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Optimal error estimate of discontinuous Galerkin methods for advection-diffusion-reaction problems with low regularity

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ABSTRACT

In this paper, we present a class of discontinuous Galerkin finite element methods for advection-diffusion-reaction problems and establish *a priori* error estimates when the solution is only in $H^{1+s}(\Omega)$ with $s \in (0, 1/2]$.

1. Introduction

Discontinuous Galerkin methods for elliptic partial boundary problems have been studied since the late 1970s. Most DG methods are derived by imposing a proper stabilization term, e.g., Ayuso and Marini in [1] derived the DG formulations by the so-called weighted residual approach which gives a linear relationship between the residual in the elements and the jump between across the element boundaries. For problems with discontinuous coefficients, this stabilization term and selection of the weights need careful treatments to be robust. Robustness means the constant in the *a priori* error estimate is independent of the jump of the coefficients. Cai, Ye and Zhang in [9] developed a non-standard DG formulation by carefully defining duality pairs on element interfaces for interface problems. For comments and remarks on various DG methods studied by many researchers, we refer readers to [1,2,9,10] and references therein.

Standard *a priori* error estimate for DG methods (see, e.g., [1,2]) requires the underlying problems to be sufficiently smooth, i.e., at least piece-wise $H^s(\Omega)$ with $s > 3/2$, so there is an error equation. The following a priori error estimate was established in [1]:

$$|||u - u_h||| \le C(\Omega) h^{s-1} C_(\epsilon, \beta, \rho) |u|_{s,\Omega} \tag{1.1}$$

provided that the solution u is at least piece-wise H^s with $s > 3/2$. For advection-diffusion-reaction problems with discontinuous coefficients, it is well known that the solutions of such problems may belong to $H^{1+s}(\Omega)$ with possibly very small positive s (see, e.g., [12]) in elements near singularities and are very smooth away from singularities. This kind of error estimate is also not optimal with respect to the local regularity since k is a global exponent.

The purposes of this paper are to present a class of DG methods and to establish optimal *a priori* error estimates of these methods when the underlying problem is not piece-wise $H^{3/2}(\Omega)$ regular. First, we derive a non-standard variational formulation for advection-diffusion-reaction problems. The formulation is defined in an appropriate function space that permits discontinuity across element interfaces and does not require piece-wise $H^s(\Omega)$, $s \ge 3/2$, smoothness. Hence, both continuous and discontinuous (including Crouzeix-Raviart) finite element spaces may be used and are conforming with respect to this variational formulation. The derivation may be regarded as the extension of the formulation in [9,10] for the interface problem, which leads an error equation naturally by carefully defining the duality pairs on element interfaces for problems with low regularity. Second, we establish the *a priori* error bound, and the constant in the estimate is independent of the parameters of the underlying problem and is optimal with respect to the local regularity. In the final section, we consider coefficient h_e^{-1} in the stabilization term may cause problems in the convergence analysis, and modified the DG finite element formulation and space by introducing the tangential derivative along edge e.

1.1. Notations

Throughout the paper, we will use the standard notations for the norms and seminorms in Sobolev Space. For a domain Ω, denote the Sobolev space by $W^{s,r}(\Omega)$ equipped with the standard Sobolev norm $\|\cdot\|_{s,r,\Omega}$ and seminorm $|\cdot|_{s,r,\Omega}$, where s is a real number and $1 \le r \le \infty$. When $r = 2$, $W^{s,2}(\Omega)$ is a Hilbert space and is denoted by $H^s(\Omega)$ with the norm $\|\cdot\|_{s,\Omega}$ and seminorm $|\cdot|_{s,\Omega}$. (We omit the subscript Ω from the inner product and norm designation when there is no risk of confusion.)

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To keep the homogeneity of dimensions, on a domain Ω with diameter L, define

$$\|v\|_{k,\Omega}^2 := \sum_{s=0}^{k} L^{2s} |v|_{s,\Omega}^2 \quad \text{for} \quad v \in H^k(\Omega),\ k \geq 0 \tag{1.2}$$

and

$$\|v\|_{k,\infty,\Omega} := \sum_{s=0}^{k} L^{s} |v|_{s,\infty,\Omega} \quad \text{for} \quad v \in W^{k,\infty}(\Omega),\ k \geq 0. \tag{1.3}$$

The paper is organized as follows. Section 2 introduces the advection-diffusion-reaction problem with discontinuous coefficients and the assumptions. Section 3 introduces the derivation of the variational formulations. In section 4, we derive the discontinuous finite element formulation and prove the stability in a strong norm. Section 5 gives the *a priori* error estimate analysis. Finally, section 6 presents a new discontinuous Galerkin methods in the modified DG finite element space.

2. Advection-diffusion-reaction problem and preliminaries

Let Ω be a bounded polygonal domain in $\Re^2$ with boundary $\partial\Omega = \bar{\Gamma}_D \cup \bar{\Gamma}_N$ and $\Gamma_D \cap \Gamma_N = \emptyset$ and let $\boldsymbol{n} = (n_1, n_2)$ be the outward unit vector normal to the boundary. Let $\boldsymbol{\beta} = (\beta_1, \beta_2)^t \in W^{1,\infty}(\Omega)^2$ be the velocity vector field defined on $\overline{\Omega}$. Define inflow and outflow boundaries of $\partial\Omega$ by

$$\Gamma^- = \{x \in \partial\Omega : \boldsymbol{\beta}(x)\cdot\boldsymbol{n}(x) < 0\} \quad \text{and} \quad \Gamma^+ = \{x \in \partial\Omega : \boldsymbol{\beta}(x)\cdot\boldsymbol{n}(x) > 0\}$$

respectively, and let

$$\Gamma_D^{\pm} = \Gamma_D \cap \Gamma^{\pm} \quad \text{and} \quad \Gamma_N^{\pm} = \Gamma_N \cap \Gamma^{\pm}.$$

Consider the following advection-diffusion-reaction problem with discontinuous diffusion coefficients:

$$-\nabla\cdot(\alpha(x)\nabla u - \boldsymbol{\beta} u) + \gamma u = f \quad \text{in } \Omega \tag{2.1}$$

with boundary conditions

$$u = g_D \quad \text{on } \Gamma_D \quad \text{and} \quad \boldsymbol{n}\cdot\left(\boldsymbol{\beta} u \chi_{\Gamma_N^-} - \alpha\nabla u\right) = g_N \quad \text{on } \Gamma_N, \tag{2.2}$$

where $f \in L^2(\Omega)$, $g_D \in H^{1/2}(\Gamma_D)$, and $g_N \in H^{-1/2}(\Gamma_N)$ are given functions; $\chi_{\Gamma_N^-}$ is the characteristic function of the set Γ_N^-; and the diffusion coefficient $\alpha(x)$ is non-negative and piece-wise constant on polygonal subdomains of Ω with possible large jumps across subdomain boundaries (interfaces):

$$\alpha(x) = \alpha_i \geq 0 \quad \text{in } \Omega_i \quad \text{for } i = 1, \ldots, n.$$

Here, $\{\Omega_i\}_{i=1}^n$ is a partition of the domain Ω with Ω_i being an open polygonal domain. For the stability and error analysis, the assumptions on the coefficients introduced in [1,11] are adopted in this paper:
(1) There exists a constant $\rho_0 \geq 0$ such that

$$\rho(x) = \frac{1}{2}\nabla\cdot\boldsymbol{\beta} + \gamma \geq \rho_0 \geq 0, \quad \text{in } \Omega; \tag{2.3}$$

(2) The advection field has no closed curves and stationary points. This implies that there exists a function $\eta \in W^{1,\infty}(\Omega)$ such that

$$\boldsymbol{\beta}\cdot\nabla\eta \geq 2b_0 := 2\frac{\|\boldsymbol{\beta}\|_{0,\infty,\Omega}}{L}, \quad \text{in } \Omega; \tag{2.4}$$

(3) There exists a constant $c_\beta > 0$ such that

$$|\boldsymbol{\beta}(x)| \geq c_\beta \|\boldsymbol{\beta}\|_{1,\infty,\Omega}, \quad a.e. \text{in } \Omega; \tag{2.5}$$

(4) There exists a constant $c_\rho > 0$ such that

$$\|\rho\|_{0,\infty,K} \leq c_\rho(\min_K \rho(x) + b_0), \quad \forall K \in \mathcal{T}_h, \tag{2.6}$$

where $\mathcal{T}_h = \{K\}$ is a given shape-regular triangulation of Ω.

Remark 2.1. Assumption (2.3) guarantees the stability of the advection-reaction part. Assumption (2.4) is based on the regularity of $\boldsymbol{\beta} \in W^{1,\infty}(\Omega)^2$ and the conditions that $\boldsymbol{\beta}$ has no closed curves and that $|\boldsymbol{\beta}(\boldsymbol{x})| \neq \boldsymbol{0}$ for almost all $x \in \Omega$. Assumptions (2.5) and (2.6) exclude the situations of a small but a highly oscillatory advection field. Also, the following useful inequality is deduced from (2.5)

$$|\boldsymbol{\beta}|_{1,\infty,\Omega} \leq \frac{\|\boldsymbol{\beta}\|_{1,\infty,\Omega}}{L} \leq \frac{1}{c_\beta}\frac{\|\boldsymbol{\beta}\|_{0,\infty,\Omega}}{L} = \frac{b_0}{c_\beta}. \tag{2.7}$$

2.1. Jumps and averages

Let $\mathcal{T}_h = \{K\}$ be a finite element triangulation of the domain Ω. Let h_K be the diameter of the element K and $h = \max_{K\in\mathcal{T}_h} h_K$. Assume that the triangulation $\mathcal{T}_h$ is regular and also the interfaces $F = \{\partial\Omega_i \cap \partial\Omega_j : i, j = 1, \ldots, n\}$ do not cut through any element $K \in \mathcal{T}_h$. Let $\mathcal{E}_K$ be the set of three edges of element $K \in \mathcal{T}_h$. Denote the set of all edges of the triangulation $\mathcal{T}_h$ by

$$\mathcal{E} := \mathcal{E}_I \cup \mathcal{E}_D \cup \mathcal{E}_N,$$

where $\mathcal{E}_I$ is the set of all interior element edges, and $\mathcal{E}_D$ and $\mathcal{E}_N$ are the sets of all boundary edges belonging to the respective boundaries Γ_D and Γ_N. And define

$$\mathcal{E}_{\Gamma^\pm} := \mathcal{E} \cap \Gamma^\pm.$$

For each $e \in \mathcal{E}$, let h_e be the length of the edge e and $\boldsymbol{n}_e$ be a unit normal vector to e. For each interior edge $e \in \mathcal{E}_I$, choose $\boldsymbol{n}_e$ such that $\boldsymbol{\beta}\cdot\boldsymbol{n}_e > 0$ and let K_e^- and K_e^+ be the two elements sharing the common edge e such that the unit outward normal vector of K_e^- coincides with $\boldsymbol{n}_e$. When $e \in \mathcal{E}_{\Gamma^\pm}$, $\boldsymbol{n}_e$ is the unit outward normal vector and denote the element by $K_e^\pm$. Denote by $v|_e^-$ and $v|_e^+$, respectively, the traces of a function v over e. Define jumps over edges by

$$[\![v]\!]_e := \begin{cases} v|_e^- - v|_e^+ & e \in \mathcal{E}_I, \\ v|_e^- & e \in \mathcal{E}_{\Gamma^-}, \\ v|_e^+ & e \in \mathcal{E}_{\Gamma^+}. \end{cases}$$

Let w_e^+ and w_e^- be weights defined on e satisfying

$$w_e^+(x) + w_e^-(x) = 1,$$

and define the following weighted averages by

$$\{v(x)\}_w^e = \begin{cases} w_e^- v_e^- + w_e^+ v_e^+ & e \in \mathcal{E}_I, \\ v|_e^- & e \in \mathcal{E}_{\Gamma^-}, \\ v|_e^+ & e \in \mathcal{E}_{\Gamma^+}, \end{cases}$$

and

$$\{v(x)\}_e^w = \begin{cases} w_e^+ v_e^- + w_e^- v_e^+ & e \in \mathcal{E}_I, \\ v|_e^+ & e \in \mathcal{E}_{\Gamma^-}, \\ v|_e^- & e \in \mathcal{E}_{\Gamma^+} \end{cases}$$

for all $e \in \mathcal{E}$. Denote by $\{v(x)\}_e$ the weighted average of v with $w_e^+ = w_e^- = \frac{1}{2}$. When there is no ambiguity, the subscript or superscript e in the designation of the jump and the weighted averages will be dropped. A simple calculation leads to the following identity:

$$[\![uv]\!]_e = \{v\}_e^w [\![u]\!]_e + \{u\}_w^e [\![v]\!]_e. \tag{2.8}$$

Let e be the sharing edge of elements K_e^+ and K_e^-, i.e., $e = \partial K_e^+ \cap \partial K_e^-$, and denote by α_e^+ and α_e^- the diffusion coefficients on K_e^+ and K_e^-, respectively. Denote by $W_e = \{\alpha\}_w^e$ the weighted average of α on edge e. For boundary edges, set

$$w_e^\pm = 1 \quad \text{and} \quad W_e = \alpha_e^\pm \quad \text{if} \quad e \in \Gamma^\pm.$$

In this paper, we take harmonic weights $w_e^{\pm} = \frac{\alpha_e^{\mp}}{\alpha_e^- + \alpha_e^+}$. Let $\alpha_{e,\min} = \min\{\alpha_e^+, \alpha_e^-\}$ and $\alpha_{e,\max} = \max\{\alpha_e^+, \alpha_e^-\}$, thus

$$W_e = \frac{2\alpha_e^+\alpha_e^-}{\alpha_e^+ + \alpha_e^-} \quad \text{and} \quad \alpha_{e,\min} \le W_e \le 2\alpha_{e,\min}. \tag{2.9}$$

3. Variational formulations

Following [9], we derive a variational formulation of problem (2.1) - (2.2) held for piece-wise smooth test functions. The key of this derivation is the introduction of a proper solution space in which integrals over inter-edges are well-defined. Moreover, the proper solution space is crucial for *a priori* error estimates of the underlying problem with low regularity.

Let u be the solution of problem (2.1) - (2.2), then it is well known from the regularity estimate in [3] that u belongs to $H^{1+s}(\Omega)$ for some positive s which could be very small. Since $f \in L^2(\Omega)$, it is easy to see that divergences of the diffusion and advection fluxes, $\alpha\nabla u$ and $\boldsymbol{\beta} u$, are square integrable, i.e.,

$$\alpha\nabla u,\ u\boldsymbol{\beta} \in H(\mathrm{div};\Omega) \equiv \{\boldsymbol{\tau} \in L^2(\Omega)^2 \,:\, \nabla\cdot\boldsymbol{\tau} \in L^2(\Omega)\}. \tag{3.1}$$

Consider the following solution space

$$V^{1+\epsilon}(\mathcal{T}_h) = \{v \in H^{1+\epsilon}(\mathcal{T}_h) \,:\, \nabla\cdot(\alpha\nabla v) \in L^2(K),\ \forall K \in \mathcal{T}_h\}$$

for $0 < \epsilon \ll 1$, where $H^s(\mathcal{T}_h)$ is the broken Sobolev space of degree $s > 0$ with respect to $\mathcal{T}_h$:

$$H^s(\mathcal{T}_h) = \{v \in L^2(\Omega) : v|_K \in H^s(K),\ \forall K \in \mathcal{T}_h\}.$$

Denote the discrete gradient and divergence operators by

$$(\nabla_h v)|_K = \nabla(v|_K) \quad \text{and} \quad (\nabla_h\cdot\boldsymbol{\tau})|_K = \nabla\cdot(\boldsymbol{\tau}|_K),$$

for all $K \in \mathcal{T}_h$, respectively.

Multiplying equation (2.1) by a test function $v \in V^{1+\epsilon}(\mathcal{T}_h)$, integrating by parts, and using boundary conditions (2.2), we have the following:

$$(f, v) = (\alpha\nabla_h u, \nabla_h v) - \sum_{e\in\mathcal{E}_I\cup\mathcal{E}_D} \int_e [\![\alpha\nabla u\cdot\boldsymbol{n}_e v]\!] + \sum_{e\in\mathcal{E}_N}\int_e g_N v$$
$$+ (u, -\boldsymbol{\beta}\cdot\nabla_h v + \gamma v) + \sum_{e\in\mathcal{E}_I\cup\mathcal{E}_{\Gamma+}}\int_e [\![\beta_e u v]\!] + \sum_{e\in\mathcal{E}_{D-}}\int_e \beta_e g_D v,$$

where $\mathcal{E}_{D-} = \mathcal{E}_D \cap \Gamma^-$ and $\beta_e = \boldsymbol{\beta}\cdot\boldsymbol{n}_e$. Note that the Dirichlet boundary condition is used on the inflow boundary. By (3.1), it is easy to see that the normal components of the diffusion and advection fluxes are continuous across the internal edges. Then for any $e \in \mathcal{E}_I$ and $v \in V^{1+\epsilon}(\mathcal{T}_h)$,

$$\int_e [\![\alpha\nabla u\cdot\boldsymbol{n}_e]\!]\,\{v\}^w\,ds = 0 \quad \text{and} \quad \int_e [\![u\boldsymbol{\beta}\cdot\boldsymbol{n}_e]\!]\,\{v\}^w\,ds = 0.$$

By identity (2.8) and the Dirichlet boundary condition in (2.2), we have that for all $v \in V^{1+\epsilon}(\mathcal{T}_h)$,

$$(\alpha\nabla_h u, \nabla_h v) + (u, -\boldsymbol{\beta}\cdot\nabla_h v + \gamma v) - \sum_{e\in\mathcal{E}_I\cup\mathcal{E}_D}\int_e \{\alpha\nabla u\cdot\boldsymbol{n}_e\}_w[\![v]\!]$$
$$+ \sum_{e\in\mathcal{E}_I\cup\mathcal{E}_{\Gamma+}}\int_e \{\beta_e u\}_w[\![v]\!] = (f, v) - \sum_{e\in\Gamma_N}\int_e g_N v - \sum_{e\in\mathcal{E}_{D-}}\int_e \beta_e g_D v. \tag{3.2}$$

Since the derivation does not make use of the continuity of the solution, one needs to impose such a continuity in order to achieve stability. To do so, it is natural and well-known to stabilize the diffusion and the advection operators by adding proper jump terms of the solution. Following the idea of [2] (also see [9]), we stabilize the diffusion operator by adding the following equation:

$$\sum_{e\in\mathcal{E}_I\cup\mathcal{E}_D}\int_e \gamma_\theta h_e^{-1} W_e [\![u]\!][\![v]\!]\,ds = \sum_{e\in\mathcal{E}_D}\gamma_\theta h_e^{-1}W_e\int_e g_D v\,ds,\ \forall v \in V^{1+\epsilon}(\mathcal{T}_h). \tag{3.3}$$

Since the diffusion operator is self-adjoint, it is natural to symmetrize the diffusion part by adding the following equation:

$$\theta\sum_{e\in\mathcal{E}_I\cup\mathcal{E}_D}\int_e \{\alpha\nabla v\cdot\boldsymbol{n}_e\}_w[\![u]\!]\,ds = \theta\sum_{e\in\mathcal{E}_D}\int_e g_D(\alpha\nabla v\cdot\boldsymbol{n}_e)\,ds,\ \forall v \in V^{1+\epsilon}(\mathcal{T}_h) \tag{3.4}$$

with $\theta = \{-1, 0, 1\}$. Both (3.3) and (3.4) follow from the continuity of $u \in H^{1+s}(\Omega)$ and the Dirichlet boundary condition. When $\theta = 1$, (3.4) plays a role of stabilization and, hence, (3.3) is not needed.

For the advection-reaction term, introduce the following general upwind average:

$$\{\beta_e u\}_{up}^e = \beta_e\xi_e^- u^- + \beta_e\xi_e^+ u^+, \quad \text{where } \xi_e^- + \xi_e^+ = 1 \text{ and } \xi_e^- > 1/2, \tag{3.5}$$

which is more general than that in [1] since ξ_e^+ could be negative. When $\xi_e^- = 1$, (3.5) is the classic upwind.

For $u, v \in V^{1+\epsilon}(\mathcal{T}_h)$, define the diffusion and advection-reaction bilinear forms by

$$a_{d,\theta}(u,v) = (\alpha\nabla_h u, \nabla_h v) + \theta\sum_{e\in\mathcal{E}_I\cup\mathcal{E}_D}\int_e \{\alpha\nabla v\cdot\boldsymbol{n}_e\}_w[\![u]\!]\,ds \tag{3.6}$$
$$- \sum_{e\in\mathcal{E}_I\cup\mathcal{E}_D}\int_e \{\alpha\nabla u\cdot\boldsymbol{n}_e\}_w[\![v]\!]\,ds + \sum_{e\in\mathcal{E}_I\cup\mathcal{E}_D}\int_e \gamma_\theta h_e^{-1}W_e[\![u]\!][\![v]\!]\,ds$$

for $\theta \in \{-1, 0, 1\}$ and

$$a_c(u,v) = (u, -\boldsymbol{\beta}\cdot\nabla_h v + \gamma v) + \sum_{e\in\mathcal{E}_I}\int_e \{\beta_e u\}_{up}[\![v]\!]\,ds + \sum_{e\in\mathcal{E}_{\Gamma+}}\int_e \beta_e u v\,ds, \tag{3.7}$$

respectively. And also for $v \in V^{1+\epsilon}(\mathcal{T}_h)$, define the linear form by

$$f_\theta(v) = (f, v) + \sum_{e\in\mathcal{E}_D}\gamma_\theta h_e^{-1}W_e\int_e g_D v\,ds + \sum_{e\in\mathcal{E}_N}\int_e g_N v\,ds$$
$$+\theta\sum_{e\in\mathcal{E}_D}\int_e g_D(k\nabla v\cdot\boldsymbol{n}_e)\,ds - \sum_{e\in\mathcal{E}_{D-}}\int_e (\boldsymbol{\beta}\cdot\boldsymbol{n}_e)g_D v\,ds.$$

The weak solution of (2.1) - (2.2) satisfies the following variational problem: to find $u \in V^{1+\epsilon}(\mathcal{T}_h)$ such that

$$a_\theta(u, v) = f_\theta(v), \quad \forall v \in V^{1+\epsilon}(\mathcal{T}_h) \tag{3.8}$$

with

$$a_\theta(u, v) = a_{d,\theta}(u, v) + a_c(u, v).$$

4. Discontinuous finite element approximation

Let $P_k(K)$ be the space of polynomials of degree at most k on element $K \in \mathcal{T}_h$. Denote the discontinuous finite element space associated with the triangulation $\mathcal{T}_h$ by

$$\mathcal{U}_h^k = \{v \in L^2(\Omega) \,:\, v|_K \in P_k(K),\ \forall K \in \mathcal{T}_h\}.$$

Discontinuous Galerkin (DG) finite element method is to find $u_h \in \mathcal{U}_h^k \subset V^{1+\epsilon}(\mathcal{T}_h)$ such that

$$a_\theta(u_h, v_h) = f_\theta(v_h), \quad \forall\, v_h \in \mathcal{U}_h^k. \tag{4.1}$$

The method corresponding to $\theta = -1$ and the classic upwind was introduced and analyzed recently in [2] for different boundary conditions. When $\alpha(x) = \epsilon$, the methods corresponding to $\theta = 0, 1$ and the classic upwind reproduce the first two methods in [1]; the third (introduced in [5]) and fourth methods in [1] are corresponding to (4.1) with the respective classic and general upwind averages for both the diffusion and advection terms. *A priori* error bounds for DG methods had been established by various researchers (see [1,2] and references therein) provided that the solution is at least piece-wise $H^{3/2+\epsilon}$ smooth and that γ_θ is large enough.

In the remainder of this section, we prove the stability that implies the well-posedness of (4.1). To this end, define the DG norms for the diffusion and advection-reaction parts by

$$|||v|||_d^2 = \|\alpha^{1/2}\nabla_h v\|_{0,\Omega}^2 + \|v\|_j^2 \tag{4.2}$$

with

$$\|v\|_j^2 = h_e^{-1}W_e\|[\![v]\!]\|_{0,e}^2$$

and

$$|||v|||_c^2 = \|(\overline{\rho}+b_0)^{1/2}v\|_{0,\Omega}^2 + \sum_{e\in\mathcal{E}}\|c_e^{1/2}[\![v]\!]\|_{0,e}^2 \tag{4.3}$$

respectively, where $b_0 = \|\boldsymbol{\beta}\|_{0,\infty}/L$, $\overline{\rho}$ is a piece-wise constant function defined as

$$\overline{\rho}_K(x) = \min_{x\in K}\rho_K(x) = \min_{x\in K}\left(\frac{1}{2}\nabla\cdot\boldsymbol{\beta}+\gamma\right)_K, \quad \forall K\in\mathcal{T}_h \tag{4.4}$$

and

$$c_e = \begin{cases}\left(\xi_e^- - \frac{1}{2}\right)\beta_e, & \text{on } e\in\mathcal{E}_I,\\ \frac{1}{2}\beta_e, & \text{on } e\in\mathcal{E}_{\Gamma^+},\\ -\frac{1}{2}\beta_e, & \text{on } e\in\mathcal{E}_{\Gamma^-}.\end{cases} \tag{4.5}$$

The DG norm is defined as

$$|||v|||_{DG} = \left(|||v|||_d^2 + |||v|||_c^2\right)^{1/2} \tag{4.6}$$

4.1. Stability

In this section, we will prove the stability of the bilinear form $a_\theta(\cdot,\cdot)$ with respect to the DG norm $|||\cdot|||_{DG}$. First, we consider the diffusion part $a_{d,\theta}(\cdot,\cdot)$ with respect to $|||\cdot|||_d$. To this end, we introduce the following lemmas.

Lemma 4.1. *For any $\mu_h\in\mathcal{V}_h^k$ and $v\in V^{1+\epsilon}(\mathcal{T}_h)$, there exists a positive constant C_g, depending only on the polynomial degree k and the triangulation $\mathcal{T}_h$, such that*

$$\sum_{e\in\mathcal{E}_I\cup\mathcal{E}_D}\int_e\left|\{\alpha\nabla_h\mu_h\cdot\boldsymbol{n}_e\}_w[\![v]\!]\right|ds \le C_g\|\alpha^{1/2}\nabla\mu_h\|_{0,\Omega}\|v\|_j$$

and $$\sum_{e\in\mathcal{E}_I\cup\mathcal{E}_D}\int_e\left|\{\alpha\mu_h\}_w[\![v]\!]\right|ds \le C_g\|\alpha^{1/2}\mu_h\|_{0,\Omega}\|v\|_j.$$

Proof. It follows from the definition of w_e and W_e in (2.9) that

$$w_e^{\pm}\sqrt{\alpha_e^{\pm}} = \frac{\alpha_e^{\mp}}{\alpha_e^+ + \alpha_e^-}\sqrt{\alpha_e^{\pm}} = \sqrt{\frac{\alpha_e^{\mp}}{\alpha_e^+ + \alpha_e^-}}\sqrt{\frac{W_e}{2}} \le \frac{\sqrt{2}}{2}\sqrt{W_e}.$$

Together with the inverse and the Cauchy-Schwarz inequalities, it gives that

$$\begin{aligned}&\sum_{e\in\mathcal{E}_I\cup\mathcal{E}_D}\int_e\left|\{\alpha\nabla\mu_h\cdot\boldsymbol{n}_e\}_w[\![v]\!]\right|ds\\ &= \sum_{e\in\mathcal{E}_I\cup\mathcal{E}_D}\int_e\left|(w_e^+\alpha_e^+\nabla\mu_h\cdot n_e^+ + w_e^-\alpha_e^-\nabla\mu_h\cdot n_e^-)[\![v]\!]\right|ds\\ &\le c_1\sum_{e\in\mathcal{E}_I\cup\mathcal{E}_D}h_e^{-1/2}W_e^{1/2}\|[\![v]\!]\|_{0,e}\sum_{\omega=+,-}\|\alpha^{1/2}\nabla\mu_h\|_{0,K^\omega}\\ &\le C_1\|\alpha^{1/2}\nabla\mu_h\|_{0,\Omega}\|v\|_j,\end{aligned}$$

where C_1 may depend on the polynomial degree k and the triangulation $\mathcal{T}_h$, and is independent of α and h. In a similar way, we obtain that

$$\begin{aligned}\sum_{e\in\mathcal{E}_I\cup\mathcal{E}_D}\int_e\left|\{\alpha\mu_h\}_w[\![v]\!]\right|ds &= \sum_{e\in\mathcal{E}_I\cup\mathcal{E}_D}\int_e\left|(w_e^+\alpha_e^+\mu_h^+ + w_e^-\alpha_e^-\mu_h^-)[\![v]\!]\right|ds\\ &\le c_2\sum_{e\in\mathcal{E}_I\cup\mathcal{E}_D}h_e^{-1/2}W_e^{1/2}\|[\![v]\!]\|_{0,e}\sum_{\omega=+,-}\|\alpha^{1/2}\mu_h\|_{0,K^\omega}\\ &\le C_2\|\alpha^{1/2}\mu_h\|_{0,\Omega}\|v\|_j,\end{aligned}$$

where C_2 may depend only on the triangulation $\mathcal{T}_h$ and the polynomial degree k. Let $C_g = \max\{C_1, C_2\}$ and this completes the proof of the lemma. □

Lemma 4.2. *For any function $\mu_h\in\mathcal{V}_h^k$, there exists a positive constant C_p, depending on the minimum angel of the triangulation $\mathcal{T}_h$ of Ω, such that*

$$\|\alpha^{1/2}\mu_h\|_{0,\Omega} \le C_pL\left(\|\alpha^{1/2}\nabla_h\mu_h\|_{0,\Omega}^2 + \|\mu_h\|_j^2\right)^{1/2}, \tag{4.7}$$

where L is the diameter of the domain Ω.

Proof. For any piece-wise H^1 function v, the following Poincaré-Friedrichs inequality is proved in [4]:

$$\|v\|_{0,\Omega} \le CL\left(\|\nabla_h v\|_{0,\Omega}^2 + \sum_{e\in\mathcal{E}_I\cup\mathcal{E}_D}h_e^{-1}\|[\![v]\!]\|_{0,e}^2\right)^{1/2}, \tag{4.8}$$

where C is a positive constant depending on the minimum angle of the triangulation $\mathcal{T}_h$ of Ω. Since the diffusion coefficient α is piece-wise constant, then for any function $\mu_h\in\mathcal{V}_h^k$, $\alpha^{1/2}\mu_h$ is a piece-wise H^1 function. So it follows from (4.8) that

$$\|\alpha^{1/2}\mu_h\|_{0,\Omega} \le CL\left(\|\alpha^{1/2}\nabla_h\mu_h\|_{0,\Omega}^2 + \sum_{e\in\mathcal{E}_I\cup\mathcal{E}_D}h_e^{-1}\|[\![\alpha^{1/2}\mu_h]\!]\|_{0,e}^2\right)^{1/2}.$$

To show the validity of (4.7), it suffices to prove that

$$\sum_{e\in\mathcal{E}_I\cup\mathcal{E}_D}h_e^{-1}\|[\![\alpha^{1/2}\mu_h]\!]\|_{0,e}^2 \le C\left(\|\alpha^{1/2}\nabla_h\mu_h\|_{0,\Omega}^2 + \|\mu_h\|_j^2\right). \tag{4.9}$$

To this end, without loss of generality, let $\alpha_{e,min} = \alpha_e^- < \alpha_e^+$. It follows from the trace inequality and (2.9) that for each $e\in\mathcal{E}_I\cup\mathcal{E}_D$,

$$\begin{aligned}\|[\![\alpha^{1/2}\mu_h]\!]\|_{0,e}^2 &= \|\sqrt{\alpha_e^-}\mu_h^- - \sqrt{\alpha_e^+}\mu_h^+\|_{0,e}^2\\ &= \|\sqrt{\alpha_e^-}(\mu_h^- - \mu_h^+) + (\sqrt{\alpha_e^-} - \sqrt{\alpha_e^+})\mu_h^+\|_{0,e}^2\\ &\le 2\left(\|\alpha_{e,min}^{1/2}[\![\mu_h]\!]\|_{0,e}^2 + \|\sqrt{\alpha_e^+}\mu_h^+\|_{0,e}^2\right)\\ &\le C\left(W_e\|[\![\mu_h]\!]\|_{0,e}^2 + h_{K_e^+}\|\sqrt{\alpha}\nabla_h\mu_h\|_{0,K^+}^2\right).\end{aligned}$$

Multiplying by h_e^{-1} and summing up over $e\in\mathcal{E}_I\cup\mathcal{E}_D$ imply (4.9). This completes the proof of the lemma. □

To establish the stability of the bilinear form $a_\theta(\cdot,\cdot)$ in the DG norm, we follow the idea in [1]. To this end, introduce the weight function

$$\varphi = e^{-\eta} + \mathcal{K} := \chi + \mathcal{K}, \tag{4.10}$$

where η is defined in (2.4) and $\mathcal{K}$ is a positive constant. Since $\eta\in W^{1,\infty}(\Omega)$, there exist positive constants χ_1, χ_2, and χ_3 such that

$$\chi_1 \le \chi \le \chi_2 \quad\text{and}\quad \|\nabla\chi\|_\infty \le \chi_3. \tag{4.11}$$

Choose the constant $\mathcal{K}$ such that

$$\chi_1 + \mathcal{K} > 6(1+C_g)C_pL\chi_3 \quad\text{and}\quad 2(\chi_1+\mathcal{K}) > \chi_2 + \mathcal{K} \tag{4.12}$$

with C_g and C_p defined in Lemma 4.1 and Lemma 4.2, respectively.

Lemma 4.3. *Let $a_{d,\theta}(\cdot,\cdot)$ and $a_c(\cdot,\cdot)$ be the bilinear forms defined in (3.6) and (3.7), respectively, with $\gamma_\theta \ge \gamma_0 > \max\{9C_g^2, 1\}$. For any $v_h \in \mathcal{V}_h^k$, the following inequalities hold:*

$$a_{d,\theta}(v_h, \varphi v_h) \ge \frac{\chi_1 + \mathcal{K}}{6} |||v_h|||_d^2, \qquad a_c(v_h, \varphi v_h) \ge \chi_1 |||v_h|||_c^2 \tag{4.13}$$

and

$$|||\varphi v_h|||_{DG} \le \sqrt{5}(\chi_1 + \mathcal{K})|||v_h|||_{DG}. \tag{4.14}$$

Proof. By the definition of the bilinear form $a_{d,\theta}$ and the continuity of φ, we have

$$\begin{aligned}
&a_{d,\theta}(v_h, \varphi v_h) \\
&= (\alpha \nabla_h v_h, \varphi \nabla_h v_h) + (\alpha \nabla_h v_h, v_h \nabla \varphi) + \theta \sum_{e \in \mathcal{E}_I \cup \mathcal{E}_D} \int_e (\nabla \varphi \cdot \boldsymbol{n}_e)\{\alpha v_h\}_w [\![v_h]\!] \\
&+ (\theta - 1) \sum_{e \in \mathcal{E}_I \cup \mathcal{E}_D} \int_e \varphi \{\alpha \nabla v_h \cdot \boldsymbol{n}_e\}_w [\![v_h]\!] + \sum_{e \in \mathcal{E}_I \cup \mathcal{E}_D} \int_e \gamma_\theta h_e^{-1} W_e \varphi [\![v_h]\!]^2.
\end{aligned}$$

It follows from the Cauchy-Schwarz inequality, (4.11), and Lemmas 4.1 and 4.2 that

$$(\alpha \nabla_h v_h, v_h \nabla \varphi) \le \chi_3 \|\alpha^{1/2} \nabla_h v_h\|_{0,\Omega} \|\alpha^{1/2} v_h\|_{0,\Omega} \le \chi_3 C_p L |||v_h|||_d^2,$$

and that

$$\begin{aligned}
\sum_{e \in \mathcal{E}_I \cup \mathcal{E}_D} \int_e (\nabla \varphi \cdot \boldsymbol{n}_e)\{\alpha v_h\}_w [\![v_h]\!] &\le \chi_3 C_g \|\alpha^{1/2} v_h\|_{0,\Omega} |||v_h|||_d \\
&\le \chi_3 C_g C_p L |||v_h|||_d^2.
\end{aligned}$$

By Lemma 4.1, (4.12), and the assumption that $\gamma_\theta \ge \gamma_0 > \max\{9C_g^2, 1\}$, we have

$$\begin{aligned}
\sum_{e \in \mathcal{E}_I \cup \mathcal{E}_D} \int_e \varphi \{\alpha \nabla v_h \cdot \boldsymbol{n}_e\}_w [\![v_h]\!] &\le (\chi_2 + \mathcal{K}) C_g \|\alpha^{1/2} \nabla_h v_h\|_{0,\Omega} \|v_h\|_j \\
&\le \frac{(\chi_1 + \mathcal{K})}{3} \left(\|\alpha^{1/2} \nabla_h v_h\|_{0,\Omega}^2 + \gamma_0 \|v_h\|_j^2 \right).
\end{aligned}$$

For $\theta \in \{-1, 0, 1\}$, combining the above equality and inequalities gives that

$$\begin{aligned}
a_{d,\theta}(v_h, \varphi v_h) &\ge (\chi_1 + \mathcal{K}) \left(\|\alpha^{1/2} \nabla_h v_h\|_{0,\Omega}^2 + \gamma_0 \|v_h\|_j^2 \right) - \chi_3 C_p L |||v_h|||_d^2 \\
&\quad - \chi_3 C_g C_p L |||v_h|||_d^2 - \frac{2(\chi_1 + \mathcal{K})}{3} \left(\|\alpha^{1/2} \nabla_h v_h\|_{0,\Omega}^2 + \gamma_0 \|v_h\|_j^2 \right) \\
&\ge \left(\frac{\chi_1 + \mathcal{K}}{3} - (1 + C_g)\chi_3 C_p L \right) |||v_h|||_d^2 \\
&\ge \frac{\chi_1 + \mathcal{K}}{6} |||v_h|||_d^2.
\end{aligned}$$

The last inequality used (4.12). This proves the first inequality in (4.13).

For the advection-reaction part, it follows from the identity that

$$v_h \nabla v_h = \frac{1}{2} \nabla_h (v_h^2),$$

integration by parts, and the continuity of ϕ and $\boldsymbol{\beta}$ that

$$\begin{aligned}
(v_h, -\boldsymbol{\beta} \cdot \nabla_h (\varphi v_h)) &= -\frac{1}{2} \int_\Omega \varphi \boldsymbol{\beta} \cdot \nabla_h (v_h^2) - \int_\Omega (\boldsymbol{\beta} \cdot \nabla \varphi) v_h^2 \\
&= \frac{1}{2} \int_\Omega v_h^2 \nabla \cdot (\varphi \boldsymbol{\beta}) - \frac{1}{2} \sum_{K \in \mathcal{T}_h} \int_{\partial K} \varphi v_h^2 \boldsymbol{\beta} \cdot \boldsymbol{n} - \int_\Omega (\boldsymbol{\beta} \cdot \nabla \varphi) v_h^2 \\
&= \frac{1}{2} \int_\Omega (\nabla \cdot \boldsymbol{\beta}) \varphi v_h^2 - \frac{1}{2} \int_\Omega (\boldsymbol{\beta} \cdot \nabla \varphi) v_h^2 - \frac{1}{2} \sum_{e \in \mathcal{E}} \int_e \beta_e \varphi [\![v_h^2]\!].
\end{aligned}$$

With the definition of c_e in (4.5), a simple computation gives that

$$\begin{aligned}
&-\frac{1}{2} \sum_{e \in \mathcal{E}} \int_e \beta_e \varphi [\![v_h^2]\!] + \sum_{e \in \mathcal{E}_I} \int_e \{\beta_e v_h\}_{up} [\![\varphi v_h]\!] + \sum_{e \in \mathcal{E}_{\Gamma+}} \int_e \beta_e \varphi v_h^2 \\
&= -\frac{1}{2} \sum_{e \in \mathcal{E}_I} \int_e \beta_e \varphi (v_h^+ + v_h^-)[\![v_h]\!] - \frac{1}{2} \sum_{e \in \mathcal{E}_{\Gamma-}} \int_e \beta_e \varphi v_h^2 + \frac{1}{2} \sum_{e \in \mathcal{E}_{\Gamma+}} \int_e \beta_e \varphi v_h^2 \\
&+ \sum_{e \in \mathcal{E}_I} \int_e \beta_e \varphi (\xi_e^+ v_h^+ + \xi_e^- v_h^-)[\![v_h]\!] = \sum_{e \in \mathcal{E}} \int_e c_e \varphi [\![v_h]\!]^2.
\end{aligned}$$

Combining these two identities gives that

$$\begin{aligned}
a_c(v_h, \varphi v_h) &= (v_h, -\boldsymbol{\beta} \cdot \nabla_h (\varphi v_h) + \gamma \varphi v_h) \\
&\quad + \sum_{e \in \mathcal{E}_I} \int_e \{\beta_e v_h\}_{up} [\![\varphi v_h]\!] + \sum_{e \in \mathcal{E}_{\Gamma+}} \int_e \beta_e \varphi v_h^2 \\
&= \int_\Omega (\gamma + \frac{1}{2} \nabla \cdot \boldsymbol{\beta}) \varphi v_h^2 - \frac{1}{2} \int_\Omega (\boldsymbol{\beta} \cdot \nabla \varphi) v_h^2 + \sum_{e \in \mathcal{E}} \int_e c_e \varphi [\![v_h]\!]^2.
\end{aligned}$$

From (2.4) and (4.11), we have

$$-\boldsymbol{\beta} \cdot \nabla \varphi = (\boldsymbol{\beta} \cdot \nabla \eta) e^{-\eta} \ge 2 b_0 e^{-\eta} \ge 2 b_0 \chi_1.$$

Together with the definition of $\bar{\rho}$ in (2.3), we obtain that

$$\begin{aligned}
a_c(v_h, \varphi v_h) &\ge (\chi_1 + \mathcal{K}) \int_\Omega \bar{\rho} v_h^2 + \chi_1 \int_\Omega b_0 v_h^2 + (\chi_1 + \mathcal{K}) \sum_{e \in \mathcal{E}} \int_e c_e [\![v_h]\!]^2 \\
&\ge \chi_1 \|(\bar{\rho} + b_0)^{1/2} v_h\|_{0,\Omega}^2 + \chi_1 \sum_{e \in \mathcal{E}} \|c_e^{1/2} [\![v_h]\!]\|_{0,e}^2 \\
&\ge \chi_1 |||v_h|||_c^2,
\end{aligned}$$

which proves the second inequality in (4.13).

To estimate the upper bound of the DG norm of φv_h, Lemma 4.2, (4.11), and (4.12) give that

$$\begin{aligned}
|||\varphi v_h|||_d^2 &= \|\alpha^{1/2} \varphi \nabla_h v_h\|_{0,\Omega}^2 + \|\alpha^{1/2} v_h \nabla \varphi\|_{0,\Omega}^2 + \sum_{e \in \mathcal{E}_I \cup \mathcal{E}_D} \int_e h_e^{-1} W_e \varphi^2 [\![v_h]\!]^2 \\
&\le ((\chi_2 + \mathcal{K})^2 + \chi_3^2 C_p^2 L^2) |||v_h|||_d^2 \\
&\le 5(\chi_1 + \mathcal{K})^2 |||v_h|||_d^2,
\end{aligned}$$

and that

$$\begin{aligned}
|||\varphi v_h|||_c^2 &= \|(\bar{\rho} + b_0)^{1/2} \varphi v_h\|_{0,\Omega}^2 + \sum_{e \in \mathcal{E}} \|c_e^{1/2} \varphi [\![v_h]\!]\|_{0,e}^2 \\
&\le (\chi_2 + \mathcal{K})^2 |||v_h|||_c^2,
\end{aligned}$$

which implies that

$$\begin{aligned}
|||\varphi v_h|||_{DG} &\le (5(\chi_1 + K)^2 |||v_h|||_d^2 + (\chi_2 + \mathcal{K})^2 |||v_h|||_c^2)^{1/2} \\
&\le \sqrt{5}(\chi_1 + \mathcal{K})|||v_h|||_{DG},
\end{aligned}$$

which proves (4.14) and, hence, completes the proof of the lemma. □

The following lemma is about the approximation results of the L_2-projection in the DG space, which have been proved in [6] and [7].

Lemma 4.4. *Let $\varphi \in W^{1,\infty}(\Omega)$ be the function defined in (4.10). For any $v_h \in \mathcal{V}_h^k$, let $\widetilde{\varphi v_h}$ be the L_2-projection of φv_h into $\mathcal{V}_h^k$, then the following estimates hold:*

$$\|\varphi v_h - \widetilde{\varphi v_h}\|_{p,2,\Omega} \le C h^{1-p} \|\chi\|_{1,\infty,\Omega} \|v_h\|_{0,\Omega} / L, \quad p = 0, 1$$

and

$$\left(\sum_{e \in \mathcal{E}} \|\varphi v_h - \widetilde{\varphi v_h}\|_{0,e}^2 \right)^{1/2} \le C h^{1/2} \|\chi\|_{1,\infty,\Omega} \|v_h\|_{0,\Omega} / L,$$

where C is a positive constant independent of $\mathcal{K}$ and L is the diameter of Ω.

With Lemma 4.4, we estimate the upper bounds of the norms $|||\varphi v_h - \widetilde{\varphi v_h}|||_d$ and $|||\varphi v_h - \widetilde{\varphi v_h}|||_c$ in the following lemma.

Lemma 4.5. *For any $v_h \in \mathcal{V}_h^k$, then the following estimates hold:*

$$|||\widetilde{\varphi v_h} - \varphi v_h|||_d \le CC_p\|\chi\|_{1,\infty}|||v_h|||_d$$

and

$$|||\widetilde{\varphi v_h} - \varphi v_h|||_c \le C\left(\frac{h}{L}\right)^{1/2}\|\chi\|_{1,\infty}\|(\bar\rho + b_0)^{1/2}v_h\|_{0,\Omega}.$$

Proof. For any function $v_h \in \mathcal{V}_h^k$, since α is a piece-wise constant function, then $\alpha^{1/2}v_h \in \mathcal{V}_h^k$ and $\alpha^{1/2}\widetilde{\varphi v_h}$ is the L_2 projection of $\alpha^{1/2}\varphi v_h$ into $\mathcal{V}_h^k$. Lemma 4.4 gives that

$$\|\alpha^{1/2}\varphi v_h - \alpha^{1/2}\widetilde{\varphi v_h}\|_{p,2,\Omega} \le Ch^{1-p}\|\chi\|_{1,\infty}\|\alpha^{1/2}v_h\|_{0,\Omega}/L, \quad p=0,1$$

and that

$$\left(\sum_{e\in\mathcal{E}}\|\alpha^{1/2}\varphi v_h - \alpha^{1/2}\widetilde{\varphi v_h}\|_{0,e}^2\right)^{1/2} \le Ch^{1/2}\|\chi\|_{1,\infty,\Omega}\|\alpha^{1/2}v_h\|_{0,\Omega}/L.$$

Together with the definition of d-norm in (4.2), the fact that $\alpha_{e,\min} \le W_e \le 2\alpha_{e,\min}$, and Lemma 4.2, we have

$$\begin{aligned}|||\varphi v_h - \widetilde{\varphi v_h}|||_d^2 &= \|\alpha^{1/2}\nabla_h(\varphi v_h - \widetilde{\varphi v_h})\|_{0,\Omega}^2 \\ &\quad + \sum_{e\in\mathcal{E}_I\cup\mathcal{E}_D} h_e^{-1}W_e\|[\![\varphi v_h - \widetilde{\varphi v_h}]\!]\|_{0,e}^2 \\ &\le C^2\|\chi\|_{1,\infty}^2\|\alpha^{1/2}v_h\|_{0,\Omega}^2/L^2 \\ &\le C^2C_p^2\|\chi\|_{1,\infty}^2|||v_h|||_d^2,\end{aligned}$$

which proves the first inequality.

In a similar way, by the fact that $\bar\rho + b_0$ is a piece-wise constant function and Lemma 4.4, we have that

$$\begin{aligned}&\|(\bar\rho + b_0)^{1/2}(\varphi v_h - \widetilde{\varphi v_h}\|_{p,2,\Omega} \\ &\le Ch^{1-p}\|\chi\|_{1,\infty}\|(\bar\rho+b_0)^{1/2}v_h\|_{0,\Omega}/L, \quad p=0,1.\end{aligned}$$

Together with the inequality that

$$|c_e| \le \|\boldsymbol\beta\|_{0,\infty} \le b_0L, \ \forall e\in\mathcal{E}$$

and the fact that $h/L \le 1$, we obtain that

$$\begin{aligned}&|||\widetilde{\varphi v_h} - \varphi v_h|||_c \\ &= \left(\|(\bar\rho+b_0)^{1/2}(\widetilde{\varphi v_h} - \varphi v_h)\|_{0,\Omega}^2 + \sum_{e\in\mathcal{E}}\|c_e^{1/2}[\![\widetilde{\varphi v_h} - \varphi v_h]\!]\|_{0,e}^2\right)^{1/2} \\ &\le \left(C^2\frac{h^2}{L^2}\|\chi\|_{1,\infty}^2\|(\bar\rho+b_0)^{1/2}v_h\|_{0,\Omega}^2 + b_0LC^2\frac{h}{L^2}\|\chi\|_{1,\infty}^2\|v_h\|_{0,\Omega}^2\right)^{1/2} \\ &\le C\left(\frac{h}{L}\right)^{1/2}\|\chi\|_{1,\infty}\|(\bar\rho+b_0)^{1/2}v_h\|_{0,\Omega},\end{aligned}$$

which proves the second inequality and, hence, completes the proof of the lemma. □

Lemma 4.6. *Under the same hypotheses of Lemma 4.3, for any $v_h \in \mathcal{V}_h^k$, there exist constants χ_4 and χ_5 independent of $\mathcal{K}$, such that*

$$a_d(v_h, \varphi v_h - \widetilde{\varphi v_h}) \le \chi_4|||v_h|||_d^2 \tag{5.15a}$$

and that

$$a_c(v_h, \varphi v_h - \widetilde{\varphi v_h}) \le \chi_5(h/L)^{1/2}|||v_h|||_c^2. \tag{5.15b}$$

Proof. By the definition of $a_{d,\theta}$ in (3.6), the Cauchy-Schwarz inequality, the assumption that $\gamma_\theta \ge \gamma_0 > \max\{9C_g^2, 1\}$, Lemma 4.1, and Lemma 4.4, we have that

$$\begin{aligned}&a_{d,\theta}(v_h, \widetilde{\varphi v_h} - \varphi v_h) \\ &= (\alpha\nabla_h v_h, \nabla_h(\widetilde{\varphi v_h} - \varphi v_h)) + \sum_{e\in\mathcal{E}_I\cup\mathcal{E}_D}\int_e \gamma_\theta h_e^{-1}W_e[\![v_h]\!][\![\widetilde{\varphi v_h} - \varphi v_h]\!]ds \\ &- \sum_{e\in\mathcal{E}_I\cup\mathcal{E}_D}\int_e\{\alpha\nabla_h v_h\cdot n_e\}_w[\![\widetilde{\varphi v_h} - \varphi v_h]\!]ds \\ &+\theta\sum_{e\in\mathcal{E}_I\cup\mathcal{E}_D}\int_e\{\alpha\nabla_h(\widetilde{\varphi v_h} - \varphi v_h)\cdot n_e\}_w[\![v_h]\!]ds \\ &\le \gamma_\theta|||v_h|||_d|||\widetilde{\varphi v_h} - \varphi v_h|||_d + C_g\|\alpha^{1/2}\nabla v_h\|_{0,\Omega}\|\widetilde{\varphi v_h} - \varphi v_h\|_j \\ &+C\frac{\|\chi\|_{1,\infty}}{L}\|\alpha^{1/2}v_h\|_{0,\Omega}\|v_h\|_j \\ &\le \left(\gamma_\theta + C_g + CC_p\|\chi\|_{1,\infty,\Omega}\right)|||v_h|||_d|||\widetilde{\varphi v_h} - \varphi v_h|||_d.\end{aligned}$$

This proves the validity of (5.15a) with $\chi_4 = \gamma_\theta + C_g + CC_p\|\chi\|_{1,\infty,\Omega}$, independent of $\mathcal{K}$.

Rewriting the advection - reaction part by integration by parts and using (2.8) give that, for any $u, v \in V^{1+\epsilon}(\mathcal{T}_h)$,

$$\begin{aligned}a_c(u,v) &= (u,\gamma v) + (\nabla_h(u\boldsymbol\beta), v) - \sum_{e\in\mathcal{E}}\int_e\beta_e[\![uv]\!] + \sum_{e\in\mathcal{E}_I}\int_e\beta_e\{u\}_{up}[\![v]\!] + \sum_{e\in\Gamma^+}\int_e\beta_e uv \\ &= (u,(\gamma+\nabla\cdot\boldsymbol\beta)v) + (\boldsymbol\beta\cdot\nabla_h u, v) - \sum_{e\in\mathcal{E}_I}\int_e\beta_e\{v\}^{up}[\![u]\!] - \sum_{e\in\Gamma^-}\int_e\beta_e uv \\ &= (u,(\gamma+\nabla\cdot\boldsymbol\beta)v) + (\boldsymbol\beta\cdot\nabla_h u, v) + \sum_{e\in\mathcal{E}_I}\int_e c_e[\![u]\!][\![v]\!] - \sum_{e\in\Gamma^-\cup\mathcal{E}_I}\int_e\beta_e[\![u]\!]\{v\}.\end{aligned}$$

Let $P\boldsymbol\beta$ be the L_2 projection of $\boldsymbol\beta$ onto $\mathcal{V}_h^0$, i.e., the space of piece wise constant with respect to $\mathcal{T}_h$ with the following approximation property holds:

$$\|\boldsymbol\beta - P\boldsymbol\beta\|_{0,\infty,\Omega} \le Ch|\boldsymbol\beta|_{1,\infty,\Omega}. \tag{4.15}$$

Since $P\boldsymbol\beta\cdot\nabla_h v_h \in \mathcal{V}_h^k$, the definition of $\widetilde{\varphi v_h}$ gives that

$$\int_\Omega P\boldsymbol\beta\cdot\nabla_h v_h(\varphi v_h - \widetilde{\varphi v_h}) = 0.$$

Combining the identities gives that

$$\begin{aligned}&a_c(v_h, \widetilde{\varphi v_h} - \varphi v_h) \\ &= \int_\Omega(\gamma+\nabla\cdot\boldsymbol\beta)v_h(\widetilde{\varphi v_h} - \varphi v_h) + \int_\Omega(\widetilde{\varphi v_h} - \varphi v_h)(\boldsymbol\beta - P\boldsymbol\beta)\cdot\nabla_h v_h \\ &\quad + \sum_{e\in\mathcal{E}_I}\int_e c_e[\![v_h]\!][\![\widetilde{\varphi v_h} - \varphi v_h]\!] - \sum_{e\in\mathcal{E}_{\Gamma^-}\cup\mathcal{E}_I}\int_e\beta_e[\![v_h]\!]\{\widetilde{\varphi v_h} - \varphi v_h\} \\ &:= I + II + III + IV.\end{aligned}$$

It follows from (2.6), (2.7) and Lemma 4.4 that

$$\begin{aligned}I &= \int_\Omega\rho v_h(\widetilde{\varphi v_h} - \varphi v_h) + \frac{1}{2}\int_\Omega\nabla\cdot\boldsymbol\beta v_h(\widetilde{\varphi v_h} - \varphi v_h) \\ &\le c_\rho\|(\bar\rho+b_0)^{1/2}v_h\|_\Omega\|(\bar\rho+b_0)^{1/2}(\widetilde{\varphi v_h} - \varphi v_h)\|_\Omega \\ &\quad + \frac{b_0}{2c_\beta}\|v_h\|_\Omega\|\widetilde{\varphi v_h} - \varphi v_h\|_\Omega \\ &\le (c_\rho + \frac{1}{2c_\beta})C\frac{h}{L}\|\chi\|_{1,\infty}\|(\bar\rho+b_0)^{1/2}v_h\|_{0,\Omega}^2.\end{aligned}$$

Using (4.15), (2.7), Lemma 4.4 and the inverse inequality gives that

$$II \le Ch|\beta|_{1,\infty}\|\nabla_h v_h\|\frac{h}{L}\|\chi\|_{1,\infty}\|v_h\| \le C\frac{h}{L}\frac{b_0}{c_\beta}\|\chi\|_{1,\infty,\Omega}\|v_h\|^2_{0,\Omega}.$$

By (2.4), Lemma 4.4 and the Cauchy-Schwarz inequality, we have

$$III + IV \le C\left(\sum_{e\in\mathcal{E}}\|c_e^{1/2}[\![v_h]\!]\|_{0,e}\right)\left(\frac{h^{1/2}}{L}\|\beta\|^{1/2}_{0,\infty}\|\chi\|_{1,\infty}\|v_h\|\right)$$
$$\le C\left(\frac{h}{L}\right)^{1/2}\|\chi\|_{1,\infty}\left(\sum_{e\in\mathcal{E}}\|c_e^{1/2}[\![v_h]\!]\|^2_{0,e} + b_0\|v_h\|^2_{0,\Omega}\right).$$

Together with the fact that $h/L < 1$, we obtain that

$$a_c(v_h, \widetilde{\varphi v_h} - \varphi v_h) \le (1 + c_\rho + \frac{2}{c_\beta})C\|\chi\|_{k+1,\infty,\Omega}\left(\frac{h}{L}\right)^{1/2}|||v_h|||_c^2,$$

which completes the proof with $\chi_5 = (1 + c_\rho + \frac{2}{c_\beta})C\|\chi\|_{k+1,\infty,\Omega}$. □

Next theorem gives the stability of the variational form.

Theorem 4.7. *Let $a_{d,\theta}(\cdot,\cdot)$ and $a_c(\cdot,\cdot)$ be the bilinear forms defined in (3.6) and (3.7), respectively, with $\gamma_\theta \ge \gamma_0 > \max\{9C_g^2, 1\}$. Then there exist positive constants a_0 and h_0 such that for all $h < h_0$ and $v_h \in \mathcal{U}_h^k$,*

$$\sup_{w_h\in\mathcal{U}_h^k}\frac{a_\theta(v_h, w_h)}{|||w_h|||_{DG}} \ge a_0|||v_h|||_{DG}. \tag{4.16}$$

Proof. For any $v_h \in \mathcal{U}_h^k$, let $w_h = \widetilde{\varphi v_h} \in \mathcal{U}_h^k$ be the L_2 projection of φv_h into $\mathcal{U}_h^k$. First it follows from the triangle inequality and Lemma 4.3 and Lemma 4.5 that

$$|||\widetilde{\varphi v_h}|||_{DG} \le (|||\widetilde{\varphi v_h} - \varphi v_h|||_{DG} + |||\varphi v_h|||_{DG}) \le C|||v_h|||_{DG}.$$

To show the validity of (4.16), it suffices to show that

$$a_\theta(v_h, w_h) \ge C|||v_h|||^2_{DG}. \tag{4.17}$$

To this end, by Lemmas 4.3 and 4.6, we have that

$$a_{d,\theta}(v_h, \widetilde{\varphi v_h}) = a_{d,\theta}(v_h, \widetilde{\varphi v_h} - \varphi v_h) + a_{d,\theta}(v_h, \varphi v_h)$$
$$\ge \left(\frac{\chi_1 + \mathcal{K}}{6} - \chi_4\right)|||v_h|||_d^2.$$

Note that in Lemma 4.6, the constant χ_4 is independent of $\mathcal{K}$, so we can choose $\mathcal{K}$ such that $\chi_1 + \mathcal{K}$ is bigger than $12\chi_4$. Then it follows that

$$a_{d,\theta}(v_h, \widetilde{\varphi v_h}) \ge \chi_4|||v_h|||_d^2.$$

In a similar way, then for $h < h_0$ we have that

$$a_c(v_h, \widetilde{\varphi v_h}) \ge c|||v_h|||_c^2,$$

with c only depending on χ_1 and χ_5. Combining the two inequalities gives (4.17) and, hence, completes the proof of the theorem. □

5. A priori error estimate

In this section, we establish *a priori* error estimate in the DG norm defined in (4.6) for the discontinuous finite element methods.

Let P_h be the L_2-projection onto $\mathcal{U}_h^k$. The standard approximation argument in [8,9] gives that: for $u \in V^{1+\epsilon}(\mathcal{T}_h) \cap H^{1+s}(\mathcal{T}_h)$ with $\epsilon \le s \le 1$,

$$\|\alpha^{1/2}\nabla(u - P_h u)\|_{\epsilon,\Omega} \le C\left(\sum_{K\in\mathcal{T}_h} h_K^{2(s-\epsilon)}\|\alpha^{1/2}\nabla u\|^2_{s,K}\right)^{1/2}, \tag{5.1}$$

$$\|u - P_h u\|_{r,p,K} \le Ch^{s+1-r}|u|_{s+1,p,K}, \quad r = 0, 1, \ 1 \le p \le \infty, \ K \in \mathcal{T}_h. \tag{5.2}$$

Together with the trace inequality, the following estimate holds:

$$\|u - P_h u\|_{0,e} \le Ch_{K_e}^{s+1/2}|u|_{s+1,K_e}, \ \forall e \in \mathcal{E}. \tag{5.3}$$

Let f_k be the L_2 projection of f onto $\mathcal{U}_h^k$, define

$$osc(f, K) = \frac{h_K}{\sqrt{\alpha_K}}\|f - f_{k-1}\|_{0,K} \quad \text{and} \quad asc(f) = \left(\sum_{K\in\mathcal{T}_h} osc(f, K)^2\right)^{1/2}.$$

Remark 5.1. The symbol $\lesssim$ used in this section denotes smaller than or equal to, up to a positive constant depending only on the triangulation $\mathcal{T}_h$, the domain Ω, the polynomial degree k, independent of the coefficients of the problem and h.

The next lemma proved in [10] gives a trace inequality of functions with low regularities.

Lemma 5.2. *For any $K \in \mathcal{T}_h$, assume that $v \in V^{1+s}(K)$ and $w_h \in P_k(K)$, then the following trace inequality holds:*

$$\int_e (\nabla v \cdot \mathbf{n})w_h ds \lesssim h_e^{-1/2}||w_h||_{0,e}(||\nabla v||_{0,K} + h_K||\triangle v||_{0,K}).$$

Lemma 5.3. *Let $u \in V^{1+s}(\mathcal{T}_h) \bigcap H^{1+\epsilon}(\Omega)$ be the solution of (2.1) with boundary conditions (2.2). For any $v \in \mathcal{U}_h^k$, let $\xi = u - v$, then on any $K \in \mathcal{T}_h$, the following estimate holds:*

$$h_K\|\alpha^{1/2}\Delta\xi\|_{0,K} \lesssim \|\alpha^{1/2}\nabla\xi\|_{0,K} + \frac{h_K}{\sqrt{\alpha_K}}\|\nabla\cdot(\beta\xi) + \gamma\xi\|_{0,K} + osc(f, K).$$

Proof. For any $v \in \mathcal{U}_h^k$, denote the element residual of v over $K \in \mathcal{T}_h$ by

$$r_K = \left[-\nabla\cdot(\alpha\nabla v) + \nabla\cdot(\beta v) + \gamma v - f_{k-1}\right]_K$$
$$= \left[\nabla\cdot(\alpha\nabla\xi) - \nabla\cdot(\beta\xi) - \gamma\xi + f - f_{k-1}\right]_K.$$

Let ψ_K be the cubic element bubble function on K, then it follows from integration by parts, and the Cauchy-Schwarz, the triangle, and the inverse inequalities that

$$\|r_K\|^2_{0,K} \lesssim \int_K r_K^2\psi_K = \int_K \left(\nabla\cdot(\alpha\nabla\xi) - \nabla\cdot(\beta\xi) - \gamma\xi + f - f_{k-1}\right) r_K\psi_K$$
$$= -\int_K \alpha\nabla\xi\cdot\nabla(r_K\psi_K) + \int_K (f - f_{k-1} - \nabla\cdot(\beta\xi) - \gamma\xi)r_K\psi_K$$
$$\lesssim \|\alpha\nabla\xi\|_{0,K}|r_K\psi_K|_{1,K}$$
$$+ \left(\|\nabla\cdot(\beta\xi) + \gamma\xi\|_{0,K} + \|f - f_{k-1}\|_{0,K}\right)\|r_K\psi_K\|_{0,K}$$
$$\lesssim \left(h_K^{-1}\|\alpha\nabla\xi\|_{0,K} + \|\nabla\cdot(\beta\xi) + \gamma\xi\|_{0,K} + \|f - f_{k-1}\|_{0,K}\right)\|r_K\|_{0,K},$$

which implies

$$\|r_K\|_{0,K} \lesssim h_K^{-1}\|\alpha\nabla\xi\|_{0,K} + \|\nabla\cdot(\beta\xi) + \gamma\xi\|_{0,K} + \|f - f_{k-1}\|_{0,K}.$$

Now, the lemma is a direct consequence of the fact that

$$\Delta\xi = \alpha_K^{-1}\left(r_K + f_{k-1} - f + \nabla\cdot(\beta\xi) + \gamma\xi\right)_K$$

and the triangle inequality. □

Theorem 5.4. *Let $u \in V^{1+s}(\mathcal{T}_h) \bigcap H^{1+\epsilon}(\Omega)$ be the solution of (2.1) with boundary conditions (2.2), and $u|_K \in H^{1+s_K}(K)$ be the restriction on $K \in \mathcal{T}_h$. Let u_h be the solution of discrete problem (4.1). There exists a positive constant C, depending on the domain, the triangulation $\mathcal{T}_h$ and the polynomial degree (but independent of mesh size h and the coefficients of the problem), such that*

$$|||u - u_h|||_{DG} \le C\sum_{K\in\mathcal{T}_h} C_{K,\alpha,\rho,\beta}h_K^{s_K}|\alpha^{1/2}\nabla u|_{s_K,K} + osc(f), \tag{5.4}$$

where

$$C_{K,\alpha,\beta,\rho} = 1 + \frac{h_K^{1/2}}{\alpha_K^{1/2}}\left(\|\beta\|_{0,\infty,\Omega}^{1/2} + h_K^{1/2}\|\rho\|_{0,\infty,\Omega}^{1/2}\right) + \frac{h_K}{\alpha_K}\left(\|\beta\|_{0,\infty,\Omega} + h_K\|\rho\|_{0,\infty,\Omega}\right).$$

Proof. Let

$$E = u - P_h u \quad \text{and} \quad E_h = u_h - P_h u.$$

By the triangle inequality and the standard approximation argument of P_h, (5.1), (5.2) and (5.3), to show the validity of (5.4), we need to prove that

$$|||E_h|||_{DG} \le C \sum_{K\in\mathcal{T}_h} C_{K,\alpha,\rho,\beta} h_K^{s_K} |\alpha^{1/2}\nabla u|_{s_K,K} + osc(f).$$

By Theorem 4.7 and the error equation, we have

$$a_0|||E_h|||_{DG} \le \frac{a_\theta(E_h, v_h)}{|||v_h|||_{DG}} = \frac{a_\theta(E, v_h)}{|||v_h|||_{DG}}.$$

Hence, it suffices to show that

$$\frac{a_{d,\theta}(E, v_h)}{|||v_h|||_{DG}} \le C \sum_{K\in\mathcal{T}_h} C_{K,\alpha,\rho,\beta} h_K^{s_K} |\alpha^{1/2}\nabla u|_{s_K,K} + osc(f) \tag{5.5}$$

and $$\frac{a_c(E, v_h)}{|||v_h|||_{DG}} \le C \sum_{K\in\mathcal{T}_h} C_{K,\alpha,\rho,\beta} h_K^{s_K} |\alpha^{1/2}\nabla u|_{s_K,K} + osc(f). \tag{5.6}$$

To this end, the definition of $a_{d,\theta}$ gives that

$$\begin{aligned} a_{d,\theta}(E, v_h) &= (\alpha\nabla_h E, \nabla_h v_h) + \theta \sum_{e\in\mathcal{E}_I\cup\mathcal{E}_D} \int_e \{\alpha\nabla v_h \cdot \boldsymbol{n}_e\}_w [\![E]\!] \\ &\quad - \sum_{e\in\mathcal{E}_I\cup\mathcal{E}_D} \int_e \{\alpha\nabla E \cdot \boldsymbol{n}_e\}_w [\![v_h]\!] + \sum_{e\in\mathcal{E}_I\cup\mathcal{E}_D} \int_e \gamma_\theta h_e^{-1} W_e [\![E]\!][\![v_h]\!] \\ &:= I1 + I2 + I3 + I4 \end{aligned}$$

It is clear that Lemma 4.1 and the Cauchy-Schwarz inequality imply

$$I1 + I2 + I4 \lesssim |||E|||_d |||v_h|||_d.$$

Using Lemma 5.2, Lemma 5.3, and the Cauchy-Schwarz inequality, we have

$$\begin{aligned} I3 &\le \sum_{e\in\mathcal{E}_I\bigcup\mathcal{E}_D} h_e^{-1/2} W_e^{1/2} \|[\![v_h]\!]\|_{0,e} \times \sum_{\omega=+,-} \left(\|\alpha^{1/2}\nabla E\|_{0,K^\omega} + h_{K^\omega}\|\alpha^{1/2} \triangle E\|_{0,K^\omega}\right) \\ &\le \|v_h\|_j \left(\|\alpha^{1/2}\nabla_h E\|_{0,\Omega} + \sum_{K\in\mathcal{T}_h} h_K \|\alpha_K^{1/2} \triangle E\|_{0,K}\right) \\ &\le |||v_h|||_d \left(|||E|||_d + \sum_{K\in\mathcal{T}_h} \frac{h_K}{\sqrt{\alpha_K}} \|\nabla\cdot(\beta E) + \gamma E\|_{0,K} + osc(f)\right). \end{aligned}$$

Summing up all the terms gives that

$$a_{d,\theta}(E, v_h) \lesssim |||v_h|||\left(|||E|||_d + \sum_{K\in\mathcal{T}_h} \frac{h_K}{\sqrt{\alpha_K}} \|\nabla\cdot(\beta E) + \gamma E\|_{0,K} + osc(f)\right).$$

It follows from (2.5), (2.7), (5.1)-(5.3) and the fact that $h/L < 1$ that

$$\begin{aligned} \|\nabla\cdot(\beta E) + \gamma E\|_{0,K} &= \|\rho E + E\nabla\cdot\beta/2 + \beta\cdot\nabla E\|_{0,K} \\ &\lesssim (\|\rho\|_{0,\infty,\Omega} + |\beta|_{1,\infty})\|E\|_{0,K} + \|\beta\|_{0,\infty,\Omega}|E|_{1,K} \\ &\lesssim h^{1+s_K}\|\rho\|_{0,\infty,\Omega}|u|_{1+s_K,K} + h^{s_K}\|\beta\|_{0,\infty,\Omega}|u|_{1+s_K,K} \end{aligned}$$

and that

$$|||E|||_d \lesssim \sum_{K\in\mathcal{T}_h} h_K^{s_K}|\alpha^{1/2}\nabla u|_{s_K,K},$$

which implies (5.5).

Next we show the validity of (5.6). The definition of a_c gives

$$a_c(E, v_h) = (E, -\beta\cdot\nabla_h v_h + \gamma v_h) + \sum_{e\in\mathcal{E}_I} \int_e \{\beta_e E\}_{up}[\![v_h]\!] + \sum_{e\in\mathcal{E}_{\Gamma^+}} \int_e \beta_e E\, v_h.$$

Since $P_h\left(\beta\cdot\nabla_h v_h\right) \in \mathcal{U}_h^k$, we have

$$\int_\Omega P_h\left(\beta\cdot\nabla_h v_h\right) E\, dx = \int_\Omega P_h\left(\beta\cdot\nabla_h v_h\right)(u - P_h u)\, dx = 0.$$

Together with (2.7), the inverse inequality, and (5.1)-(5.2), we have

$$\begin{aligned} \int_\Omega -\beta\cdot\nabla_h v_h E &= \int_\Omega (P_h\beta - \beta)\cdot\nabla_h v_h E \\ &\lesssim h|\beta|_{1,\infty,\Omega}\|\nabla_h v_h\|_{0,\Omega}\|E\|_{0,\Omega} \\ &\lesssim \|b_0^{1/2} v_h\|_{0,\Omega}\|b_0^{1/2}E\|_{0,\Omega} \\ &\lesssim |||v_h|||_c \sum_{K\in\mathcal{T}_h} h_K^{1+s_K}\|\beta\|_{0,\infty,\Omega}^{1/2}|u|_{1+s_K,K}. \end{aligned}$$

Applying $\gamma = \rho - \frac{1}{2}\nabla\cdot\beta$, (2.6), (2.7), and (5.1)-(5.3) gives that

$$\begin{aligned} (E, \gamma v_h) &= \int_\Omega (\rho - \frac{1}{2}\nabla\cdot\beta)\, E\, v_h\, dx \\ &\lesssim c_\rho\|E\|_{0,\Omega}\|(\overline{\rho} + b_0)v_h\|_{0,\Omega} + \frac{b_0}{c_\beta}\|E\|_{0,\Omega}\|v_h\|_{0,\Omega} \\ &\lesssim |||v_h|||_c \sum_{K\in\mathcal{T}_h} \left(\|\overline{\rho}\|_{0,\Omega} + \|\beta\|_{0,\infty,\Omega}\right)^{1/2} h_K^{1+s_K}|u|_{1+s_K,K} \end{aligned}$$

and that

$$\begin{aligned} &\sum_{e\in\mathcal{E}_I} \int_e \{\beta_e E\}_{up}[\![v_h]\!] + \sum_{e\in\mathcal{E}_{\Gamma^+}} \int_e \beta_e E v_h \\ &\lesssim |||v_h|||_c \sum_{K\in\mathcal{T}_h} h_K^{1/2+s_K}\|\beta\|_{0,\infty,\Omega}^{1/2}|u|_{1+s_K,K}. \end{aligned}$$

Now, (5.6) is a direct consequence of the above three inequalities. This completes the proof of the theorem. □

6. A new discontinuous Galerkin method

In section 4, we stabilize the diffusion operator by adding the following equation:

$$\sum_{e\in\mathcal{E}_I\cup\mathcal{E}_D} \int_e \gamma_\theta h_e^{-1} W_e [\![u]\!][\![v]\!]\, ds = \sum_{e\in\mathcal{E}_D} \gamma_\theta h_e^{-1} W_e \int_e g_D v\, ds,\ \forall\, v \in V^{1+\epsilon}(\mathcal{T}_h).$$

The order h_e^{-1} may lead to the difficulty in the convergence analysis.

Considering this, for any $v \in V^{1+\epsilon}(\mathcal{T}_h)$, denote the tangential derivative along edge e by

$$\gamma_e(\nabla v) = \frac{\partial v}{\partial t}.$$

And for any $v \in V^{1+\epsilon}(\mathcal{T}_h)$, we add the following term to stabilize:

$$\sum_{e\in\mathcal{E}_I\cup\mathcal{E}_D} \int_e \gamma_\theta h_e W_e [\![\gamma_e(\nabla u)]\!][\![\gamma_e(\nabla v)]\!]\, ds = \sum_{e\in\mathcal{E}_D} \gamma_\theta h_e W_e \int_e \gamma_e(\nabla g_D)\gamma_e(\nabla v)\, ds.$$

Now, define the new bilinear form for $u, v \in V^{1+\epsilon}(\mathcal{T}_h)$ by

$$\begin{aligned} \hat{a}_{d,\theta}(u, v) &= (\alpha\nabla_h u, \nabla_h v) + \sum_{e\in\mathcal{E}_I\cup\mathcal{E}_D} \int_e \gamma_\theta h_e W_e [\![\gamma_e(\nabla u)]\!][\![\gamma_e(\nabla v)]\!]\, ds \\ &\quad + \theta \sum_{e\in\mathcal{E}_I\cup\mathcal{E}_D} \int_e \{\alpha\nabla v\cdot\boldsymbol{n}_e\}_w [\![u]\!]\, ds - \sum_{e\in\mathcal{E}_I\cup\mathcal{E}_D} \int_e \{\alpha\nabla u\cdot\boldsymbol{n}_e\}_w [\![v]\!]\, ds \end{aligned}$$

for $\theta \in \{-1, 0, 1\}$. And define the new linear form for $v \in V^{1+\epsilon}(\mathcal{T}_h)$ by

$$\hat{f}_\theta(v) = (f, v) + \sum_{e\in\mathcal{E}_D} \gamma_\theta h_e W_e \int_e \gamma_e(\nabla g_D)\gamma_e(\nabla v)\,ds + \sum_{e\in\mathcal{E}_N} \int_e g_N v\,ds$$
$$+\theta \sum_{e\in\mathcal{E}_D} \int_e g_D(k\nabla v\cdot \boldsymbol{n}_e)\,ds - \sum_{e\in\mathcal{E}_{D^-}} \int_e (\boldsymbol{\beta}\cdot\boldsymbol{n}_e) g_D\, v\,ds.$$

The new variational formulation is to find $\hat{u} \in V^{1+\epsilon}(\mathcal{T}_h)$ such that

$$\hat{a}_\theta(\hat{u}, v) \equiv \hat{a}_{d,\theta}(\hat{u}, v) + a_c(\hat{u}, v) = \hat{f}_\theta(v), \quad \forall v \in V^{1+\epsilon}(\mathcal{T}_h).$$

To discretize the problem, modify the DG finite element space associated with the triangulation $\mathcal{T}_h$ as

$$\widehat{\mathcal{U}}_h^k = \{v \in L^2(\Omega) : v|_K \in P_k(K),\ \forall K \in \mathcal{T}_h \text{ and } [\![\overline{v}]\!]_e = 0,\ \forall e \in \mathcal{E}_I\},$$

where $\overline{v}_e = \frac{1}{|e|}\int_e v\,ds$ is the average of v on e.

The new DG finite element method is to find $\hat{u}_h \in \widehat{\mathcal{U}}_h^k$ such that

$$\hat{a}_\theta(\hat{u}_h, v) = \hat{f}_\theta(v), \quad \forall\ v \in \widehat{\mathcal{U}}_h^k.$$

For any $v \in \widehat{\mathcal{U}}_h^k$, define the norm for the modified DG space by

$$|||v|||_{dg}^2 = \|\alpha^{1/2}\nabla_h v\|_{0,\Omega}^2 + \|v\|_{dj}^2 + |||v|||_c^2,$$

where

$$\|v\|_{dj}^2 := \sum_{e\in\mathcal{E}_I\cup\mathcal{E}_D} h_e W_e \|[\![\gamma_e(\nabla v)]\!]\|_{0,e}^2.$$

The following lemma implies the equivalence between $\|[\![u]\!]\|$ and $h_e\|[\![\gamma_e(\nabla u)]\!]\|$ in the DG finite element space.

Lemma 6.1. *For any $v \in \widehat{\mathcal{U}_h^k}$ and any $e \in \mathcal{E}_I$, $\|[\![v]\!]\|_{0,e}$ and $h_e\|[\![\gamma_e(\nabla u)]\!]\|$ are equivalent, i.e., there exist positive constants c_m and c_M such that*

$$c_m\|[\![v]\!]\|_{0,e} \le h_e\|[\![\gamma_e(\nabla u)]\!]\| \le c_M\|[\![v]\!]\|_{0,e}.$$

Proof. By a scaling argument, it suffices to prove that $\|[\![\gamma_e(\nabla v)]\!]\| = 0$ implies that $v \equiv 0$ on e. It follows that

$$[\![\gamma_e(\nabla v]\!]_e = [\![\frac{\partial v}{\partial \boldsymbol{t}_e}]\!]_e = \frac{\partial}{\partial \boldsymbol{t}_e}[\![v]\!]_e = 0.$$

Hence, $[\![v]\!]_e$ is a constant, which implies that

$$[\![v]\!]_e = \overline{[\![v]\!]}_e = \frac{1}{|e|}\int_e [\![v]\!]_e\,ds = [\![\overline{v}]\!]_e = 0.$$

This completes the proof of the lemma. □

Corollary 6.2. *For any $v \in \widehat{\mathcal{U}}_h^k$, $a_{d,\theta}(v,v)$ and $\hat{a}_{d,\theta}(v,v)$ are equivalent.*

Data availability

No data was used for the research described in the article.

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