi, \nabla v) = (A\nabla_h(u-w), \nabla v), \quad \forall v \in H^1_D(\Omega). \tag{3.6}$$

we then have the classical Helmholtz decomposition [28, 5]:

$$A\nabla_h(u-w) = \begin{cases} A\nabla\phi + \nabla^\perp\psi & \text{in } 2D, \\ A\nabla\phi + \nabla\times\boldsymbol{\psi} & \text{in } 3D \end{cases} \quad \text{with } \psi \in \mathcal{H}. \tag{3.7}$$

The decomposition is orthogonal, i.e.,

$$\|A^{1/2}\nabla_h(u-w)\|^2 = \begin{cases} \|A^{1/2}\nabla\phi\|^2 + \|A^{-1/2}\nabla^\perp\psi\|^2 & \text{in } 2D, \\ \|A^{1/2}\nabla\phi\|^2 + \|A^{-1/2}\nabla\times\boldsymbol{\psi}\|^2 & \text{in } 3D. \end{cases} \tag{3.8}$$

**Lemma 3.3.** *Let $w$ be a fixed function in $H^1(\mathcal{T})$ and $\phi$ and $\psi$ in 2D ($\boldsymbol{\psi}$ in 3D) be the corresponding Helmholtz decomposition of $w$ given in* (3.7)*. We have*

$$\inf_{v\in H^1_D(\Omega)} \|A^{1/2}\nabla(v-w)\| = \|A^{-1/2}\nabla^\perp\psi\| \textit{ in } 2D \quad \textit{or} \quad \|A^{-1/2}\nabla\times\boldsymbol{\psi}\| \textit{ in } 3D, \tag{3.9}$$

*and*

$$\inf_{\boldsymbol{\tau}\in\Sigma_f(\Omega)} \|A^{-1/2}\boldsymbol{\tau} + A^{1/2}\nabla_h w\| = \|A^{1/2}\nabla\phi\| \tag{3.10}$$

*Proof.* We firstly prove (3.9) in two dimensions. Note the following orthogonality condition holds:

$$(\nabla v, \nabla^\perp w) = 0 \quad \forall v \in H^1_D(\Omega) \text{ and } \forall\, w\in\mathcal{H}.$$

Then applying (3.7) and Cauchy-Schwartz inequality gives

$$\|A^{-1/2}\nabla^\perp\psi\|^2 = (\nabla(u-w), \nabla^\perp\psi) = (\nabla(v-w), \nabla^\perp\psi) \le \inf_{v\in H^1_D(\Omega)} \|A^{1/2}\nabla(v-w)\|\|A^{-1/2}\nabla^\perp\psi\|.$$

A special choice of $v = u-\phi$ gives (3.9). Three dimensional case can be proved in the same way.

To prove (3.10), for any $\boldsymbol{\tau}\in\Sigma_f(\Omega)$, (3.6) and integration by parts give

$$\|A^{1/2}\nabla\phi\|^2 = (A\nabla_h(u-w), \nabla\phi) = (A\nabla u + \boldsymbol{\tau}, \nabla\phi) - (\boldsymbol{\tau} + A\nabla_h w, \nabla\phi) = -(\boldsymbol{\tau} + A\nabla_h w, \nabla\phi).$$

Applying Cauchy-Schwartz inequality gives that

$$\|A^{1/2}\nabla\phi\| \le \inf_{\boldsymbol{\tau}\in\Sigma_f(\Omega)} \|A^{-1/2}(\boldsymbol{\tau} + A\nabla_h w)\|.$$

Taking the special choice $\boldsymbol{\tau} = \nabla^\perp\psi - A\nabla u \in \Sigma_f(\Omega)$ in 2D and $\boldsymbol{\tau} = \nabla\times\boldsymbol{\psi} - A\nabla u \in \Sigma_f(\Omega)$ in 3D, yields the first equality in (3.10) as follows:

$$\|A^{1/2}\nabla\phi\| \le \inf_{\boldsymbol{\tau}\in\Sigma_f(\Omega)} \|A^{-1/2}\boldsymbol{\tau} + A^{1/2}\nabla_h w\| = \|A^{1/2}\nabla\phi\|.$$

This completes the proof of the lemma. □

**Theorem 3.4.** *Let $u\in H^1_D(\Omega)$ be the solution of* (2.1)*. In two and three dimensions, for all $w\in H^1(\mathcal{T})$, we have*

$$\|A^{1/2}\nabla_h(u-w)\|^2 = \inf_{\boldsymbol{\tau}\in\Sigma_f(\Omega)} \|A^{-1/2}\boldsymbol{\tau} + A^{1/2}\nabla_h w\|^2 + \inf_{v\in H^1_D(\Omega)} \|A^{1/2}\nabla_h(v-w)\|^2. \tag{3.11}$$

*Proof.* The identity (3.11) is a direct consequence of (3.8) and Lemma 3.3. $\square$

Let $H(\mathrm{curl};\Omega) \subset L^2(\Omega)^d$ ($d = 2, 3$) be the space of all vector-valued functions whose curl are in $L^2(\Omega)$, and denote its curl free subspace by

$$\mathring{H}_D(\mathrm{curl};\Omega) = \{\boldsymbol{\tau} \in H(\mathrm{curl};\Omega) : \nabla\times\boldsymbol{\tau} = 0 \text{ in } \Omega \text{ and } \boldsymbol{\tau} \times \mathbf{n} = \mathbf{0} \text{ on } \Gamma_D\}\,.$$

**Corollary 3.5.** *Let $u \in H^1_D(\Omega)$ be the solution of (2.1). In both two and three dimensions, for all $w \in H^1(\mathcal{T})$, we have*

$$\|A^{1/2}\nabla_h(u-w)\|^2 = \inf_{\boldsymbol{\tau}\in\Sigma_f(\Omega)} \|A^{-1/2}\boldsymbol{\tau} + A^{1/2}\nabla_h w\|^2 + \inf_{\boldsymbol{\gamma}\in\mathring{H}_D(\mathrm{curl};\Omega)} \|A^{1/2}(\boldsymbol{\gamma} - \nabla_h w)\|^2. \quad (3.12)$$

*Proof.* The result of (3.12) is an immediate consequence of Theorem 3.4 and the fact that $\nabla H^1_D(\Omega) = \mathring{H}_D(\mathrm{curl};\Omega)$. $\square$

**Remark 3.6.** *It is easy to see that if $w \in H^1_D(\Omega)$ in* Lemma 3.4, i.e., $w$ is conforming, the second part on the right of (3.11) vanishes. It is thus natural to refer $\inf_{\boldsymbol{\tau}\in\Sigma_f(\Omega)} \|A^{-1/2}\boldsymbol{\tau} + A^{1/2}\nabla_h w\|^2$ as the conforming error and $\inf_{v\in H^1_D(\Omega)} \|A^{1/2}\nabla_h(v-w)\|^2$ as the nonconforming error.

For each $K \in \mathcal{T}$, denote by $\Lambda_K$ and $\lambda_K$ the maximal and minimal eigenvalues of $A_K = A|_K$, respectively. For each $F \in \mathcal{E}$, let $\Lambda_F^{\pm} = \Lambda_{K_F^{\pm}}$, $\lambda_F^{\pm} = \lambda_{K_F^{\pm}}$, and $\lambda_F = \min\{\lambda_F^+, \lambda_F^-\}$ if $F \in \mathcal{E}_I$ and $\lambda_F = \lambda_F^-$ if $F \in \mathcal{E}_D \cup \mathcal{E}_N$. To this end, let

$$\Lambda_{\mathcal{T}} = \max_{K\in\mathcal{T}} \Lambda_K \quad \text{and} \quad \lambda_{\mathcal{T}} = \min_{K\in\mathcal{T}} \lambda_K.$$

Assume that each local matrix $A_K$ is similar to the identity matrix in the sense that its maximal and minimal eigenvalues are almost of the same size. More precisely, there exists a moderate size constant $\kappa > 0$ such that

$$\frac{\Lambda_K}{\lambda_K} \le \kappa, \quad \forall\, K \in \mathcal{T}.$$

Nevertheless, the ratio of global maximal and minimal eigenvalues, $\Lambda_{\mathcal{T}}/\lambda_{\mathcal{T}}$, is allowed to be very large.

For a function $w \in H^1(\mathcal{T})$ , denote its traces on $F$ by $w|_F^- := (w|_{K_F^-})|_F$ and $w|_F^+ := (w|_{K_F^+})|_F$ and the jump of $w$ across the facet $F$ by

$$[\![w]\!]|_F = \begin{cases} w|_F^- - w|_F^+, & \forall\, F \in \mathcal{E}_I, \\ w|_F^-, & \forall\, F \in \mathcal{E}_D \cup \mathcal{E}_N. \end{cases}$$

For future conveniences, in the following lemma we show the relationship between the nonconforming error and the residual based error of solution jump on facets. It is noted that the constant is robust with respect to the diffusion coefficient jump.

**Lemma 3.7.** *Let $w$ be a fixed function in $H^1(\mathcal{T})$. In two and three dimensions, there exists a constant $C_r$ that is independent of the jump of the coefficient such that*

$$\inf_{v\in H^1_D(\Omega)} \|A^{1/2}\nabla_h(v-w)\| \le C_r \left( \sum_{F\in\mathcal{E}_I\cup\mathcal{E}_D} \lambda_F h_F^{-1} \|[\![w]\!]\|^2_{0,F} \right)^{1/2}. \quad (3.13)$$

*Proof.* We firstly prove (3.13) in two dimensions. Let $\psi$ be given in the Helmholtz decomposition in (3.7). From (3.12) we have

$$\inf_{v\in H_D^1(\Omega)} \|A^{1/2}\nabla_h(v-w)\| = \|A^{-1/2}\nabla^\perp\psi\|.$$

Now applying the fact that $(\nabla\phi, \nabla^\perp\psi) = 0$ and integration by parts gives

$$\|A^{-1/2}\nabla^\perp\psi\|^2 = (\nabla_h(u-w), \nabla^\perp\psi) = -\sum_{F\in\mathcal{E}_I\cup\mathcal{E}_D} \int_F [\![w]\!] \left(\nabla^\perp\psi\cdot\mathbf{n}_F\right) ds. \tag{3.14}$$

Without loss of generality, assume that $\lambda_F^- \le \lambda_F^+$ for each $F\in\mathcal{E}_I$. It follows from Lemma 2.4 in [17] and the Cauchy-Schwarz inequality that

$$\begin{aligned}\sum_{F\in\mathcal{E}_I\cup\mathcal{E}_D} \int_F [\![w]\!] \left(\nabla^\perp\psi\cdot\mathbf{n}_F\right) ds &\le C \sum_{F\in\mathcal{E}_I\cup\mathcal{E}_D} h_F^{-1/2}\|[\![w]\!]\|_{0,F}\|\nabla^\perp\psi\|_{0,K_F^-}\\ &\le C\left(\sum_{F\in\mathcal{E}_I\cup\mathcal{E}_D}\lambda_F h_F^{-1}\|[\![w]\!]\|_{0,F}^2\right)^{1/2}\|A^{-1/2}\nabla^\perp\psi\|,\end{aligned}$$

which, together with (3.14), yields

$$\|A^{-1/2}\nabla^\perp\psi\| \le C\left(\sum_{F\in\mathcal{E}_I\cup\mathcal{E}_D}\lambda_F h_F^{-1}\|[\![w]\!]\|_{0,F}^2\right)^{1/2}.$$

In three dimensions, (3.13) can be proved similarly. □

# 4 Error estimators and indicators

## 4.1 NC finite element approximation

For the convenience of readers, in this subsection we introduce the nonconforming finite element space in two dimensions and its properties. For clarity, we refer the facet as edge in this subsection.

Let $\mathbb{P}_k(K)$ and $\mathbb{P}_k(F)$ be the spaces of polynomials of degree less than or equal to $k$ on the element $K$ and edge $F$, respectively. Define the nonconforming finite element space of order $k(k\ge 1)$ on the triangulation $\mathcal{T}$ by

$$\mathcal{U}^k(\mathcal{T}) = \left\{v\in L^2(\Omega) \colon v|_K\in\mathbb{P}_k(K), \forall\, K\in\mathcal{T} \text{ and } \int_F [\![v]\!]\, p\, ds = 0, \forall\, p\in\mathbb{P}_{k-1}(F),\ \forall\, F\in\mathcal{E}_I\right\} \tag{4.1}$$

and its subspace by

$$\mathcal{U}_D^k(\mathcal{T}) = \left\{v\in\mathcal{U}^k(\mathcal{T})\colon \int_F v\, p\, ds = 0, \quad \forall\, p\in\mathbb{P}_{k-1}(F),\ \forall\, F\in\mathcal{E}_D\right\}.$$

The spaces defined above are exactly the same as those defined in [21] for $k=1$, [27] for $k=2$, [20] for $k=4$ and 6, [4] for general odd order, and [36, 6] for general order. Then the nonconforming finite element approximation of order $k$ is to find $u_\mathcal{T}\in\mathcal{U}_D^k(\mathcal{T})$ such that

$$a_h(u_\mathcal{T}, v) := (A\nabla_h u_\mathcal{T}, \nabla_h v) = (f, v) - \langle g, v\rangle_{\Gamma_N}, \quad \forall\, v\in\mathcal{U}_D^k(\mathcal{T}). \tag{4.2}$$

Below we describe basis functions of $\mathcal{U}^k(\mathcal{T})$ and their properties. To this end, for each $K \in \mathcal{T}$, let $m_k = \dim(\mathbb{P}_{k-3}(K))$ for $k > 3$ and $m_k = 0$ for $k \le 3$. Denote by $\{\mathbf{x}_j, j = 1, \cdots, m_k\}$ the set of all interior Lagrange points in $K$ with respect to the space $\mathbb{P}_k(K)$ and by $P_{j,K} \in \mathbb{P}_{k-3}(K)$ the nodal basis function corresponding to $\mathbf{x}_j$, i.e.,

$$P_{j,K}(\mathbf{x}_i) = \delta_{ij} \quad \text{for } i = 1, \cdots, m_k,$$

where $\delta_{ij}$ is the Kronecker delta function. For each $0 \le j \le k-1$, let $L_{j,F}$ be the $j$th order Gauss-Legendre polynomial on $F$ such that $L_{j,F}(\mathbf{e}_F) = 1$. Note that $L_{j,F}$ is an odd or even function when $j$ is odd or even. Hence, $L_{j,F}(\mathbf{s}_F) = -1$ for odd $j$ and $L_{j,F}(\mathbf{s}_F) = 1$ for even $j$.

For odd $k$, the set of degrees of freedom of $\mathcal{U}^k(\mathcal{T})$ (see Lemma 2.1 in [4]) can be given by

$$\int_K v\, P_{j,K}\, dx, \quad j = 1, \cdots, m_k \tag{4.3}$$

for all $K \in \mathcal{T}$ and

$$\int_F v\, L_{j,F}\, ds, \quad j = 0, \cdots, k-1 \tag{4.4}$$

for all $F \in \mathcal{E}$. Define the basis function $\phi_{i,K} \in \mathcal{U}^k(\mathcal{T})$ satisfying

$$\begin{cases} \int_{K'} \phi_{i,K}\, P_{j,K'}\, dx = \delta_{ij}\delta_{KK'}, & \forall\, j = 1, \cdots, m_k, \quad \forall\, K' \in \mathcal{T}, \\ \int_F \phi_{i,K}\, L_{j,F}\, ds = 0, & \forall\, j = 0, \cdots, k-1, \quad \forall\, F \in \mathcal{E}, \end{cases} \tag{4.5}$$

for $i = 1, \cdots, m_k$ and $K \in \mathcal{T}$, and the basis function $\phi_{i,F} \in \mathcal{U}^k(\mathcal{T})$ satisfying

$$\begin{cases} \int_K \phi_{i,F}\, P_{j,K}\, dx = 0, & \forall\, j = 1, \cdots, m_k, \quad \forall\, K \in \mathcal{T}, \\ \int_{F'} \phi_{i,F}\, L_{j,F'}\, ds = \delta_{ij}\delta_{FF'}, & \forall\, j = 0, \cdots, k-1, \quad \forall\, F' \in \mathcal{E}, \end{cases} \tag{4.6}$$

for $i = 0, \cdots, k-1$ and $F \in \mathcal{E}$. Then the nonconforming finite element space is the space spanned by all these basis functions, i.e.,

$$\mathcal{U}^k(\mathcal{T}) = \text{span}\,\{\phi_{i,K} :\ K \in \mathcal{T}\}_{i=1}^{m_k} \oplus \text{span}\,\{\phi_{i,F} :\ F \in \mathcal{E}\}_{i=0}^{k-1}.$$

**Lemma 4.1.** *For all $K \in \mathcal{T}$, the basis functions $\{\phi_{j,K}\}_{j=1}^{m_k}$ have support on $K$ and vanish on the boundary of $K$, i.e.,*

$$\phi_{j,K} \equiv 0 \quad \text{on } \partial K.$$

*Proof.* Obviously, (4.5) implies that support$\{\phi_{j,K}\} \in \overline{K}$. To show that $\phi_{j,K}|_{\partial K} \equiv 0$, considering each edge $F \in \mathcal{E}_K$, the second equation of (4.5) indicates that there exists $a_F \in R$ such that

$$\phi_{j,K}|_F = a_F L_{k,F}.$$

Note that $L_{k,F}$ is an odd function on $F$ and that values of $L_{k,F}$ at two end-points of $F$ are $-1$ and 1, respectively. Now the continuity of $\phi_{j,K}$ in $K$ implies that $a_F = 0$ and, hence, $\phi_{j,K} \equiv 0$ on $\partial K$. □

For each $K$, denote by $\mathcal{E}_K$ the set of all edges of $K$. For each $F \in \mathcal{E}$, denote by $\omega_F$ the union of all elements that share the common edge $F$; and define a sign function $\chi_F$ on the set $\mathcal{E}_{K_F^+} \cup \mathcal{E}_{K_F^-} \setminus \{F\}$ (when $F$ is a boundary edge, let $\mathcal{E}_{K_F^+} = \emptyset$) such that

$$\chi_F(F') = \begin{cases} 1, & \text{if } \ \mathbf{e}_{F'} = \bar{F} \cap \bar{F}', \\ -1, & \text{if } \ \mathbf{s}_{F'} = \bar{F} \cap \bar{F}'. \end{cases}$$

**Lemma 4.2.** *For all $F \in \mathcal{E}$, the basis functions $\{\phi_{j,F}\}_{j=0}^{k-1}$ have support on $\overline{\omega}_F$, and their restrictions on $\mathcal{E}_{K_F^+} \cup \mathcal{E}_{K_F^-}$ has the following representation:*

$$\phi_{j,F} = \begin{cases} \dfrac{1}{\|L_{j,F}\|_{0,F}^2} \left(L_{j,F} - L_{k,F}\right), & on \quad F, \\ 0, & on \quad \mathcal{E}_{K_F^+} \cup \mathcal{E}_{K_F^-} \setminus \{F\} \end{cases} \tag{4.7}$$

*when $j$ is odd, and*

$$\phi_{j,F} = \begin{cases} \dfrac{1}{\|L_{j,F}\|_{0,F}^2} L_{j,F}, & on \quad F, \\ \dfrac{\chi_F(F')}{\|L_{j,F}\|_{0,F}^2} L_{k,F'}, & on \quad F' \in \mathcal{E}_{K_F^+} \cup \mathcal{E}_{K_F^-} \setminus \{F\} \end{cases} \tag{4.8}$$

*when $j$ is even.*

*Proof.* By (4.6), it is easy to see that support of $\phi_{j,F}$ is $\overline{\omega}_F$. Since $\phi_{j,F}|_F^{\pm} \in \mathbb{P}_k(F)$, there exist constants $a_{i,F}^{\pm}$ such that

$$\phi_{j,F}|_F^{\pm} = \sum_{i=0}^{k} a_{i,F}^{\pm} L_{i,F}.$$

Using (4.6) and the orthogonality of $\{L_{i,F}\}_{i=0}^{k}$, it is obvious that

$$a_{i,F}^{\pm} = \begin{cases} \|L_{j,F}\|_{0,F}^{-2}, & \text{for } i = j, \\ 0, & \text{for } 0 \le i \le k-1 \text{ and } i \ne j \end{cases}$$

and, hence,

$$\phi_{j,F}|_F^{\pm} = \frac{1}{\|L_{j,F}\|_{0,F}^2} L_{j,F} + a_{k,F}^{\pm} L_{k,F}. \tag{4.9}$$

By (4.6), it is also easy to see that there exists constant $a_{j,F,F'}$ for each $F' \in \mathcal{E}_{K_F^+} \cup \mathcal{E}_{K_F^-} \setminus \{F\}$ such that

$$\phi_{j,F}|_{F'} = a_{j,F,F'} L_{k,F'}. \tag{4.10}$$

Since $L_{k,F'}$ is an odd function for all $F' \in \mathcal{E}_{K_F^+} \cup \mathcal{E}_{K_F^-} \setminus \{F\}$ and $\phi_{j,F}$ is continuous in $K_F^+$ and $K_F^-$, (4.10) implies that

$$\phi_{j,F}|_K(\mathbf{s}_F) = \phi_{j,F}|_K(\mathbf{e}_F), \quad K \in \{K_F^+, K_F^-\}. \tag{4.11}$$

Combining the facts that $L_{j,F}(\mathbf{e}_F) = -L_{j,F}(\mathbf{s}_F) = 1$ for odd $j$ and that $L_{j,F}(\mathbf{e}_F) = L_{j,F}(\mathbf{s}_F) = 1$ for even $j$, (4.9), and (4.11), we have

$$a_{k,F}^{\pm} = \begin{cases} -\dfrac{1}{\|L_{j,F}\|_{0,F}^2}, & \text{for odd } j, \\ 0, & \text{for even } j, \end{cases}$$

which, together with (4.9), leads to the formulas of $\phi_{j,F}|_F$ in (4.7) and (4.8). Finally, for each $F' \in \mathcal{E}_{K_F^+} \cup \mathcal{E}_{K_F^-} \setminus \{F\}$, $a_{j,F,F'}$ in (4.10) can be directly computed based on the continuity of $\phi_{j,F}$ in $K_F^+$ and $K_F^-$. This completes the proof of the lemma. ∎

**Remark 4.3.** *As a consequence of* Lemma 4.2*, the basis function* $\phi_{j,F}$ *is continuous on the edge* $F$*, i.e.,* $[\![\phi_{j,F}]\!]|_F = 0$ *for all* $j = 0, \cdots, k-1$*; moreover,* $\phi_{j,F}$ *vanishes at end points of* $F$*, i.e.,* $\phi_{j,F}(\mathbf{s}_F) = \phi_{j,F}(\mathbf{e}_F) = 0$*, for odd* $j$*.*

**Lemma 4.4.** *Let* $F$ *be an edge of* $K$*. Assume that* $p \in \mathbb{P}_{k-1}(K)$*. Then we have that*

$$\int_{\partial K} p\,\phi_{j,F}\,ds = \int_F p\,\phi_{j,F}\,ds. \tag{4.12}$$

*Moreover, if* $\int_F p\,\phi_{j,F}\,ds = 0$ *for all* $j = 0, \cdots, k-1$*, then* $p \equiv 0$ *on* $F$*.*

*Proof.* Since $\{L_{j,F}\}_{j=0}^k$ are orthogonal polynomials on $F$, (4.12) is a direct consequence of Lemma 4.2. □

## 4.2 Equilibrated flux recovery

In this subsection, we introduce a fully explicit post-processing procedure for recovering an equilibrated flux. To this end, define $f_{k-1} \in L^2(\Omega)$ by

$$f_{k-1}|_K = \Pi_K(f), \quad \forall\, K \in \mathcal{T},$$

where $\Pi_K$ is the $L^2$ projection onto $\mathbb{P}_{k-1}(K)$. For simplicity, assume that the Neumann data $g$ is a piecewise polynomial of degree less than or equal to $k-1$, i.e., $g|_F \in \mathbb{P}_{k-1}(F)$ for all $F \in \mathcal{E}_N$.

Denote the $H(\text{div};\Omega)$ conforming Raviart-Thomas (RT) space of index $k-1$ with respect to $\mathcal{T}$ by

$$RT^{k-1}(\mathcal{T}) = \left\{\boldsymbol{\tau} \in H(\text{div};\Omega) \,:\, \boldsymbol{\tau}|_K \in RT^{k-1}(K),\ \forall\, K \in \mathcal{T}\right\},$$

where $RT^{k-1}(K) = \mathbb{P}_{k-1}(K)^d + \mathbf{x}\,\mathbb{P}_{k-1}(K)$. Let

$$\Sigma_f^{k-1}(\mathcal{T}) = \left\{\boldsymbol{\tau} \in RT^{k-1} : \nabla\cdot\boldsymbol{\tau} = f_{k-1} \text{ in } \Omega \quad \text{and} \quad \boldsymbol{\tau}\cdot\mathbf{n}_F = g \text{ on } \Gamma_N\right\}.$$

On a triangular element $K \in \mathcal{T}$, a vector-valued function $\boldsymbol{\tau}$ in $RT^{k-1}(K)$ is characterized by the following degrees of freedom (see Proposition 2.3.4 in [10]):

$$\int_K \boldsymbol{\tau}\cdot\boldsymbol{\zeta}\,dx, \quad \forall\,\boldsymbol{\zeta} \in \mathbb{P}_{k-2}(K)^d,$$

and

$$\int_F (\boldsymbol{\tau}\cdot\mathbf{n}_F)\,p\,ds, \quad \forall\, p \in \mathbb{P}_{k-1}(F) \text{ and } \forall\, F \in \mathcal{E}_K.$$

For each $K \in \mathcal{T}$, define a sign function $\mu_K$ on $\mathcal{E}_K$ such that

$$\mu_K(F) = \begin{cases} 1, & \text{if} \quad \mathbf{n}_K|_F = \mathbf{n}_F, \\ -1, & \text{if} \quad \mathbf{n}_K|_F = -\mathbf{n}_F. \end{cases} \tag{4.13}$$

Define the numerical flux

$$\tilde{\boldsymbol{\sigma}}_{\mathcal{T}} = -A\nabla_h u_{\mathcal{T}} \quad \text{and} \quad \tilde{\boldsymbol{\sigma}}_K = -A\nabla(u_{\mathcal{T}}|_K), \quad \forall\, K \in \mathcal{T}. \tag{4.14}$$

With the numerical flux $\tilde{\boldsymbol{\sigma}}_\mathcal{T}$ given in (4.14), for each element $K \in \mathcal{T}$, we recover a flux $\hat{\boldsymbol{\sigma}}_K \in RT^{k-1}(K)$ such that:

$$\int_K \hat{\boldsymbol{\sigma}}_K \cdot \boldsymbol{\tau}\, dx = \int_K \tilde{\boldsymbol{\sigma}}_\mathcal{T} \cdot \boldsymbol{\tau}\, dx, \quad \forall\, \boldsymbol{\tau} \in \mathbb{P}_{k-2}(K)^d \tag{4.15}$$

and that

$$\int_F \hat{\boldsymbol{\sigma}}_K \cdot \mathbf{n}_F\, L_{i,F}\, ds = \begin{cases} \mu_K(F)\|L_{i,F}\|^2_{0,F}\left(\int_K \tilde{\boldsymbol{\sigma}}_\mathcal{T} \cdot \nabla\phi_{i,F}\, dx + \int_K f\, \phi_{i,F}\, dx\right), & \forall\, F \in \mathcal{E}_K \setminus \mathcal{E}_N, \\ \mu_K(F)\|L_{i,F}\|^2_{0,F}\left(\int_F g\, \phi_{i,F}\, ds\right), & \forall\, F \in \mathcal{E}_K \cap \mathcal{E}_N \end{cases} \tag{4.16}$$

for $i = 0, \cdots, k-1$. Now the global recovered flux $\hat{\boldsymbol{\sigma}}_\mathcal{T}$ is defined by

$$\hat{\boldsymbol{\sigma}}_\mathcal{T}\big|_K = \hat{\boldsymbol{\sigma}}_K, \quad \forall\; K \in \mathcal{T}. \tag{4.17}$$

**Remark 4.5.** *We emphasize that the above flux recovery procedure is fully explicit. To our knowledge, the existing methods for recovery equilibrate flux for higher order nonconforming elements are implicit and requires to solve local problems, see e.g.*[1, 26, 9]. Our recovered flux appears to be the same as the one in [9] for odd order nonconforming elements.Due to the fundamental differences between the odd and even order nonconforming elements, we are currently not able to extend the explicit approach to the even orders.

**Lemma 4.6.** *Let $u_\mathcal{T}$ be the finite element solution in* (4.2) and $\hat{\boldsymbol{\sigma}}_\mathcal{T}$ be the recovered flux defined in (4.17). Then for any $K \in \mathcal{T}$, the following equality

$$\int_{\partial K} \hat{\boldsymbol{\sigma}}_\mathcal{T} \cdot \mathbf{n}_K\, q\, dx = \int_K \tilde{\boldsymbol{\sigma}}_\mathcal{T} \cdot \nabla q\, dx + \int_K f\, q\, dx \tag{4.18}$$

holds for all $q \in \mathbb{P}_k(K)$.

*Proof.* Without loss of generality, assume that $K \in \mathcal{T}$ is an interior element. For each $q \in \mathbb{P}_k(K)$, there exist $a_{j,F}$ and $a_{j,K}$ such that

$$q = \sum_{F\in\mathcal{E}_K} \sum_{j=0}^{k-1} a_{j,F}\, \phi_{j,F} + \sum_{j=1}^{m_k} a_{j,K}\, \phi_{j,K} \equiv \sum_{F\in\mathcal{E}_K} q_F + q_K.$$

It follows from Lemma 4.1, (4.12), Lemma 4.2, and the definition of the recovered flux $\hat{\boldsymbol{\sigma}}_\mathcal{T}$ in (4.16) that

$$\begin{aligned} \int_{\partial K} \hat{\boldsymbol{\sigma}}_K \cdot \mathbf{n}_K\, q\, ds &= \sum_{F\in\mathcal{E}_K} \sum_{j=0}^{k-1} a_{j,F} \int_F \hat{\boldsymbol{\sigma}}_K \cdot \mathbf{n}_K\, \phi_{j,F}\, ds \\ = \quad & \sum_{F\in\mathcal{E}_K} \sum_{j=0}^{k-1} \frac{a_{j,F}\, \mu_K(F)}{\|L_{j,F}\|^2_F} \int_F \hat{\boldsymbol{\sigma}}_K \cdot \mathbf{n}_F\, L_{j,F}\, ds = \sum_{F\in\mathcal{E}_K} \sum_{j=0}^{k-1} a_{j,F} \left(\int_K \tilde{\boldsymbol{\sigma}}_\mathcal{T} \cdot \nabla\phi_{j,F}\, dx + \int_K f\, \phi_{j,F}\, dx\right) \\ = \quad & \sum_{F\in\mathcal{E}_K} \left(\int_K \tilde{\boldsymbol{\sigma}}_\mathcal{T} \cdot \nabla q_F\, dx + \int_K f\, q_F\, dx\right). \end{aligned} \tag{4.19}$$

Choosing $v = \phi_{j,K}$ in (4.2) gives

$$\int_K \tilde{\boldsymbol{\sigma}}_{\mathcal{T}} \cdot \nabla \phi_{j,K}\, dx + \int_K f\, \phi_{j,K}\, dx = 0$$

for $j = 1, \cdots, m_k$. Multiplying the above equality by $a_{j,K}$ and summing over $j$ imply

$$\int_K \tilde{\boldsymbol{\sigma}}_{\mathcal{T}} \cdot \nabla q_K\, dx + \int_K f\, q_K\, dx = 0. \tag{4.20}$$

Now (4.18) is the summation of (4.19) and (4.20). This completes the proof of the lemma. □

**Theorem 4.7.** *Let $u_{\mathcal{T}}$ be the finite element solution in* (4.2)*. Then the recovered flux $\hat{\boldsymbol{\sigma}}_{\mathcal{T}}$ defined in* (4.17) *belongs to $\Sigma_f^{k-1}(\mathcal{T})$.*

*Proof.* First we prove that $\hat{\boldsymbol{\sigma}}_{\mathcal{T}} \in H(\mathrm{div};\Omega)$. For each $F \in \mathcal{E}_I$, note that $\hat{\boldsymbol{\sigma}}_{\mathcal{T}}|_F^{\pm} \in \mathbb{P}_{k-1}(F)$. Then it follows from Lemma 4.2, (4.16), the assumption that $g|_F \in \mathbb{P}_{k-1}(F)$, and (4.2) with $v = \phi_{j,F}$ that

$$\begin{aligned}
\int_F [\![\hat{\boldsymbol{\sigma}} \cdot \mathbf{n}_F]\!]\, \phi_{j,F}\, ds &= \sum_{K \in \{K_F^+, K_F^-\}} \frac{\mu_K(F)}{\|L_{k,F}\|_F^2} \int_F \hat{\boldsymbol{\sigma}}_K \cdot \mathbf{n}_F\, L_{j,F}\, ds \\
&= \sum_{K \in \{K_F^+, K_F^-\}} \left( \int_K \tilde{\boldsymbol{\sigma}}_{\mathcal{T}} \cdot \nabla \phi_{j,F}\, ds + \int_K f\, \phi_{j,F}\, ds \right) \\
&= \int_{\omega_F} \tilde{\boldsymbol{\sigma}}_{\mathcal{T}} \cdot \nabla \phi_{j,F}\, ds + \int_{\omega_F} f\, \phi_{j,F}\, ds - \int_{\Gamma_N \cap \partial\omega_F} g\, \phi_{j,F}\, ds \\
&= 0
\end{aligned}$$

for $j = 0, \cdots, k-1$. Now Lemma 4.4 implies that $[\![\hat{\boldsymbol{\sigma}}_{\mathcal{T}} \cdot \mathbf{n}_F]\!]|_F = 0$ and, hence, $\hat{\boldsymbol{\sigma}}_{\mathcal{T}} \in H(\mathrm{div}, \Omega)$. Second, for each $K \in \mathcal{T}$ and for any $p \in \mathbb{P}_{k-1}(K)$, note that $\nabla p \in \mathbb{P}_{k-2}(K)^d$. By integration by parts, (4.15), and Lemma 4.6, we have

$$\begin{aligned}
\int_K \nabla \cdot \hat{\boldsymbol{\sigma}}_K\, p\, dx &= -\int_K \hat{\boldsymbol{\sigma}}_K \cdot \nabla p\, dx + \int_{\partial K} \hat{\boldsymbol{\sigma}}_K \cdot \mathbf{n}_K\, p\, ds \\
&= -\int_K \tilde{\boldsymbol{\sigma}}_{\mathcal{T}} \cdot \nabla p\, dx + \left( \int_K \tilde{\boldsymbol{\sigma}}_{\mathcal{T}} \cdot \nabla p\, dx + \int_K f\, p\, dx \right) = \int_K f\, p\, dx,
\end{aligned}$$

which implies that $\nabla \cdot \hat{\boldsymbol{\sigma}}_{\mathcal{T}} = f_{k-1}$ in $\Omega$.

Finally, for $F \in \mathcal{E}_N$, Lemma 4.4 and (4.16) gives

$$\int_F \hat{\boldsymbol{\sigma}}_{\mathcal{T}} \cdot \mathbf{n}_F \phi_{j,F}\, ds = \|L_{j,F}\|_{0,F}^{-2} \int_F \hat{\boldsymbol{\sigma}}_{\mathcal{T}} \cdot \mathbf{n}_F L_{j,F}\, ds = \int_F g\, \phi_{j,F}\, ds,$$

for $j = 0, \cdots, k-1$, which, together with Lemma 4.4, implies that $\hat{\boldsymbol{\sigma}}_{\mathcal{T}} \cdot \mathbf{n}_F = g|_F$ for all $F \in \mathcal{E}_N$. This completes the proof of the theorem. □

### 4.3 Gradient recovery

In this subsection, we demonstrate the gradient recovery procedure in the space of $H(\mathrm{curl};\Omega)$ for the nonconforming finite element solutions of odd orders in the two dimensions. We note that such recovery is fully explicit through a simple weighted average on each edge. The recovery technique can be easily extended to other discontinuous finite element solutions and to three dimensional problems with the similar averaging technique on facets. For the first order nonconforming Crouzeix-Raviart element, the weighted average approach is first introduced in [18]. Define

$$H_D(\mathrm{curl};\Omega) = \{\boldsymbol{\tau} \in H(\mathrm{curl};\Omega) : \boldsymbol{\tau}\cdot\mathbf{t} = 0 \text{ on } \Gamma_N.\}$$

To this end, denote the $H_D(\mathrm{curl};\Omega)$ conforming Nédélec (NE) space of index $k-1$ with respect to $\mathcal{T}$ by

$$NE^{k-1}(\mathcal{T}) = \left\{\boldsymbol{\tau} \in H_D(\mathrm{curl};\Omega) \,:\, \boldsymbol{\tau}|_K \in NE^{k-1}(K),\ \forall\, K \in \mathcal{T}\right\},$$

where $NE^{k-1}(K) = \mathbb{P}_{k-1}(K)^2 + (-y,x)\,\mathbb{P}_{k-1}(K)$. On a triangular element $K \in \mathcal{T}$, a vector valued function $\boldsymbol{\tau} \in NE^{k-1}(K)$ is characterized by the following degrees of freedom (see Proposition 2.3.1 in [10]):

$$\int_K \boldsymbol{\tau}\cdot\boldsymbol{\zeta}\,dx, \quad \forall\,\boldsymbol{\zeta} \in \mathbb{P}_{k-2}(K)^2 \quad \text{and} \quad \int_F (\boldsymbol{\tau}\cdot\mathbf{t})\,p\,dx, \quad \forall\, p \in \mathbb{P}_{k-1}(F) \text{ and } \forall F \in \mathcal{E}_K.$$

Define the numerical gradient

$$\tilde{\boldsymbol{\rho}}_{\mathcal{T}} = \nabla_h u_{\mathcal{T}} \quad \text{and} \quad \tilde{\boldsymbol{\rho}}_K = \nabla u_{\mathcal{T}}|_K, \quad \forall\, K \in \mathcal{T}. \tag{4.21}$$

For each edge $F \in \mathcal{E}$, denote the $i$-th moment of a weighted average of the tangential components of the numerical gradient by

$$S_{i,F} = \begin{cases} \theta_F \displaystyle\int_F \left(\tilde{\boldsymbol{\rho}}_{K_F^-}\cdot\mathbf{t}_F\right) L_{i,F}\,ds + (1-\theta_F)\int_F \left(\tilde{\boldsymbol{\rho}}_{K_F^+}\cdot\mathbf{t}_F\right) L_{i,F}\,ds, & \text{if} \quad F \in \mathcal{E}_I, \\ 0, & \text{if} \quad F \in \mathcal{E}_D, \\ \displaystyle\int_F \left(\tilde{\boldsymbol{\rho}}_{K_F^-}\cdot\mathbf{t}_F\right) L_{i,F}\,ds, & \text{if} \quad F \in \mathcal{E}_N \end{cases}$$

with the weight $\theta_F = \dfrac{\Lambda_F^-}{\Lambda_F^- + \Lambda_F^+}$ for $i = 0, \cdots, k-1$. For each $K \in \mathcal{T}$, define $\hat{\boldsymbol{\rho}}_K \in NE^{k-1}(K)$ by

$$\begin{cases} \displaystyle\int_F (\hat{\boldsymbol{\rho}}_K\cdot\mathbf{t}_F) L_{i,F}\,ds = S_{i,F}, & \text{for } i = 0,\cdots,k-1 \text{ and } \forall\ F \in \mathcal{E}_K, \\ \displaystyle\int_K \hat{\boldsymbol{\rho}}_K\cdot\boldsymbol{\zeta}\,dx = \int_K \tilde{\boldsymbol{\rho}}_K\cdot\boldsymbol{\zeta}\,dx, & \forall\,\boldsymbol{\zeta} \in \mathbb{P}_{k-2}(K)^2. \end{cases} \tag{4.22}$$

Then the recovered gradient $\hat{\boldsymbol{\rho}}_{\mathcal{T}}$ is defined in $NE^{k-1}(\mathcal{T})$ such that

$$\hat{\boldsymbol{\rho}}_{\mathcal{T}}\big|_K = \hat{\boldsymbol{\rho}}_K, \quad \forall\ K \in \mathcal{T}. \tag{4.23}$$

### 4.4 Equilibrated a posteriori error estimator for nonconforming solutions

In section 4.2, we introduce an equilibrated flux recovery for the nonconforming elements of odd order. Let $\hat{\boldsymbol{\sigma}}_{\mathcal{T}} \in \Sigma_f(\Omega)$ be the recovered flux defined in (4.17), we define the local indicator and the global estimator for the conforming error by

$$\eta_{\sigma,K} = \|A^{-1/2}(\hat{\boldsymbol{\sigma}}_{\mathcal{T}} - \tilde{\boldsymbol{\sigma}}_{\mathcal{T}})\|_{0,K}, \quad \forall K \in \mathcal{T} \tag{4.24}$$

and

$$\eta_{\sigma} = \left(\sum_{K\in\mathcal{T}} \eta_{\sigma,K}^2\right)^{1/2} = \|A^{-1/2}(\hat{\boldsymbol{\sigma}}_{\mathcal{T}} - \tilde{\boldsymbol{\sigma}}_{\mathcal{T}})\|, \tag{4.25}$$

respectively.

In section 4.3, we recover the gradient in $H_D(\mathrm{curl};\Omega)$ through averaging on each edge. Let $\hat{\boldsymbol{\rho}}_{\mathcal{T}} \in H_D(\mathrm{curl};\Omega)$ be the recovered gradient defined in (4.23), then the local indicator and the global estimator for the nonconforming error are defined by

$$\eta_{\rho,K} = \|A^{1/2}(\hat{\boldsymbol{\rho}}_{\mathcal{T}} - \tilde{\boldsymbol{\rho}}_{\mathcal{T}})\|_{0,K}, \quad \forall K \in \mathcal{T} \tag{4.26}$$

and

$$\eta_{\rho} = \left(\sum_{K\in\mathcal{T}} \eta_{\rho,K}^2\right)^{1/2} = \|A^{1/2}(\hat{\boldsymbol{\rho}}_{\mathcal{T}} - \tilde{\boldsymbol{\rho}}_{\mathcal{T}})\|, \tag{4.27}$$

respectively.

The local indicator and the global estimator for the nonconforming elements are then defined by

$$\eta_K = \left(\eta_{\sigma,K}^2 + \eta_{\rho,K}^2\right)^{1/2} \quad \text{and} \quad \eta = \left(\sum_{K\in\mathcal{T}} \eta_K^2\right)^{1/2} = \left(\eta_\sigma^2 + \eta_\rho^2\right)^{1/2}, \tag{4.28}$$

respectively.

**Remark 4.8.** *To estimate the nonconforming error, one may simply use the weighted solution jump given in* Lemma 3.8 *(see* [16] *for the residual error estimator). Comparing with the recovery estimator defined in* (4.27)*, the weighted solution jump requires location of physical interfaces; moreover, our numerical results show that the recovered estimator is more accurate than the residual estimator.*

## 5 Global reliability and local efficiency

In this section, we establish the global reliability and efficiency for the error indicators and estimator defined in in (4.24)–(4.28) for the NC elements of the odd orders.

Let

$$\mathrm{osc}\,(f,K) = \frac{h_K}{\sqrt{\lambda_K}}\|f - f_{k-1}\|_{0,K} \quad \text{and} \quad \mathrm{osc}\,(f,\mathcal{T}) = \left(\sum_{K\in\mathcal{T}} \mathrm{osc}\,(f,K)^2\right)^{1/2}.$$

**Theorem 5.1.** (Global Reliability) *Let $u_{\mathcal{T}}$ be the nonconforming solution to* (4.2). There exist constants $C_r$ and $C$ that is independent of the jump of the coefficient such that

$$\|A^{1/2}\nabla_h(u - u_{\mathcal{T}})\|_{0,\Omega} \le \eta_\sigma + C_r\,\eta_\rho + C\,\mathrm{osc}\,(f,\mathcal{T}). \tag{5.29}$$

*Proof.* The theorem is a direct result of Lemmas 5.2 and 5.3. □

Note that the global reliability bound in (5.29) does not require the quasi-monotonicity assumption on the distribution of the diffusion coefficient $A(x)$. The reliability constant $C_r$ for the nonconforming error is independent of the jump of $A(x)$, but not equal to one. This is due to the fact that the explicitly recovered gradient $\hat{\boldsymbol{\rho}}_{\mathcal{T}}$ is not curl free.

In the following, we bound the conforming error above by the estimator $\eta_\sigma$ given in (4.25).

**Lemma 5.2.** *The global conforming error estimator, $\eta_\sigma$, given in* (4.25) *is reliable, i.e., there exists a constant $C$ such that*

$$\inf_{\boldsymbol{\tau}\in\Sigma_f(\Omega)} \|A^{-1/2}(\boldsymbol{\tau}-\tilde{\boldsymbol{\sigma}}_{\mathcal{T}})\| \le \eta_\sigma + C\,\mathrm{osc}\,(f,\mathcal{T}). \tag{5.30}$$

*Proof.* Let $\phi \in H^1_D(\Omega)$ be the conforming part of the Helmholtz decomposition of $u-u_{\mathcal{T}}$. By (3.10), integration by parts, and the assumption that $g|_F \in \mathbb{P}_{k-1}(F)$, we have

$$\begin{aligned} &\inf_{\boldsymbol{\tau}\in\Sigma_f(\Omega)} \|A^{-1/2}\boldsymbol{\tau} + A^{1/2}\nabla_h u_{\mathcal{T}}\|^2_{0,\Omega} \\ =&\|A^{1/2}\nabla\phi\|^2 = (A\nabla(u-u_{\mathcal{T}}),\nabla\phi) = (A\nabla u + \hat{\boldsymbol{\sigma}}_{\mathcal{T}},\nabla\phi) - (\hat{\boldsymbol{\sigma}}_{\mathcal{T}} - \tilde{\boldsymbol{\sigma}}_{\mathcal{T}},\nabla\phi) \\ =& (f-f_{k-1},\phi) - (\hat{\boldsymbol{\sigma}}_{\mathcal{T}} - \tilde{\boldsymbol{\sigma}}_{\mathcal{T}},\nabla\phi). \end{aligned} \tag{5.31}$$

Let $\bar{\phi}_K = \dfrac{1}{|K|}\displaystyle\int_K \phi\,dx$. It follows from the definitions of $f_{k-1}$ and the Cauchy-Schwarz and the Poincaré inequalities that

$$\begin{aligned} \sum_{K\in\mathcal{T}} (f-f_{k-1},\phi)_K &= \sum_{K\in\mathcal{T}} (f-f_{k-1},\phi-\bar{\phi}_K)_K \\ \le\ & C\sum_{K\in\mathcal{T}} \frac{h_K}{\lambda_K^{1/2}} \|f-f_{k-1}\|_{0,K}\|A^{1/2}\nabla\phi\|_{0,K} \\ \le\ & C\,\mathrm{osc}\,(f,\mathcal{T})\|A^{1/2}\nabla\phi\|, \end{aligned}$$

which, together with (5.31) and the Cauchy-Schwartz inequality, leads to (5.30). This completes the proof of the lemma. □

Since our recovered gradient is not in $\mathring{H}_D(\mathrm{curl};\Omega)$, it is not straightforward to verify the reliability bound by Theorem 3.1. However, it still plays a role in our reliability analysis.

**Lemma 5.3.** *The global nonconforming error estimator, $\eta_\rho$, given in* (4.27) *is reliable, i.e., there exists a constant $C_r$ such that*

$$\inf_{v\in H^1_D(\Omega)} \|A^{1/2}(\nabla v - \nabla_h u_{\mathcal{T}})\| \le C_r\,\eta_\rho. \tag{5.32}$$

*Proof.* By Lemma 3.7, to show the validity of (5.32), it then suffices to prove that

$$\lambda_F^{1/2} h_F^{-1/2} \|[\![u_{\mathcal{T}}]\!]\|_{0,F} \le C\|A^{1/2}(\hat{\boldsymbol{\rho}}_{\mathcal{T}} - \tilde{\boldsymbol{\rho}}_{\mathcal{T}})\|_{0,\omega_F} \tag{5.33}$$

for all $F \in \mathcal{E}_I \cup \mathcal{E}_D$. Note that $[\![u_{\mathcal{T}}]\!]|_F$ is an odd function for all $F \in \mathcal{E}_I$. Hence, $\left\|[\![\tilde{\boldsymbol{\rho}}_{\mathcal{T}} \cdot \mathbf{t}_F]\!]\right\|_{0,F} = 0$ implies $\|[\![u_{\mathcal{T}}]\!]\|_{0,F} = 0$. By the equivalence of norms in a finite dimensional space and the scaling argument, we have that

$$h_F^{-1/2}\|[\![u_{\mathcal{T}}]\!]\|_{0,F} \le C\, h_F^{1/2} \left\|[\![\tilde{\boldsymbol{\rho}}_{\mathcal{T}} \cdot \mathbf{t}_F]\!]\right\|_{0,F}. \tag{5.34}$$

Since $\hat{\boldsymbol{\rho}}_{\mathcal{T}} \in H_D(\mathrm{curl};\Omega)$, it then follows from the triangle, the trace, and the inverse inequalities that

$$\begin{aligned}
\left\|[\![\tilde{\boldsymbol{\rho}}_{\mathcal{T}} \cdot \mathbf{t}_F]\!]\right\|_{0,F} &= \left\|[\![(\tilde{\boldsymbol{\rho}}_{\mathcal{T}} - \hat{\boldsymbol{\rho}}_{\mathcal{T}}) \cdot \mathbf{t}_F]\!]\right\|_{0,F} \le \left\|(\tilde{\boldsymbol{\rho}}_{\mathcal{T}} - \hat{\boldsymbol{\rho}}_{\mathcal{T}})|_{K_F^+} \cdot \mathbf{t}_F\right\|_{0,F} + \left\|(\tilde{\boldsymbol{\rho}}_{\mathcal{T}} - \hat{\boldsymbol{\rho}}_{\mathcal{T}})|_{K_F^-} \cdot \mathbf{t}_F\right\|_{0,F} \\
&\le C\, h_F^{-1/2}\left(\left\|\tilde{\boldsymbol{\rho}}_{\mathcal{T}} - \hat{\boldsymbol{\rho}}_{\mathcal{T}}\right\|_{0,\omega_F} + h_F\|\nabla\times(\hat{\boldsymbol{\rho}}_{\mathcal{T}} - \tilde{\boldsymbol{\rho}}_{\mathcal{T}})\|_{0,\omega_F}\right) \\
&\le C\, h_F^{-1/2}\left\|\tilde{\boldsymbol{\rho}}_{\mathcal{T}} - \hat{\boldsymbol{\rho}}_{\mathcal{T}}\right\|_{0,\omega_F} \le C\,\lambda_F^{-1/2} h_F^{-1/2}\left\|A^{1/2}\left(\tilde{\boldsymbol{\rho}}_{\mathcal{T}} - \hat{\boldsymbol{\rho}}_{\mathcal{T}}\right)\right\|_{0,\omega_F}
\end{aligned}$$

for all $F \in \mathcal{E}_I$, which, together with (5.34), implies (5.33) and, hence, (5.32). In the case that $F \in \mathcal{E}_D$, (5.33) can be proved in a similar fashion. This completes the proof of the lemma. □

## 5.1 Local Efficiency

In this section, we establish local efficiency of the indicators $\eta_{\sigma,K}$ and $\eta_{\rho,K}$ defined in (4.24) and (4.26), respectively.

**Theorem 5.4.** (Local Efficiency) *For each $K \in \mathcal{T}$, there exists a positive constant $C_e$ that is independent of the mesh size and the jump of the coefficient such that*

$$\eta_K \le C_e \left(\|A^{1/2}\nabla_h(u - u_{\mathcal{T}})\|_{0,\omega_K} + \mathrm{osc}\,(f, K)\right), \tag{5.35}$$

*where $\omega_K$ is the union of all elements that shares at least an edge with $K$.*

*Proof.* (5.35) is a direct consequence of Lemmas 5.6 and 5.7. □

Note that the local efficiency bound in (5.35) holds regardless the distribution of the diffusion coefficient $A(x)$.

## 5.2 Local Efficiency for $\eta_{\sigma,K}$

To establish local efficiency bound of $\eta_{\sigma,K}$, we introduce some auxiliary functions defined locally in $K$. To this end, for each edge $F \in \mathcal{E}_K$, denote by $F'$ and $F''$ the other two edges of $K$ such that $F, F'$, and $F''$ form counter-clockwise orientation. Without loss of generality, assume that $\mu_K \equiv 1$ on $\mathcal{E}_K$. Let

$$w_{F} = (\hat{\boldsymbol{\sigma}}_K - \tilde{\boldsymbol{\sigma}}_K) \cdot \mathbf{n}_K|_{F} \in \mathbb{P}_{k-1}(F), \quad a_{F} = w_{F}(\mathbf{s}_{F}), \quad \text{and} \quad b_{F} = w_{F}(\mathbf{e}_{F}). \tag{5.36}$$

Define the auxiliary function corresponding to $F$, $\tilde{w}_{F} \in \mathbb{P}_k(K)$, such that

$$\int_K \tilde{w}_{F}\, P_{j,K}\, dx = 0, \quad \forall\, j = 1, \cdots, m_k$$

and

$$\tilde{w}_{F}|_{F} = w_{F} + \gamma_{F} L_{k,F}, \quad \tilde{w}_{F}|_{F'} = -\beta_{F} L_{k,F'}, \quad \text{and} \quad \tilde{w}_{F}|_{F''} = \beta_{F} L_{k,F''},$$

where $\gamma_{F} = \dfrac{a_{F} - b_{F}}{2}$ and $\beta_{F} = \dfrac{a_{F} + b_{F}}{2}$.

**Lemma 5.5.** *For each $F \in \mathcal{E}_K$, there exists a positive constant $C$ such that*

$$\|\tilde{w}_{_F}\|_{0,K} \le C\, h_F^{1/2} \|w_{_F}\|_{0,F}. \tag{5.37}$$

*Proof.* By the Cauchy-Schwarz and the inverse inequalities, we have

$$|\gamma_{_F}| = \Big|\frac{1}{2}\int_F w'_{_F}\, ds\Big| \le \frac{h_F^{1/2}}{2}\|w'_{_F}\|_{0,F} \le C h_F^{-1/2}\|w_{_F}\|_{0,F}. \tag{5.38}$$

Approximation property and the inverse inequality give

$$\|w_{_F} - \beta_{_F}\|_{0,F} \le C h_F \|w'_{_F}\|_{0,F} \le C\|w_{_F}\|_{0,F},$$

which, together with the triangle inequality, gives

$$|\beta_{_F}| = h_F^{-1/2}\|\beta_{_F}\|_{0,F} \le h_F^{-1/2}(\|w_{_F} - \beta_{_F}\|_{0,F} + \|w_{_F}\|_{0,F}) \le C\, h_F^{-1/2}\|w_{_F}\|_{0,F}. \tag{5.39}$$

Since $\|L_{k,F}\|_{0,F} \le h_F^{1/2}$ for all $F \in \mathcal{E}_K$, by (5.38) and (5.39), we have that

$$\|\tilde{w}_{_F}\|_{0,F} = \Big(\|w_{_F}\|^2_{0,F} + \gamma^2_{_F}\|L_{k,F}\|^2_{0,F}\Big)^{1/2} \le C\,\|w_{_F}\|_{0,F}$$

and that

$$\|\tilde{w}_{_F}\|_{0,F'} \le h^{1/2}_{_{F'}}|\beta_{_F}| \le C\,\|w_{_F}\|_{0,F} \quad \text{and} \quad \|\tilde{w}_{_F}\|_{0,F''} \le h^{1/2}_{_{F''}}|\beta_{_F}| \le C\,\|w_{_F}\|_{0,F}.$$

Now (5.37) is a direct consequence of the fact that

$$\|\tilde{w}_{_F}\|_{0,K} \le C \sum_{F' \in \mathcal{E}_K} h^{1/2}_{_{F'}}\|\tilde{w}_{_F}\|_{0,F'}$$

which follows from the equivalence of norms in a finite dimensional space, and the fact that $\|\tilde{w}_{_F}\|_{\partial K} = 0$ implies $\|\tilde{w}_{_F}\|_K = 0$. This completes the proof of the lemma. □

**Lemma 5.6.** *There exists a positive constant $C$ such that*

$$\eta_{\sigma,K} \le C\,\Big(\|A^{1/2}\nabla_h(u - u_{\mathcal{T}})\|_{0,K} + \mathrm{osc}\,(f,K)\Big), \quad \forall\, K \in \mathcal{T}. \tag{5.40}$$

*Proof.* According to (4.15), it is easy to see that if $\|\,(\hat{\boldsymbol{\sigma}}_K - \tilde{\boldsymbol{\sigma}}_K)\cdot\mathbf{n}_F\|_{0,F} = 0$ for all $F \in \mathcal{E}_K$ implies that $\|\hat{\boldsymbol{\sigma}}_K - \tilde{\boldsymbol{\sigma}}_K\|_{0,K} = 0$. Hence, by the equivalence of norms in a finite dimensional space, we have that

$$\|\hat{\boldsymbol{\sigma}}_K - \tilde{\boldsymbol{\sigma}}_K\|_{0,K} \le C \sum_{F\in\mathcal{E}_K} h_F^{1/2}\|\,(\hat{\boldsymbol{\sigma}}_K - \tilde{\boldsymbol{\sigma}}_K)\cdot\mathbf{n}_F\|_{0,F} \le C \sum_{F\in\mathcal{E}_K} h_F^{1/2}\|w_{_F}\|_{0,F}, \tag{5.41}$$

where $w_{_F}$ is defined in (5.36). By the orthogonality property of $\{L_{j,F}\}_{j=0}^k$ and the definition of $\tilde{w}_{_F}$, we have

$$\|w_{_F}\|^2_{0,F} = \int_{\partial K}(\hat{\boldsymbol{\sigma}}_K - \tilde{\boldsymbol{\sigma}}_K)\cdot\mathbf{n}\,\tilde{w}_{_F}\, ds.$$

It then follows from (4.18), integration by parts, the Cauchy-Schwarz inequality, and (5.37) that

$$\begin{aligned}\|w_{_F}\|^2_{0,F} &= \int_K \tilde{\boldsymbol{\sigma}}_K\cdot\nabla\tilde{w}_{_F}\, dx + \int_K f\,\tilde{w}_{_F}\, dx - \int_K \tilde{\boldsymbol{\sigma}}_K\cdot\nabla\tilde{w}_{_F}\, dx - \int_K (\nabla\cdot\tilde{\boldsymbol{\sigma}}_K)\ \tilde{w}_{_F} dx \\ &= \int_K \big(f - \nabla\cdot\tilde{\boldsymbol{\sigma}}_K\big)\,\tilde{w}_{_F}\, dx \le C\, h_F^{1/2}\|f - \nabla\cdot\tilde{\boldsymbol{\sigma}}_K\|_{0,K}\|w_{_F}\|_{0,F},\end{aligned}$$

which implies

$$\|w_F\|_{0,F} \le C h_F^{1/2} \|f - \nabla \cdot \tilde{\boldsymbol{\sigma}}_K\|_{0,K}.$$

Together with (5.41), we have

$$\eta_{\sigma,K} \le \lambda_K^{-1/2} \|\hat{\boldsymbol{\sigma}}_K - \tilde{\boldsymbol{\sigma}}_K\|_{0,K} \le C \frac{h_K}{\sqrt{\lambda_K}} \|f - \nabla \cdot \tilde{\boldsymbol{\sigma}}_K\|_{0,K}.$$

Now (5.40) is a direct consequence of the following efficiency bound of the element residual (see, e.g., [8]):

$$\frac{h_K}{\sqrt{\lambda_K}} \|f - \nabla \cdot \tilde{\boldsymbol{\sigma}}_K\|_K \le C \left( \left\| A^{1/2} \nabla (u - u_{\mathcal{T}}) \right\|_{0,K} + \frac{h_K}{\sqrt{\lambda_K}} \left\| f - f_{k-1} \right\|_{0,K} \right).$$

This completes the proof of the theorem. □

## 5.3 Local Efficiency for $\eta_{\rho,K}$

In this section, we establish local efficiency bound for the nonconforming error indicator $\eta_{\rho,K}$ defined in (4.26).

**Lemma 5.7.** *There exists a positive constant $C$ that is independent of the mesh size and the jump of the coefficient such that*

$$\eta_{\rho,K} \le C \|A^{1/2} \nabla_h (u - u_{\mathcal{T}})\|_{0,\omega_K}, \quad \forall K \in \mathcal{T}. \tag{5.42}$$

*Proof.* By (4.22), it is easy to see that $\|(\hat{\boldsymbol{\rho}}_K - \tilde{\boldsymbol{\rho}}_K) \cdot \mathbf{t}_F\|_{0,F} = 0$ for all $F \in \mathcal{E}_K$ implies that $\|\hat{\boldsymbol{\rho}}_K - \tilde{\boldsymbol{\rho}}_K\|_{0,K} = 0$. By the equivalence of norms in a finite dimensional space and the scaling argument, we have

$$\|\hat{\boldsymbol{\rho}}_K - \tilde{\boldsymbol{\rho}}_K\|_{0,K} \le C \sum_{F \in \mathcal{E}_K} h_F^{1/2} \|(\hat{\boldsymbol{\rho}}_K - \tilde{\boldsymbol{\rho}}_K) \cdot \mathbf{t}_F\|_{0,F}. \tag{5.43}$$

Without loss of generality, assume that $K$ is an interior element. By (4.22), a direct calculation gives

$$(\hat{\boldsymbol{\rho}}_K - \tilde{\boldsymbol{\rho}}_K)\big|_F \cdot \mathbf{t}_F = \begin{cases} (\theta_F - 1)[\![\tilde{\boldsymbol{\rho}} \cdot \mathbf{t}_F]\!]|_F, & \text{if} \quad K = K_F^-, \\ \theta_F [\![\tilde{\boldsymbol{\rho}} \cdot \mathbf{t}_F]\!]|_F, & \text{if} \quad K = K_F^+ \end{cases} \tag{5.44}$$

for all $F \in \mathcal{E}_K$. It is also easy to verify that

$$\left(\Lambda_F^-\right)^{1/2} (1 - \theta_F) \le \left( \frac{\Lambda_F^- \Lambda_F^+}{\Lambda_F^- + \Lambda_F^+} \right)^{1/2} \quad \text{and} \quad \left(\Lambda_F^+\right)^{1/2} \theta_F \le \left( \frac{\Lambda_F^- \Lambda_F^+}{\Lambda_F^- + \Lambda_F^+} \right)^{1/2}. \tag{5.45}$$

Combining (5.43), (5.44), and (5.45) gives

$$\eta_{\rho,K} \le \Lambda_K^{1/2} \|\hat{\boldsymbol{\rho}}_K - \tilde{\boldsymbol{\rho}}_K\|_K \le C \sum_{F \in \mathcal{E}_K} \left( \frac{\Lambda_F^- \Lambda_F^+}{\Lambda_F^- + \Lambda_F^+} \right)^{1/2} h_F^{1/2} \|[\![\tilde{\boldsymbol{\rho}}_{\mathcal{T}} \cdot \mathbf{t}_F]\!]\|_{0,F}. \tag{5.46}$$

Now, (5.42) is a direct consequence of (5.46) and the following efficiency bound for the jump of tangential derivative on edges

$$\left( \frac{\Lambda_F^- \Lambda_F^+}{\Lambda_F^- + \Lambda_F^+} \right)^{1/2} h_F^{1/2} \|[\![\tilde{\boldsymbol{\rho}} \cdot \mathbf{t}_F]\!]\|_{0,F} \le C \|A^{1/2} \nabla (u - u_{\mathcal{T}})\|_{0,\omega_F}$$

for all $F \in \mathcal{E}_I$. This completes the proof of the lemma. □

# 6 Numerical Result

In this section, we report numerical results on two test problems. The first one is on the Crouziex-Raviart nonconforming finite element approximation to the Kellogg benchmark problem [29]. This is an interface problem in (2.1) with $\Omega = (-1, 1)^2$, $\Gamma_N = \emptyset$, $f = 0$,

$$A(x) = \begin{cases} 161.4476387975881, & \text{in } (0,1)^2 \cup (-1,0)^2, \\ 1, & \text{in } \Omega \setminus ([0,1]^2 \cup [-1,0]^2), \end{cases}$$

and the exact solution in the polar coordinates is given by $u(r, \theta) = r^{0.1}\mu(\theta)$, where $\mu(\theta)$ is a smooth function of $\theta$.

Starting with a coarse mesh, Figure 1 depicts the mesh when the relative error is less than 10%. Here the relative error is defined as the ratio between the energy norm of the true error and the energy norm of the exact solution. Clearly, the mesh is centered around the singularity (the origin) and there is no over-refinement along interfaces. Figure 2 is the log-log plot of the energy norm of the true error and the global error estimator $\eta$ versus the total number of degrees of freedom. It can be observed that the error converges in an optimal order (very close to $-1/2$) and that the efficiency index, i.e.,

$$\frac{\eta}{\|A^{1/2}\nabla_h(u - u_{\mathcal{T}})\|}$$

is close to one when the mesh is fine enough.

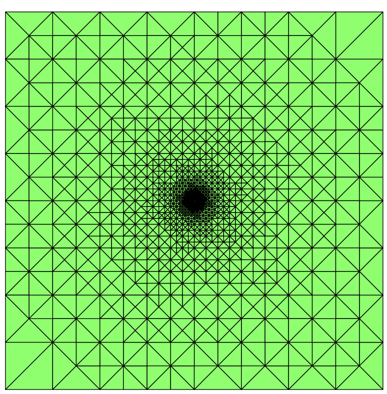

Figure 1: Kellogg problem: final mesh.

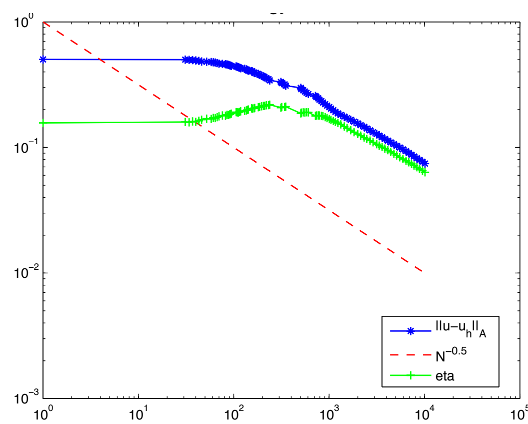


Figure 2: Error comparison.

With $f = 0$ for the Kellogg problem, we note that $\eta_\sigma = 0$, therefore, $\eta = \eta_\rho$. Even though for the nonconforming error we recover a gradient that is not curl free, (thus we were not be able to prove that the reliability constant is 1 for the nonconforming error) the numerics still shows the behavior of asymptotic exactness, i.e., when the mesh is fine enough the efficiency index is close to 1.

For the second test problem, we consider a Poisson L-shaped problem that has a nonzero conforming error $\eta_\sigma$. On the L-shaped domain $\Omega = [-1,\, 1]^2 \setminus [0,\, 1] \times [-1,\, 0]$, the Poisson problem ($A = I$) has the following exact solution

$$u(r, \theta) = r^{2/3} \sin((2\theta + \pi)/3) + r^2/2.$$

The numerics is based on the Crouziex-Raviart finite element approximation. With the relative error being less than 0.75%, the final mesh generated the adaptive mesh refinement algorithm is depicted in Figure 3. Clearly, the mesh is relatively centered around the singularity (origin). Comparison of the true error and the estimator is presented in Figure 4. It is obvious that the error converges in an optimal order (very close to $-1/2$) and that the efficiency index is very close to 1 for all iterations.

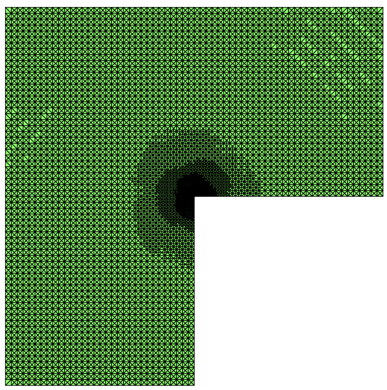

Figure 3: L-shape problem: final mesh.

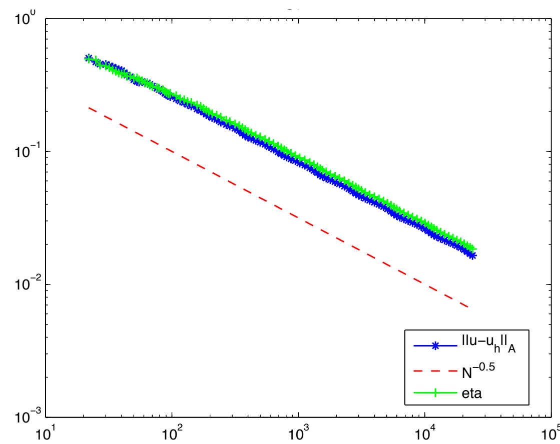


Figure 4: Error comparison.

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