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Stabilized finite element methods with fast iterative solution algorithms for the Stokes problem

Zhiqiang Cai, Jim Douglas, Jr.*

Department of Mathematics, Purdue University, 1395 Mathematical Science Building, West Lafayette, IN 47907-1395, USA

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Abstract

This paper studies a new absolutely stabilized formulation for the Stokes problem that is a modification of the method of [13]. It is shown that the bilinear form is elliptic and continuous with respect to the H^1-norm for the velocity and the L^2-norm for the pressure. Optimal order error estimates for the finite element approximation of both the velocity and pressure in L^2 are established, as well as one in H^1 for the velocity. The formulation is nonsymmetric. We then introduce two symmetrized forms which retain ellipticity and continuity with respect to the same norm. It will be shown that the preconditioned conjugate gradient method and other existing iterative approaches can be applied with a convergence rate uniform in the number of unknowns. Also, modifications of other stabilized finite element methods are considered. © 1998 Elsevier Science S.A. All rights reserved.

1. Introduction

In recent years, so-called 'stabilization' techniques have been used extensively to stabilize unstable numerical methods for partial differential equations (see Baiocchi and Brezzi [2] for a general framework). While existing results indicate that such methods have great promise, a fast solver for the resulting algebraic equations has been missing for many such methods, possibly because of a too simple treatment of the stabilization term and the lack of symmetry of the schemes. The effect of the former is that the bilinear form is either non-elliptic or non-continuous with respect to norms separating velocity and pressure. The effect of the latter is that existing iterative methods cannot be applied directly. In this paper, we first describe a new absolutely stabilized finite element method for the Stokes problem, with the method being related to the approach of Douglas and Wang [13]. In it, a weighted L^2-inner product is replaced by the H^{-1}-inner product, which is further replaced by a discrete H^{-1}-inner product for feasible computation. Our bilinear form is then elliptic and continuous with respect to the H^1-norm for the velocity and the L^2-norm for the pressure, while that in [13] is not elliptic. Optimal order error estimates for the finite element approximation for both the velocity in the L^2- and H^1-norms and the pressure in the L^2-norm follow in a standard way. Then, we introduce two symmetrized forms of the method. One gives an augmented symmetric form which is indefinite; however, its Schur complement is elliptic and continuous with respect to the same norms. Hence, we can efficiently use existing iterative techniques with any effective elliptic preconditioner associated with velocity, including one of multigrid or domain-decomposition type, along with a simple preconditioner associated with pressure, such as one of diagonal matrix type. The other gives a symmetric and positive definite problem which is spectrally equivalent to the Schur complement of the augmented symmetric form. This can be solved by a preconditioned conjugate gradient method with the

* Corresponding author.

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same preconditioner. The condition numbers of both preconditioned problems are then uniform in the number of unknowns.

The paper is organized as follows. The Stokes problem is introduced in Section 2, along with some notation. We describe a new absolutely stabilized formulation and establish its ellipticity and continuity in Section 3. Its discrete counterpart and the corresponding finite element approximation are discussed in Section 4. Symmetrizations of this stabilized formulation and comments on existing iterative methods for such systems are presented in Section 5. Finally, we discuss implementation issues in Section 6, other stabilized finite element methods in Section 7, and the time-dependent Stokes problem in Section 8.

2. The Stokes problem and preliminaries

Let Ω be a bounded, open subset in $\mathscr{R}^d$ ($d = 2$ or 3) with Lipschitz boundary $\partial\Omega$. The stationary Stokes equations in dimensionless variables can be written as

$$\begin{cases} -\nu\,\Delta \boldsymbol{u} + \nabla p = \boldsymbol{f} & \text{in } \Omega\,, \\ \nabla\cdot \boldsymbol{u} = 0 & \text{in } \Omega\,, \\ \boldsymbol{u} = \boldsymbol{0} & \text{on } \partial\Omega\,, \end{cases} \tag{2.1}$$

where the symbols Δ, ∇ and $\nabla\cdot$ denote the Laplacian, gradient, and divergence operators, respectively, and $\boldsymbol{f}(x)$ is the external force per unit volume acting on the fluid at $x \in \Omega$. For simplicity, but without loss of generality, we assume that the viscosity ν is equal to one.

Let $\mathscr{D}(\Omega)$ be the linear space of infinitely differentiable functions with compact support on Ω. We use the standard notation and definitions for the Sobolev spaces $H^s(\Omega)^d$, the associated inner products $(\cdot\,,\cdot)_s$ and their respective norms $\|\cdot\|_s$, $s \geq 0$. (We suppress the designations d and Ω on the inner products and norms because the dependence on dimension and region will be clear by context.) The space $H^0(\Omega)^d$ coincides with $L^2(\Omega)^d$, in which case the norm and inner product are denoted by $\|\cdot\|$ and $(\cdot\,,\cdot)$, respectively. As usual, $H^s_0(\Omega)$ is the closure of $\mathscr{D}(\Omega)$ with respect to the norm $\|\cdot\|_s$, and $H^{-1}(\Omega)$ is the dual of $H^1_0(\Omega)$ with norm defined by

$$\|\varphi\|_{-1} = \sup_{0\neq\phi\in H^1_0(\Omega)} \frac{(\varphi, \phi)}{\|\phi\|_1}\,.$$

Finally, let $L^2_0(\Omega)$ denote the subspace of $L^2(\Omega)$ consisting of all such functions in $L^2(\Omega)$ having mean value zero.

It is well known that (2.1) has a unique solution in

$$\mathscr{V} \equiv H^1_0(\Omega)^d \times L^2_0(\Omega)\,. \tag{2.2}$$

It will be convenient below to define the operator $A : H^{-1}(\Omega)^d \to H^1_0(\Omega)^d$ as the solution operator for the Poisson problem

$$\begin{cases} -\Delta\boldsymbol{\varphi} + \boldsymbol{\varphi} = \boldsymbol{v} & \text{in } \Omega\,, \\ \boldsymbol{\varphi} = \boldsymbol{0} & \text{on } \partial\Omega\,; \end{cases} \tag{2.3}$$

i.e., $A\boldsymbol{v} = \boldsymbol{\varphi}$ for a given $\boldsymbol{v} \in H^{-1}(\Omega)^d$ is the solution of (2.3). It is well known that $(A\cdot\,,\cdot)^{1/2}$ defines a norm that is equivalent to the H^{-1}-norm.

3. Stabilized formulation

We describe here a stabilized formulation of (2.1) that is essentially a continuous counterpart of that in [13] (see also [2]). The ellipticity of the associated bilinear form and the continuity of the associated bilinear and linear forms on $\mathscr{V}$ are established in Lemma 3.1. This will in turn imply the well-posedness of the stabilized formulation and its equivalence to (2.1).

For any $(\boldsymbol{u}, p)$ and any $(\boldsymbol{v}, q)$ in $\mathscr{V}$, we define the bilinear form

$$b(\boldsymbol{u}, p; \boldsymbol{v}, q) = b_s(\boldsymbol{u}, p; \boldsymbol{v}, q) + b_a(\boldsymbol{u}, p; \boldsymbol{v}, q) \tag{3.1}$$

and the linear form

$$f(\boldsymbol{v}, q) = (\boldsymbol{f}, \boldsymbol{v}) + \alpha(A\boldsymbol{f}, -\Delta \boldsymbol{v} + \nabla q).$$

Then, $b_s(\cdot\,,\cdot)$ and $b_a(\cdot\,;\cdot)$ are, respectively, the symmetric and skew-symmetric parts of $b(\cdot\,;\cdot)$:

$$b_s(\boldsymbol{u}, p; \boldsymbol{v}, q) = (\nabla \boldsymbol{u}, \nabla \boldsymbol{v}) + \alpha(A(-\Delta \boldsymbol{u} + \nabla p), -\Delta \boldsymbol{v} + \nabla q), \tag{3.2}$$

$$b_a(\boldsymbol{u}, p; \boldsymbol{v}, q) = -(p, \nabla \cdot \boldsymbol{v}) + (\nabla \cdot \boldsymbol{u}, q); \tag{3.3}$$

$\alpha > 0$ is the stabilization parameter.

The stabilized formulation to be considered in this section is to find $(\boldsymbol{u}, p) \in \mathcal{V}$ such that

$$b(\boldsymbol{u}, p; \boldsymbol{v}, q) = f(\boldsymbol{v}, q), \quad (\boldsymbol{v}, q) \in \mathcal{V}. \tag{3.4}$$

The bilinear form $b(\boldsymbol{v}, q; \boldsymbol{v}, q)$ with $\alpha = 0$ is not stable, since it does not control q. Hence, the α-term in $b(\boldsymbol{v}, q; \boldsymbol{v}, q)$ adds stability to the bilinear form while that in $f(\boldsymbol{v}, q)$ maintains consistency. The description 'absolutely stabilized finite element method' refers here to allowing α to be an arbitrary positive number. Conditionally stabilized finite element methods are discussed in Section 6.

The bilinear and linear forms in (3.4) involve the H^{-1}-inner product, which requires solution of a boundary value problem for its evaluation. In the literature of stabilized finite element methods, this inner product has often been replaced by a simple, weighted L^2-inner product based on a local inverse inequality on the finite element space. The effect of this, too simple, treatment is that the resulting bilinear form is either not elliptic or not continuous with respect to any norm separating the velocity, $\boldsymbol{u}$, and the pressure, p. We will, instead, make use of a computationally feasible discrete H^{-1}-inner product to be defined in the next section.

Below, we will use C with or without subscripts to denote a generic positive constant, possibly different at different occurrences, which is independent of the mesh size h introduced in the subsequent section but may depend on the domain Ω and stabilization parameters.

LEMMA 3.1. The bilinear form $b(\cdot\,;\cdot)$ is elliptic and continuous in $\mathcal{V}$; i.e. there exists a positive constant C such that

$$\frac{1}{C}(\|\boldsymbol{v}\|_1^2 + \|q\|^2) \leq b(\boldsymbol{v}, q; \boldsymbol{v}, q) \tag{3.5}$$

and

$$b(\boldsymbol{u}, p; \boldsymbol{v}, q) \leq C(\|\boldsymbol{u}\|_1^2 + \|p\|^2)^{1/2}(\|\boldsymbol{v}\|_1^2 + \|q\|^2)^{1/2}, \tag{3.6}$$

for all $(\boldsymbol{u}, p)$ and $(\boldsymbol{v}, q)$ in $\mathcal{V}$. Moreover, the bilinear form $b_s(\cdot\,;\cdot)$ is elliptic and continuous and the linear form $f(\cdot)$ continuous on $\mathcal{V}$.

PROOF. By the Pôincaré–Friedrichs inequality, to show the validity of (3.5), it suffices to prove that, for any $(\boldsymbol{v}, q) \in \mathcal{V}$,

$$\|q\| \leq C(\|\nabla \boldsymbol{v}\| + \|-\Delta \boldsymbol{v} + \nabla q\|_{-1}). \tag{3.7}$$

Since

$$\|\Delta \boldsymbol{v}\|_{-1} = \sup_{\boldsymbol{\varphi} \in H_0^1(\Omega)^d} \frac{|(\Delta \boldsymbol{v}, \boldsymbol{\varphi})|}{\|\boldsymbol{\varphi}\|_1} = \sup_{\boldsymbol{\varphi} \in H_0^1(\Omega)^d} \frac{|(\nabla \boldsymbol{v}, \nabla \boldsymbol{\varphi})|}{\|\boldsymbol{\varphi}\|_1} \leq \|\nabla \boldsymbol{v}\|, \tag{3.8}$$

(3.7) is an immediate consequence of the triangle inequality and the fact (see [17]) that

$$\|q\| \leq C\|\nabla q\|_{-1}, \quad q \in L_0^2(\Omega). \tag{3.9}$$

To show (3.6), first note that, for $\boldsymbol{v}, \boldsymbol{\psi} \in H^{-1}(\Omega)^d$,

$$|(A\boldsymbol{v}, \boldsymbol{\psi})| \leq \|A\boldsymbol{v}\|_1 \|\boldsymbol{\psi}\|_{-1} \leq C\|\boldsymbol{v}\|_{-1}\|\boldsymbol{\psi}\|_{-1}. \tag{3.10}$$

The second inequality above follows from the H^1-regularity bound of the Poisson problem. Now, (3.6) is straightforward from (3.10) and the Cauchy–Schwarz and triangle inequalities. Continuity of the bilinear form $b_s(\cdot\,;\cdot)$ and the linear form $f(\cdot)$ in $\mathcal{V}$ can be shown by a similar argument. Ellipticity of the bilinear form $b_s(\cdot\,;\cdot)$ follows from (3.5) and the fact that $b_s(\boldsymbol{v}, q; \boldsymbol{v}, q) = b(\boldsymbol{v}, q; \boldsymbol{v}, q)$. This completes the proof of the lemma. □

An immediate consequence of the Lax–Milgram Theorem and Lemma 3.1 is the following theorem.

THEOREM 3.1. Problem (3.4) has a unique solution in $\mathcal{V}$.

Problems (2.1) and (3.4) are equivalent (see [2]).

4. Finite element approximation

This section presents an absolutely stabilized finite element formulation for the Stokes problem based on (3.4). We first discuss the well-posedness of the discrete problem and then establish optimal order error estimates in the L^2- and H^1-norms for the velocity and the L^2-norm for the pressure. For convenience of argument, we assume that the domain Ω is a polygonal domain in R^d and that the finite element space for the pressure is continuous. However, extensions to more general domains and discontinuous finite element spaces for the pressure can be made without difficulty; the appropriate modification of the method for discontinuous pressure spaces is addressed at the end of this section.

We will approximate the solution of (3.4) by using a Galerkin-type finite element method. Let $\mathcal{T}_h$ be a partition of Ω into finite elements; i.e., let $\Omega = \cup_{K\in\mathcal{T}_h} K$ with $h = \max\{\mathrm{diam}(K) : K \in \mathcal{T}_h\}$. Assume the triangulation $\mathcal{T}_h$ to be quasiuniform (see [12]). Let $\mathscr{C}^0(\Omega)$ be the space of continuous functions on Ω and let $\mathcal{V}^h = \boldsymbol{U}^h \times P^h$ be a finite-dimensional subspace of $H_0^1(\Omega)^d \times (L_0^2(\Omega) \cap \mathscr{C}^0(\Omega))$ such that, for any $(\boldsymbol{v}, q) \in (H^{r+1}(\Omega)^d \times H^r(\Omega)) \cap \mathcal{V}$, there exists an interpolant of $(\boldsymbol{v}, q)$, denoted by $(\boldsymbol{v}^I, q^I)$, in $\mathcal{V}^h$ such that

$$\|\boldsymbol{v} - \boldsymbol{v}^I\| + h\|\boldsymbol{v} - \boldsymbol{v}^I\|_1 \le Ch^{r+1}\|\boldsymbol{v}\|_{r+1}\,, \tag{4.1}$$

$$\sum_{K\in\mathcal{T}_h} h_K\|\Delta(\boldsymbol{v} - \boldsymbol{v}^I)\|_{0,K} \le Ch^r\|\boldsymbol{v}\|_{r+1}\,, \tag{4.2}$$

$$\|q - q^I\| + h\|q - q^I\|_1 \le Ch^r\|q\|_r\,, \tag{4.3}$$

where $r \ge 0$ for (4.1) and $r \ge 1$ for (4.2) and (4.3). It is well known that (4.1), (4.2), and (4.3) hold for typical finite element spaces consisting of continuous piecewise polynomials with respect to quasiuniform triangulations (cf. [12]).

As mentioned earlier, we need to replace the H^{-1}-inner product in (3.4) by a computationally feasible discrete inner product that ensures the equivalence on $\mathcal{V}^h$ between the standard norm in $\mathcal{V}$ and that induced by the discrete bilinear form. (A discrete H^{-1} approach was introduced in [4] for scalar elliptic equations and was extended to the Stokes problem in [11] in the context of first-order system least-squares methods.) So, let $A_h : H^{-1}(\Omega)^d \to \boldsymbol{U}^h$ be the discrete solution operator $\boldsymbol{\varphi} = A_h\boldsymbol{v} \in \boldsymbol{U}^h$ for the Poisson problem (2.3) defined by the relations

$$\int_\Omega (\nabla\boldsymbol{\varphi}\cdot\nabla\boldsymbol{\psi} + \boldsymbol{\varphi}\cdot\boldsymbol{\psi})\,\mathrm{d}x = (\boldsymbol{v}, \boldsymbol{\psi})\,, \quad \boldsymbol{\psi} \in \boldsymbol{U}^h\,. \tag{4.4}$$

It is easy to check that $(A_h\,\cdot\,, \cdot)^{1/2}$ defines a semi-norm on $H^{-1}(\Omega)^d$ which is equivalent to the discrete H^{-1} semi-norm

$$\|\cdot\|_{-1,h} \equiv \sup_{\boldsymbol{\phi}\in\boldsymbol{U}^h} \frac{(\cdot, \boldsymbol{\phi})}{\|\boldsymbol{\phi}\|_1}\,.$$

Assume that there is a preconditioner $B_h : H^{-1}(\Omega)^d \to \boldsymbol{U}^h$ that is symmetric with respect to the $L^2(\Omega)^d$-inner product and spectrally equivalent to A_h; i.e., there exists a positive constant C, independent of the mesh size h, such that

$$\frac{1}{C}(A_h\boldsymbol{v},\boldsymbol{v}) \le (B_h\boldsymbol{v},\boldsymbol{v}) \le C(A_h\boldsymbol{v},\boldsymbol{v}), \quad \boldsymbol{v} \in \boldsymbol{U}^h. \tag{4.5}$$

Finally, we introduce a 'discrete' Laplacian operator, $\Delta_h : H_0^1(\Omega)^d \to \boldsymbol{U}^h$; let $\boldsymbol{\varphi} = \Delta_h\boldsymbol{v} \in \boldsymbol{U}^h$ satisfy

$$(\boldsymbol{\varphi}, \boldsymbol{w}) = -(\nabla\boldsymbol{v}, \nabla\boldsymbol{w}), \quad \forall\, \boldsymbol{w} \in \boldsymbol{U}^h. \tag{4.6}$$

Define the discrete counterparts of the bilinear and linear forms as follows:

$$b^h(\boldsymbol{u}, p; \boldsymbol{v}, q) = b_s^h(\boldsymbol{u}, p; \boldsymbol{v}, q) + b_a(\boldsymbol{u}, p; \boldsymbol{v}, q), \tag{4.7}$$

where

$$\begin{aligned} b_s^h(\boldsymbol{u}, p; \boldsymbol{v}, q) = {} & (\nabla\boldsymbol{u}, \nabla\boldsymbol{v}) + \alpha_1(B_h(-\Delta_h\boldsymbol{u} + \nabla p), -\Delta_h\boldsymbol{v} + \nabla q) \\ & + \alpha_2 \sum_{K\in\mathcal{T}} h_K^2(-\Delta\boldsymbol{u} + \nabla p, -\Delta\boldsymbol{v} + \nabla q)_{0,K}, \end{aligned} \tag{4.8}$$

and

$$f^h(\boldsymbol{v}, q) = (\boldsymbol{f}, \boldsymbol{v}) + \alpha_1(B_h\boldsymbol{f}, -\Delta_h\boldsymbol{v} + \nabla q) + \alpha_2 \sum_{K\in\mathcal{T}_h} h_K^2(\boldsymbol{f}, -\Delta\boldsymbol{v} + \nabla q)_{0,K}. \tag{4.9}$$

Here, $(\cdot\,,\cdot)_{0,K}$ indicates the inner product in $L^2(K)$ and $\alpha_i > 0$ $(i = 1, 2)$ are stabilization parameters. For simplicity, we consider only the case when $\alpha_1 = \alpha_2$. The modified absolutely-stabilized finite element method is then to find an approximation, $(\boldsymbol{u}^h, p^h) \in \boldsymbol{U}^h \times P^h$, of (3.4) such that

$$b^h(\boldsymbol{u}^h, p^h; \boldsymbol{v}, q) = f^h(\boldsymbol{v}, q), \quad (\boldsymbol{v}, q) \in \boldsymbol{U}^h \times P^h. \tag{4.10}$$

The only difference between our approach and that in [13] is the addition of the B_h-terms which ensure the uniform ellipticity of $b^h(\cdot\,;\cdot)$ with respect to the standard norm of $\mathcal{V}$ (see Lemma 4.1), a property not shared by the bilinear form in [13].

LEMMA 4.1. The bilinear form $b^h(\cdot\,;\cdot)$ is elliptic in $\mathcal{V}^h$; i.e., there exists a positive constant C independent of h such that, for any $(\boldsymbol{v}, q)$ in $\mathcal{V}^h$,

$$\|\boldsymbol{v}\|_1^2 + \|q\|^2 \le C b^h(\boldsymbol{v}, q; \boldsymbol{v}, q). \tag{4.11}$$

Moreover, for any $(\boldsymbol{v}, q) \in \mathcal{V}^h$, there exists a positive constant C, independent of h, such that

$$b^h(\boldsymbol{u}, p; \boldsymbol{v}, q) \le C(\|\boldsymbol{u}\|_1^2 + \|p\|^2)^{1/2}(\|\boldsymbol{v}\|_1^2 + \|\boldsymbol{q}\|^2)^{1/2} \tag{4.12}$$

for all $(\boldsymbol{u}, p)$ in $\mathcal{V}^h$ and

$$|f^h(\boldsymbol{v}, q)| \le C(\|\boldsymbol{v}\|_1 + \|q\|). \tag{4.13}$$

PROOF. Again by the Poincaré–Friedrichs inequality, to show the validity of (4.11), it suffices to prove that, for any $(\boldsymbol{v}, q) \in \mathcal{V}^h$,

$$\|q\|^2 \le C b^h(\boldsymbol{v}, q; \boldsymbol{v}, q). \tag{4.14}$$

To do so, let $Q_h : L^2(\Omega)^d \to \boldsymbol{U}^h$ denote the $L^2(\Omega)^d$ projection onto $\boldsymbol{U}^h$. Then, (4.1) implies that, for all $\boldsymbol{v}$ in $H_0^1(\Omega)^d$,

$$\|\boldsymbol{v} - Q_h\boldsymbol{v}\| \le Ch\|\boldsymbol{v}\|_1 \quad \text{and} \quad \|Q_h\boldsymbol{v}\|_1 \le C\|\boldsymbol{v}\|_1. \tag{4.15}$$

For any $\boldsymbol{\varphi} \in L^2(\Omega)^d$, a standard duality argument implies that

$$\|\boldsymbol{\varphi} - Q_h\boldsymbol{\varphi}\|_{-1} \le Ch\|\boldsymbol{\varphi}\| \le C\Big(\sum_{K\in\mathcal{T}_h} h_K^2\|\boldsymbol{\varphi}\|_{0,K}^2\Big)^{1/2}$$

and

$$\|Q_h\boldsymbol{\varphi}\|_{-1} = \sup_{\boldsymbol{v}\in H_0^1(\Omega)^d} \frac{(Q_h\boldsymbol{\varphi}, \boldsymbol{v})}{\|\boldsymbol{v}\|_1} \leqslant C \sup_{\boldsymbol{v}\in H_0^1(\Omega)^d} \frac{(\boldsymbol{\varphi}, Q_h\boldsymbol{v})}{\|Q_h\boldsymbol{v}\|_1} = C\|\boldsymbol{\varphi}\|_{-1,h}\,.$$

Hence,

$$\|\boldsymbol{\varphi}\|_{-1}^2 \leqslant C\Big(\sum_{K\in\mathcal{T}_h} h_K^2\|\boldsymbol{\varphi}\|_{0,K}^2 + \|\boldsymbol{\varphi}\|_{-1,h}^2\Big),$$

which, together with the choice $\boldsymbol{\varphi} = \nabla q$ and the inequality (3.9), implies that

$$\|q\|^2 \leqslant C\Big(\sum_{K\in\mathcal{T}_h} h_K^2\|\nabla q\|_{0,K}^2 + \|\nabla q\|_{-1,h}^2\Big), \quad q\in P^h\,. \tag{4.16}$$

It follows from (4.6) that

$$\|\Delta_h\boldsymbol{v}\|_{-1,h} = \sup_{\boldsymbol{\varphi}\in U^h} \frac{|(\Delta_h\boldsymbol{v}, \boldsymbol{\varphi})|}{\|\boldsymbol{\varphi}\|_1} = \sup_{\boldsymbol{\varphi}\in U^h} \frac{|(\nabla\boldsymbol{v}, \nabla\boldsymbol{\varphi})|}{\|\boldsymbol{\varphi}\|_1} \leqslant \|\nabla\boldsymbol{v}\|\,, \tag{4.17}$$

so that the triangle and inverse inequalities (see [12]) and the assumption (4.5) imply that

$$\begin{aligned}\|q\|^2 &\leqslant C\Big(\sum_{K\in\mathcal{T}_h} h_K^2(\|\nabla q - \Delta\boldsymbol{v}\|_{0,K}^2 + \|\Delta\boldsymbol{v}\|_{0,K}^2) + \|\nabla q - \Delta_h\boldsymbol{v}\|_{-1,h}^2 + \|\Delta_h\boldsymbol{v}\|_{-1,h}^2\Big)\\ &\leqslant Cb^h(\boldsymbol{v}, q; \boldsymbol{v}, q)\,.\end{aligned}$$

This establishes the validity of (4.14) and, hence, (4.11). Since $B_h = B_hQ_h$ and $A_h = A_hQ_h$, (4.5) is valid for all $\boldsymbol{v}\in L^2(\Omega)^d$. Also, since, for any $\boldsymbol{\varphi}$ and $\boldsymbol{\psi}$ in $L^2(\Omega)^d$,

$$(B_h\boldsymbol{\varphi}, \boldsymbol{\psi}) \leqslant C(B_h\boldsymbol{\varphi}, \boldsymbol{\varphi})^{1/2}(B_h\boldsymbol{\psi}, \boldsymbol{\psi})^{1/2}$$

(4.12) and (4.13) then follow from (4.5), the Cauchy–Schwarz, triangle, and inverse inequalities, and a proof similar to that given for Lemma 3.1. This finishes the proof of the lemma.

The following theorem is an immediate consequence of Lemma 4.1 and the Lax–Milgram Lemma.

THEOREM 4.1. Problem (4.10) has a unique solution in $\boldsymbol{U}^h \times P^h$.

Let us turn to the convergence of the discrete solution.

THEOREM 4.2. Let $(\boldsymbol{u}, p)$ *and* $(\boldsymbol{u}^h, p^h)$ *be the solutions of (2.1) and (4.10), respectively. Assume that* $(\boldsymbol{u}, p)$ *is in* $H^{r+1}(\Omega)^d \times H^r(\Omega)$ *with* $r \geqslant 1$. *Then,*

$$\|\boldsymbol{u} - \boldsymbol{u}^h\|_1 + \|p - p^h\| \leqslant Ch^r(\|\boldsymbol{u}\|_{r+1} + \|p\|_r)\,. \tag{4.18}$$

If, in addition $(\boldsymbol{u}, p)$ *is* H^2*-regular (i.e.,* $\|\boldsymbol{u}\|_2 + \|p\|_1 \geqslant C\|\boldsymbol{f}\|$; *see [21]), then*

$$\|\boldsymbol{u} - \boldsymbol{u}^h\| \leqslant Ch^{r+1}(\|\boldsymbol{u}\|_{r+1} + \|p\|_r)\,. \tag{4.19}$$

PROOF. It is easy to check that the approximation error, $(\boldsymbol{u} - \boldsymbol{u}^h, p - p^h)$, satisfies the error equation

$$b^h(\boldsymbol{u} - \boldsymbol{u}^h, p - p^h; \boldsymbol{v}, q) = 0\,, \quad (\boldsymbol{v}, q)\in \mathcal{V}^h\,. \tag{4.20}$$

Let $(\boldsymbol{u}^I, p^I)\in\mathcal{V}^h$ be an interpolant of $(\boldsymbol{u}, p)$ satisfying (4.1), (4.2) and (4.3). Then, the triangle inequality gives

$$\|\boldsymbol{u} - \boldsymbol{u}^h\|_1 + \|p - p^h\| \leqslant Ch^r(\|\boldsymbol{u}\|_{r+1} + \|p\|_r) + \|\boldsymbol{u}^I - \boldsymbol{u}^h\|_1 + \|p^I - p^h\|\,.$$

Hence, it is left for us to show that

$$\|\boldsymbol{u}^I - \boldsymbol{u}^h\|_1 + \|p^I - p^h\| \leqslant Ch^r(\|\boldsymbol{u}\|_{r+1} + \|p\|_r)\,. \tag{4.21}$$

It follows from the stability inequality (4.11), the error equation (4.20), (4.5), (4.6) and the Cauchy–Schwarz, triangle, and inverse inequalities that

$$
\begin{aligned}
\|\boldsymbol{u}^I - \boldsymbol{u}^h\|_1^2 + \|p^I - p^h\|^2 &\leq Cb^h(\boldsymbol{u}^h, p^I - p^h; \boldsymbol{u}^I - \boldsymbol{u}^h, p^I - p^h) \\
&= Cb^h(\boldsymbol{u}^I - \boldsymbol{u}, p^I - p; \boldsymbol{u}^I - \boldsymbol{u}^h, p^I - p^h) \\
&\leq C\|\boldsymbol{u}^I - \boldsymbol{u}\|_1 \|\boldsymbol{u}^I - \boldsymbol{u}^h\|_1 \\
&\quad + C\|-\Delta_h(\boldsymbol{u}^I - \boldsymbol{u}) + \nabla(p^I - p)\|_{-1,h} \| - \Delta_h(\boldsymbol{u}^I - \boldsymbol{u}^h) + \nabla(p^I - p^h)\|_{-1,h} \\
&\quad + C \sum_{K \in \mathcal{T}_h} h_K^2 \| - \Delta(\boldsymbol{u}^I - \boldsymbol{u}) + \nabla(p^I - p)\|_{0,K} \| - \Delta(\boldsymbol{u}^I - \boldsymbol{u}^h) + \nabla(p^I - p^h)\|_{0,K} \\
&\quad + C\|\nabla \cdot (\boldsymbol{u}^I - \boldsymbol{u}^h)\| \, \|p^I - p\| + C\|\nabla \cdot (\boldsymbol{u}^I - \boldsymbol{u})\| \, \|p^I - p^h\| \\
&\leq C(\|\boldsymbol{u}^I - \boldsymbol{u}^h\|_1 + \|p^I - p^h\|)\Big(\|\boldsymbol{u}^I - \boldsymbol{u}\|_1 + \|p^I - p\| + h\|p^I - p\|_1 + \sum_{K \in \mathcal{T}_h} h_K \|\Delta(\boldsymbol{u}^I - \boldsymbol{u})\|_{0,K}\Big),
\end{aligned}
$$

which, together with (4.1), (4.2) and (4.3), implies (4.21) and hence (4.18). The L^2-error estimate (4.19) can be established by an argument similar to that in [13]. This completes the proof of the theorem. □

The continuity assumption on P^h can be removed if we modify and amplify the bilinear form $b^h(\cdot\,;\cdot)$ and the linear form $f^h(\cdot)$ as follows (see [13]):

$$\tilde{b}^h(\boldsymbol{u}, p; \boldsymbol{v}, q) = \tilde{b}_s^h(\boldsymbol{u}, p; \boldsymbol{v}, q) + b_a(\boldsymbol{u}, p; \boldsymbol{v}, q) + \alpha_3 \sum_{E \in \mathcal{E}^h} h_E \int_E [\![p]\!][\![q]\!] \, \mathrm{d}s$$

and

$$\tilde{f}^h(\boldsymbol{v}, q) = (f, \boldsymbol{v}) + \alpha_1(B_h f, -\Delta_h \boldsymbol{v} + \nabla_h q) + \alpha_2 \sum_{K \in \mathcal{T}_h} h_K^2 (f, -\Delta \boldsymbol{v} + \nabla q)_{0,K}$$

where

$$\tilde{b}_s^h(\boldsymbol{u}, p; \boldsymbol{v}, q) = (\nabla \boldsymbol{u}, \nabla \boldsymbol{v}) + \alpha_1(B_h(-\Delta_h \boldsymbol{u} + \nabla_h p), -\Delta_h \boldsymbol{v} + \nabla_h q) + \alpha_2 \sum_{K \in \mathcal{T}_h} h_K^2 (-\Delta \boldsymbol{u} + \nabla p, -\Delta \boldsymbol{v} + \nabla q)_{0,K} .$$

Here, $\mathcal{E}^h$ is the collection of interior edges of the triangulation $\mathcal{T}_h$, h_E is the diameter of the edge E in $\mathcal{E}^h$, $[\![p]\!]$ denotes the jump of p across edges, $\alpha_3 > 0$ is another stabilization parameter, and $\nabla_h : L^2(\Omega) \to \boldsymbol{U}^h$ is a 'discrete' gradient operator defined by

$$(\nabla_h q, \boldsymbol{w}) = -(q, \nabla \cdot \boldsymbol{w}), \quad \forall\, \boldsymbol{w} \in \boldsymbol{U}^h .$$

5. Symmetrization procedures

The discrete, stabilized formulation (4.10) is nonsymmetric since the bilinear forms $b_s^h(\cdot\,;\cdot)$ and $b_a(\cdot\,;\cdot)$ are symmetric and skew-symmetric, respectively. Furthermore, the pseudo-differential order of the skew-symmetric form is the same as that of the symmetric form. Hence, perturbation arguments for iterative methods for nonsymmetric problems arising from discretizations of scalar elliptic equations do not apply. These remarks are true, in general, for systems of linear equations arising from stabilized finite element discretizations (see Section 6). In this section, we first symmetrize the problem by a procedure (not least squares) which has been used in other fields of mathematics for different purposes (for example, see [16]) and show that the Schur complement retains ellipticity and continuity with respect to the norm in $\mathcal{V}$, thus guaranteeing the efficiency of existing iterative methods. The second symmetrization procedure is a least-squares-like approach and gives a symmetric, positive-definite problem spectrally equivalent to the symmetric part of the original problem. Hence, there exist preconditioned conjugate gradient methods that can be applied with convergence uniform in the number of unknowns.

Introducing the adjoint bilinear form of $b^h(\cdot\,;\cdot)$ by

$$b_h^*(\boldsymbol{u}, p; \boldsymbol{v}, q) = b_s^h(\boldsymbol{u}, p; \boldsymbol{v}, q) - b_a(\boldsymbol{u}, p; \boldsymbol{v}, q) \quad \text{for } (\boldsymbol{u}, p), (\boldsymbol{v}, q) \in \mathcal{V}^h .$$

The problem dual to (4.10) is to find $(\boldsymbol{u}^*, p^*) \in \mathcal{V}^h$ such that

$$b_h^*(\boldsymbol{u}^*, p^*; \boldsymbol{v}, q) = f^h(\boldsymbol{v}, q), \quad (\boldsymbol{v}, q) \in \mathcal{V}^h . \tag{5.1}$$

If $(\boldsymbol{u}^h, p^h)$ and $(\boldsymbol{u}^*, p^*)$ be the solutions of (4.10) and (5.1), respectively, we define $(\boldsymbol{u}_+, p_+)$ and $(\boldsymbol{u}_-, p_-)$ by

$$(\boldsymbol{u}_+, p_+) = \frac{1}{2}(\boldsymbol{u}^h + \boldsymbol{u}^*, p^h + p^*) \quad \text{and} \quad (\boldsymbol{u}_-, p_-) = \frac{1}{2}(\boldsymbol{u}^h - \boldsymbol{u}^*, p^h - p^*) .$$

By adding and subtracting (4.10) and (5.1), we see that $(\boldsymbol{u}_+, p_+)$ and $(\boldsymbol{u}_-, p_-)$ satisfy the symmetric, coupled system

$$\begin{cases} b_s^h(\boldsymbol{u}_+, p_+; \boldsymbol{v}, q) + b_a(\boldsymbol{u}_-, p_-; \boldsymbol{v}, q) = f^h(\boldsymbol{v}, q), \\ b_s^h(\boldsymbol{u}_-, p_-; \boldsymbol{v}, q) + b_a(\boldsymbol{u}_+, p_+; \boldsymbol{v}, q) = 0, \end{cases} \tag{5.2}$$

for $(\boldsymbol{v}, q) \in \mathcal{V}^h$. Let B_s and B_a be operators associated with the bilinear forms $b_s^h(\cdot\,;\cdot)$ and $b_a(\cdot\,;\cdot)$; i.e., for $(\boldsymbol{v}, q)$ and $(\boldsymbol{w}, r)$ in $\mathcal{V}^h$,

$$(B_s(\boldsymbol{v}, q); \boldsymbol{w}, r) = b_s^h(\boldsymbol{v}, q; \boldsymbol{w}, r), \qquad (B_a(\boldsymbol{v}, q); \boldsymbol{w}, r) = b_a(\boldsymbol{v}, q; \boldsymbol{w}, r) . \tag{5.3}$$

Then, (5.2) can be rewritten as

$$\begin{cases} B_s(\boldsymbol{u}_+, p_+) + B_a(\boldsymbol{u}_-, p_-) = F, \\ B_s(\boldsymbol{u}_-, p_-) + B_a(\boldsymbol{u}_+, p_+) = 0, \end{cases} \tag{5.4}$$

where the right-hand-side F is defined by

$$(F; \boldsymbol{v}, q) = f^h(\boldsymbol{v}, q), \quad (\boldsymbol{v}, q) \in \mathcal{V}^h .$$

Let $B_a^* = -B_a$ denote the adjoint operator of B_a with respect to the L^2-inner product. For convenience, let us write it in the block 2×2 matrix form

$$\begin{pmatrix} B_s & B_a \\ B_a^* & -B_s \end{pmatrix} \begin{pmatrix} (\boldsymbol{u}_+, p_+) \\ (\boldsymbol{u}_-, p_-) \end{pmatrix} = \begin{pmatrix} F \\ 0 \end{pmatrix}, \tag{5.5}$$

which is symmetric but indefinite. By block Gauss elimination,

$$\begin{cases} B_s(\boldsymbol{u}_+, p_+) = F - B_a(\boldsymbol{u}_-, p_-), \\ (B_s + B_a^* B_s^{-1} B_a)(\boldsymbol{u}_-, p_-) = B_a^* B_s^{-1} F, \end{cases} \tag{5.6}$$

where B_s^{-1} is the inverse operator of B_s. The operator $B_s + B_a^* B_s^{-1} B_a$ is the so-called Schur complement. One could solve the second equation for $(\boldsymbol{u}_-, p_-)$ and then solve the first equation for $(\boldsymbol{u}_+, p_+)$ by preconditioned conjugate gradient methods with possibly different preconditioners. The solution of (4.7) can be recovered by averaging $(\boldsymbol{u}_+, p_+)$ and $(\boldsymbol{u}_-, p_-)$; i.e.,

$$(\boldsymbol{u}^h, p) = \frac{1}{2}(\boldsymbol{u}_+ + \boldsymbol{u}_-, p_+ + p_-) .$$

This is a two-stage algorithm since each preconditioned conjugate gradient iteration for the second equation in (5.6) requires the solution of the linear system $B_s x = b$, which could also be replaced by an iterative method. There exist single-stage algorithms (see Remark 5.1); nevertheless, efficiency of those single-stage algorithms depends on the existence of good preconditioners for the operators B_s and $B_s + B_a^* B_s^{-1} B_a$; i.e., the same preconditioners as required by the simple algorithm above. Lemma 4.1 indicates that for the operator B_s we can use any effective elliptic preconditioner associated with velocity, including one of multigrid or domain-decomposition type, along with a simple preconditioner associated with pressure, such as one of diagonal matrix type. The condition number of the resulting preconditioned operator is independent of the mesh size h. The same or similar preconditioner can be used for $B_s + B_a^* B_s^{-1} B_a$ with the condition number again being uniformly bounded, as a consequence of the following theorem.

THEOREM 5.1. The operator $B_s + B_a^* B_s^{-1} B_a$ is elliptic and continuous in $\mathcal{V}^h$; i.e., there exists a positive constant C such that

$$\frac{1}{C}(\|\boldsymbol{v}\|_1^2 + \|q\|^2) \le ((B_s + B_a^* B_s^{-1} B_a)(\boldsymbol{v}, q); \boldsymbol{v}, q) \tag{5.7}$$

and

$$((B_s + B_a^* B_s^{-1} B_a)(\boldsymbol{v}, q); \boldsymbol{w}, r) \le C(\|\boldsymbol{v}\|_1^2 + \|q\|^2)^{1/2}(\|\boldsymbol{w}\|_1^2 + \|r\|^2)^{1/2}, \tag{5.8}$$

for all $(\boldsymbol{v}, q)$ and $(\boldsymbol{w}, r)$ in $\mathcal{V}^h$.

PROOF. By the definition of B_a and integration by parts,

$$B_a(\boldsymbol{v}, q) = (\nabla q, \nabla \cdot \boldsymbol{v}).$$

Hence,

$$\begin{aligned}((B_s + B_a^* B_s^{-1} B_a)(\boldsymbol{v}, q); \boldsymbol{w}, r) &= (B_s(\boldsymbol{v}, q); \boldsymbol{w}, r) + (B_s^{-1} B_a(\boldsymbol{v}, q); B_a(\boldsymbol{w}, r)) \\ &= b_s^h(\boldsymbol{v}, q; \boldsymbol{w}, r) + (B_s^{-1}(\nabla q, \nabla \cdot \boldsymbol{v}); \nabla r, \nabla \cdot \boldsymbol{w}).\end{aligned}$$

Since the second term above for $\boldsymbol{w} = \boldsymbol{v}$ and $r = q$ is always nonnegative, the ellipticity of the bilinear form $b_s^h(\boldsymbol{v}, q; \boldsymbol{v}, q) = b^h(\boldsymbol{v}, q; \boldsymbol{v}, q)$ in Lemma 4.1 implies the inequality (5.7). It follows from the Cauchy–Schwarz inequality that

$$(B_s^{-1}(\nabla q, \nabla \cdot \boldsymbol{v}); \nabla r, \nabla \cdot \boldsymbol{w}) \le C(B_s^{-1}(\nabla q, \nabla \cdot \boldsymbol{v}); \nabla q, \nabla \cdot \boldsymbol{v})^{1/2}(B_s^{-1}(\nabla r, \nabla \cdot \boldsymbol{w}); \nabla r, \nabla \cdot \boldsymbol{w})^{1/2}.$$

Hence, to show the validity of (5.8), it suffices to prove that

$$(B_s^{-1}(\nabla q, \nabla \cdot \boldsymbol{v}); \nabla q, \nabla \cdot \boldsymbol{v}) \le C(\|\boldsymbol{v}\|_1^2 + \|q\|^2). \tag{5.9}$$

To do so, let

$$(\boldsymbol{\varphi}, \phi) = B_s^{-1}(\nabla q, \nabla \cdot \boldsymbol{v}).$$

Then,

$$B_s(\boldsymbol{\varphi}, \phi) = (\nabla q, \nabla \cdot \boldsymbol{v});$$

its solution in $\mathcal{V}^h$ satisfies the standard H^1-regularity bound

$$\|\boldsymbol{\varphi}\|_1^2 + \|\phi\|^2 \le C(\|\nabla q\|_{-1}^2 + \|\nabla \cdot \boldsymbol{v}\|^2).$$

Now, it follows from Lemma 3.1 and the inequality $\|\nabla q\|_{-1}\| \le \|q\|$ that

$$\begin{aligned}(B_s^{-1}(\nabla q, \nabla \cdot \boldsymbol{v}); \nabla q, \nabla \cdot \boldsymbol{v}) &= (\boldsymbol{\varphi}, \phi; B_s(\boldsymbol{\varphi}, \phi)) = b_s^h(\boldsymbol{\sigma}, \phi; \boldsymbol{\varphi}, \phi) \\ &\le C(\|\boldsymbol{\varphi}\|_1^2 + \|\phi\|^2) \le C(\|\boldsymbol{v}\|_1^2 + \|q\|^2).\end{aligned}$$

This proves (5.9) and, hence, (5.8). This completes the proof of the theorem. □

REMARK 5.1. Iterative methods for similar systems arising from different applications than (5.5) have been studied by many researchers. The traditional Uzawa method (cf. [1]) can be regarded as a variant of the block Gauss–Seidel method in our application. Each iteration requires the solution of the system of linear equations with the coefficient matrix B_s; hence, it is again a two-stage algorithm. This motivates the inexact Uzawa method which either iteratively approximates the solution of the system with matrix B_s or replaces B_s by a preconditioner (see e.g. [15] and the references therein). Other approaches precondition the indefinite system either symmetric, positive-definitely (see [5]) or symmetric, indefinitely (see e.g. [23,24]) with possibly good condition numbers. One can then apply the conjugate gradient method to the definite system and the conjugate residual method to the indefinite system. The efficiency of these methods depends on the existence of good preconditioners for B_s and the Schur complement $B_a^* B_s^{-1} B_a + B_s$. For a comparison of the numerical performance of those approaches for some examples arising from mixed finite element discretization of the Stokes problem, see [14].

In the remainder of this section, we present a least-squares-like symmetrization. Problem (4.10) can be rewritten as

$$(B_s + B_a)(\boldsymbol{u}^h, p^h) = F . \tag{5.10}$$

According to Lemma 4.1, we can assume that there exists a preconditioner $B_0 : H^{-1}(\Omega)^{d+1} \to \mathcal{V}^h$ that is symmetric with respect to the $L^2(\Omega)^{d+1}$ inner product and spectrally equivalent to B_s; i.e., there exist positive constants β_0 and β_1, independent of the mesh size h, such that, for $(\boldsymbol{v}, q) \in \mathcal{V}^h$,

$$\beta_0(B_s(\boldsymbol{v}, q), \boldsymbol{v}, q) \leq (B_0(\boldsymbol{v}, q), \boldsymbol{v}, q) \leq \beta_1(B_s(\boldsymbol{v}, q), \boldsymbol{v}, q) . \tag{5.11}$$

Now, (5.10) can be symmetrized to

$$(B_s + B_a)^* B_0^{-1}(B_s + B_a)(\boldsymbol{u}^h, p^h) = (B_s + B_a)^* B_0^{-1} F . \tag{5.12}$$

THEOREM 5.2. The operator $(B_s + B_a)^* B_0^{-1}(B_s + B_a)$ *is spectrally equivalent to* $B_s + B_a^* B_s^{-1} B_a$; *i.e., for any* $(\boldsymbol{v}, q) \in \mathcal{V}^h$,

$$\begin{aligned} \frac{1}{\beta_1}((B_s + B_a^* B_s^{-1} B_a)(\boldsymbol{v}, q), \boldsymbol{v}, q) &\leq ((B_s + B_a)^* B_0^{-1}(B_s + B_a)(\boldsymbol{v}, q), \boldsymbol{v}, q) \\ &\leq \frac{1}{\beta_0}((B_s + B_a^* B_s^{-1} B_a)(\boldsymbol{v}, q), \boldsymbol{v}, q) . \end{aligned} \tag{5.13}$$

PROOF. The inequalities (5.13) follow immediately from (5.11) and the identity

$$(B_s + B_a)^* B_s^{-1}(B_s + B_a) = B_s + B_a^* B_s^{-1} B_a .$$

Theorems 5.1 and 5.2, Lemma 4.1 and (5.11) indicate that (5.10) can be efficiently preconditioned by the symmetric operator B_0 with the condition number bounded uniformly in the number of unknowns. Hence, the preconditioned conjugate gradient method converges uniformly in the number of unknowns.

6. Implementations

Implementations of the stabilized finite element method based on (4.10) and preconditioned iterations in Section 5 require mainly applications of the preconditioner and the operator to a given vector. We describe them in detail in this section.

For simplicity, let $\boldsymbol{U}^h$ and P^h consist of continuous, piecewise-linear functions in two dimensions. Extensions to higher order finite element subspaces and to three dimensions can be obtained in a similar fashion. Let φ_i be the usual nodal basis function associated with the ith node; then the nodal basis for the finite element product space $\mathcal{V}^h = \boldsymbol{U}^h \times P^h$ is the form

$$\mathrm{Span}\{(\varphi_{i_1}, 0, 0), (0, \varphi_{i_2}, 0), (0, 0, \varphi_{i_3}) : i_k = 1, \ldots N_k\} = \mathrm{Span}\{\psi^1_{i_1}, \psi^2_{i_2}, \psi^3_{i_3} : i_k = 1, \ldots N_k\} ,$$

where $N_1 = N_2$ is the number of internal nodes and N_3 is the total number of nodes (including boundary nodes). Also, set

$$N = N_1 + N_2 + N_3 .$$

If the solution of (4.10) is represented by

$$(\boldsymbol{u}^h, p^h) = \sum_{j=1}^{3} \sum_{i=1}^{N_j} U^j_i \boldsymbol{\psi}^j_i , \tag{6.1}$$

(4.10) has the following block matrix form:

$$\mathscr{B}\mathscr{U} \equiv \begin{pmatrix} \mathscr{B}_{11} & \mathscr{B}_{12} & \mathscr{B}_{13} \\ \mathscr{B}_{21} & \mathscr{B}_{22} & \mathscr{B}_{23} \\ \mathscr{B}_{31} & \mathscr{B}_{32} & \mathscr{B}_{33} \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix} = \begin{pmatrix} \mathscr{F}_1 \\ \mathscr{F}_2 \\ \mathscr{F}_3 \end{pmatrix} \equiv \mathscr{F}, \tag{6.2}$$

where the blocks correspond to the ordering of the basis functions given above. Thus, $\mathscr{B}_{kl} = (B_{ij}^{kl})$, $U_k = (U_i^k)$, and $\mathscr{F}_k = (F_i^k)$ for $k, l = 1, 2, 3$, where

$$B_{ij}^{kl} = b^h(\boldsymbol{\psi}_j^l; \boldsymbol{\psi}_i^k) \quad \text{and} \quad F_i^k = f^h(\boldsymbol{\psi}_i^k).$$

More specifically, the right-hand-side vector has the following components:

$$F_i^1 = (f_1, \varphi_i) + \alpha_1((B_h\boldsymbol{f})_1, -\Delta_h\varphi_i) + \alpha_2 \sum_{K\in\mathcal{T}^h} h_K^2(f_1, -\Delta\varphi_i)_{0,K},$$

$$F_i^2 = (f_2, \varphi_i) + \alpha_1((B_h\boldsymbol{f})_2, -\Delta_h\varphi_i) + \alpha_2 \sum_{K\in\mathcal{T}^h} h_K^2(f_2, -\Delta\varphi_i)_{0,K},$$

$$F_i^3 = \alpha_1(B_h\boldsymbol{f}, \nabla\varphi_i) + \alpha_2 \sum_{K\in\mathcal{T}^h} h_K^2(\boldsymbol{f}, \nabla\varphi_i)_{0,K}.$$

The third terms in F_i^1 and F_i^2 are zero in the case that φ_i is linear or bilinear on each element. The second term in F_i^1 and F_i^2 can be computed by the definition of the discrete Laplacian operator in (4.6); i.e.,

$$((B_h\boldsymbol{f})_1, -\Delta_h\varphi_i) = (\nabla(B_h\boldsymbol{f})_1, \nabla\varphi_i) \quad \text{and} \quad ((B_h\boldsymbol{f})_2, -\Delta_h\varphi_i) = (\nabla(B_h\boldsymbol{f})_2, \nabla\varphi_i),$$

and the remaining terms can be computed as usual, provided that $B_h\boldsymbol{f}$ is known. Hence, it is clear that the computational complexity for computing the right-hand side vector $\mathscr{F}$ is $\mathcal{O}(N)$, provided that $B_h\boldsymbol{f}$ is known.

In Section 4, the preconditioner B_h was required to be a symmetric, positive-definite operator spectrally equivalent to A_h, the discrete Dirichlet solver given by (4.4). Since the action of a symmetric preconditioner applied to a given vector for an algebraic equation can be considered as one step of a symmetric iteration for that algebraic equation starting from a zero initial value, $B_h\boldsymbol{f} \in \boldsymbol{U}^h$ can be computed as follows:

- $B_h\boldsymbol{f} = \Sigma_i\, G_i^1(\varphi_i, 0) + \Sigma_i\, G_i^2(0, \varphi_i)$;
- $\mathscr{G} = (\mathscr{G}_1 / \mathscr{G}_2)$; $\mathscr{G}_1 = (G_i^1)$ and $\mathscr{G}_2 = (G_i^2)$ are then the result of one step of a symmetric iteration on (4.4) with the right-hand side $(\mathscr{F}_1 / \mathscr{F}_2)$ and with zero initial values. Here, $\mathscr{F}_1 = ((f_1, \varphi_i))$ and $\mathscr{F}_2 = ((f_2, \varphi_i))$.

For the symmetric iteration required in the second step above, multigrid or domain decomposition methods spectrally equivalent to A_h can be used. For example, the multigrid $V(1, 1)$-cycle with the Gauss–Seidel smoothing gives such a preconditioner if the directions of the sweeps are reversed in the pre- and post-smoothings. In this case, the evaluation of the preconditioner requires only $\mathcal{O}(N)$ operations.

The matrix $\mathscr{B}$ is dense because of the involvement of B_h in the bilinear form, but it is never applied in assembled form. Iterative methods solving (6.2) are usually based on computing the action of the matrix $\mathscr{B}$ applied to arbitrary vector $\mathscr{H} = (\mathscr{H}_1, \mathscr{H}_2, \mathscr{H}_3)^t$, where $\mathscr{H}_k = (H_i^k)$. Let $\mathscr{H}$ represent the coefficients of a function pair

$$(\boldsymbol{v}, q) = \sum_{j=1}^{3} \sum_{i=1}^{N_j} H_i^j \psi_i^j,$$

and let $\mathscr{V}_l^k$ represent the coefficients of the basis function ψ_l^k for $k = 1, 2, 3$. Then,

$$\mathscr{B}\mathscr{H} = ((\mathscr{B}\mathscr{H})_1, (\mathscr{B}\mathscr{H})_2, (\mathscr{B}\mathscr{H})_3)^t$$

has the components

$$(\mathscr{B}\mathscr{H})_l^k = \langle \mathscr{B}\mathscr{H}, \mathscr{V}_l^k \rangle = b^h(\boldsymbol{v}, q; \psi_l^k),$$

where $\langle \cdot\,, \cdot \rangle$ is the ℓ^2-inner product. More specifically,

$$(\mathcal{B}\mathcal{H})_l^1 = (\nabla v_1, \nabla\varphi_l) + \alpha_1(B_h(-\Delta_h \boldsymbol{v} + \nabla q)_1, -\Delta_h\varphi_l) + \alpha_2 \sum_{K\in\mathcal{T}_h} h_K^2(-\Delta v_1 + \partial_1 q, -\Delta\varphi_l)_{0,K} - (q, \partial_1\varphi_l)\,,$$

$$(\mathcal{B}\mathcal{H})_l^1 = (\nabla v_2, \nabla\varphi_l) + \alpha_1(B_h(-\Delta_h \boldsymbol{v} + \nabla q)_2, -\Delta_h\varphi_l) + \alpha_2 \sum_{K\in\mathcal{T}_h} h_K^2(-\Delta v_2 + \partial_2 q, -\Delta\varphi_l)_{0,K} - (q, \partial_2\varphi_l)\,,$$

$$(\mathcal{B}\mathcal{H})_l^3 = \alpha_1(B_h(-\Delta_h \boldsymbol{v} + \nabla q), \nabla\varphi_l) + (\nabla\cdot\boldsymbol{v}, \varphi_l) + \alpha_2 \sum_{K\in\mathcal{T}_h} h_K^2(-\Delta\boldsymbol{v} + \nabla q, \nabla\varphi_l)_{0,K}\,.$$

The terms involving the discrete Laplacian operator can be computed through the definition; for example,

$$((B_h(-\Delta_h\boldsymbol{v} + \nabla q))_1, -\Delta_h\varphi_l) = (\nabla(B_h(-\Delta_h\boldsymbol{v} + \nabla q))_1, \nabla\varphi_l)\,,$$

while $B_h(-\Delta_h\boldsymbol{v} + \nabla q)$ is computed as described above in the evaluation of the preconditioner. Also,

$$\mathcal{F}_1 = ((-\Delta_h v_1 + \partial_1 q, \varphi_i)) = ((\nabla v_1, \nabla\varphi_i) + (\partial_1 q, \varphi_i))\,,$$
$$\mathcal{F}_2 = ((-\Delta_h v_2 + \partial_2 q, \varphi_i)) = ((\nabla v_2, \nabla\varphi_i) + (\partial_2 q, \varphi_i))\,.$$

The remaining terms can be computed as usual. It is then easy to see that the matrix and vector multiplications, $\mathcal{B}\mathcal{H}$, take $\mathcal{O}(N)$ operations. As we shall see, the iterative method for (6.3) also requires computing $\mathcal{B}^{\mathrm{t}}\mathcal{H}$, where

$$(\mathcal{B}^{\mathrm{t}}\mathcal{H})_l^k = \langle \mathcal{B}^{\mathrm{t}}\mathcal{H}, \mathcal{V}_l^k\rangle = b^h(\psi_l^k; \boldsymbol{v}, q)\,.$$

These can be computed similarly to those above and again require $\mathcal{O}(N)$ operations.

In Section 5, two kinds of iterative methods were introduced for solving (4.10). We describe the implementation only of the preconditioned conjugate gradient (PCG) method based on the least-squares-like symmetrization (5.12). The implementation of the iterative method mentioned in Remark 5.1 can be made in a similar fashion. We define B_0^{-1} in terms of a matrix $\mathcal{B}_0^{-1}$. The assumptions on B_0 and Lemma 4.1 indicate that we can choose

$$\mathcal{B}_0^{-1} = \begin{pmatrix} \mathcal{N}^{-1} & 0 \\ 0 & \mathcal{D}^{-1} \end{pmatrix},$$

where $\mathcal{N}^{-1}$ is the matrix form of the preconditioner B_h and $\mathcal{D}$ is the diagonal of the mass matrix $((\varphi_j, \varphi_i))$ corresponding to the L^2-inner product. The blocks above correspond to the ordering of the basis functions into those from $\boldsymbol{U}^h$ and P^h, respectively. Applying $\mathcal{B}_0^{-1}$ to a given vector $\mathcal{G}$ involves applying the preconditioning process $\mathcal{N}^{-1}$ to the $\boldsymbol{U}^h$ components of $\mathcal{G}$ and multiplying the P^h components by d_{ii}^{-1}. This requires $\mathcal{O}(N)$ operations. Then, (5.12) has the matrix form

$$\mathcal{B}^{\mathrm{t}}\mathcal{B}_0^{-1}\mathcal{B}\mathcal{U} = \mathcal{B}^{\mathrm{t}}\mathcal{B}_0^{-1}\mathcal{F}\,. \tag{6.3}$$

Theorems 5.1 and 5.2 indicate that the PCG method with the preconditioner $\mathcal{B}_0^{-1}$ for (6.3) converges at a rate that is uniform in the mesh size h. Since the main cost of each iteration in the PCG method is the action of $\mathcal{B}_0^{-1}\mathcal{B}^{\mathrm{t}}\mathcal{B}_0^{-1}\mathcal{B}$ applied to a given vector, the computational complexity of each iteration is then proportional to the number of coefficients as discussed above.

7. Other stabilized finite element methods

In this section, we discuss other stabilized finite element methods. We will consider a general stabilized formulation similar to that in [2] in which an extra parameter is introduced.

For $(\boldsymbol{u}, p)$ and $(\boldsymbol{v}, q)$ in $\mathcal{V}^h$, define the bilinear form

$$b_t^h(\boldsymbol{u}, p; \boldsymbol{v}, q) = (\nabla\boldsymbol{u}, \nabla\boldsymbol{v}) + b_a(\boldsymbol{u}, p; \boldsymbol{v}, q) + a_t^h(-\Delta_h\boldsymbol{u} + \nabla p, -t\,\Delta_h\boldsymbol{v} + \nabla q)\,, \tag{7.1}$$

where

$$a_t^h(-\Delta_h \boldsymbol{u} + \nabla p, -t\,\Delta_h \boldsymbol{v} + \nabla q) = \alpha(B_h(-\Delta_h \boldsymbol{u} + \nabla p), -t\,\Delta_h \boldsymbol{v} + \nabla q) + \alpha \sum_{K \in \mathcal{T}_h} h_K^2(-\Delta \boldsymbol{u} + \nabla p, -t\,\Delta \boldsymbol{v} + \nabla q)_{0,K},$$

and the linear form

$$f_t^h(\boldsymbol{v}, q) = (\boldsymbol{f}, \boldsymbol{v}) + a_t^h(\boldsymbol{f}, -t\,\Delta_h \boldsymbol{v} + \nabla q).$$

The associated stabilized finite element method is to find an approximation $(\boldsymbol{u}^h, p^h) \in \boldsymbol{U}^h \times P^h$ of (3.4) such that

$$b_t^h(\boldsymbol{u}, p; \boldsymbol{v}, q) = f_t^h(\boldsymbol{v}, q), \quad (\boldsymbol{v}, q) \in \boldsymbol{U}^h \times P^h. \tag{7.2}$$

When $t = 1$, (7.2) reduces to the method defined in (4.11); for $t = 0$ and $t = -1$, (7.2) represents modifications of those introduced in [19] and [18].

THEOREM 7.1. For any real number t, there exists an $\alpha_0(t) > 0$ such that, for $\alpha \in (0, \alpha_0(t))$, the bilinear form $b_t^h(\cdot\,;\cdot)$ is elliptic and continuous on $\mathcal{V}^h$, the linear form $f_t^h(\cdot)$ is continuous on $\mathcal{V}^h$, and (7.2) has a unique solution in $\mathcal{V}^h$.

PROOF. Continuity of the bilinear and linear forms can be shown by an argument similar to that employed in the proof of Lemma 4.1. To prove the ellipticity of the bilinear form, we use an argument analogous to that in [2]. By Cauchy–Schwarz,

$$\begin{aligned}(B_h(-\Delta_h \boldsymbol{v} + \nabla q), -t\,\Delta_h \boldsymbol{v} + \nabla q) &= (B_h \nabla q, \nabla q) - (1+t)(B_h \nabla q, \Delta_h \boldsymbol{v}) + t(B_h\,\Delta_h \boldsymbol{v}, \Delta_h \boldsymbol{v}) \\ &\geq (B_h \nabla q, \nabla q) - |1+t|(B_h \nabla q, \nabla q)^{1/2}(B_h\,\Delta_h \boldsymbol{v}, \Delta_h \boldsymbol{v})^{1/2} - |t|(B_h\,\Delta_h \boldsymbol{v}, \Delta_h \boldsymbol{v}) \\ &\geq \frac{1}{2}(B_h \nabla q, \nabla q) - C_t(B_h\,\Delta_h \boldsymbol{v}, \Delta_h \boldsymbol{v})\end{aligned}$$

with $C_t = \frac{1}{2}(|1+t|^2 + |t|)$. Similarly,

$$\sum_{K \in \mathcal{T}_h} h_K^2(-\Delta \boldsymbol{v} + \nabla q, -t\,\Delta \boldsymbol{v} + \nabla q)_{0,K} \geq \frac{1}{2} \sum_{K \in \mathcal{T}_h} h_K^2 \|\nabla q\|_{0,K}^2 - C_t \sum_{K \in \mathcal{T}_h} h_K^2 \|\Delta \boldsymbol{v}\|_{0,K}^2.$$

Now, it follows from (4.5), the inverse inequality, (4.16), and (4.17) that

$$\begin{aligned}b_t^h(\boldsymbol{v}, q; \boldsymbol{v}, q) &\geq (1 - \alpha C_1 C_t)\|\nabla \boldsymbol{v}\|^2 + \frac{\alpha}{2}\left((B_h \nabla q, \nabla q) + \sum_{K \in \mathcal{T}_h} h_K^2 \|\nabla q\|_{0,K}^2\right) \\ &\geq (1 - \alpha C_1 C_t)\|\nabla \boldsymbol{v}\|^2 + \frac{\alpha}{2} C_2 \|q\|^2.\end{aligned}$$

Choosing

$$\alpha_0(t) = \frac{1}{2C_1 C_t}$$

establishes the ellipticity of the bilinear form. Then, existence and uniqueness are immediate consequences of continuity, ellipticity, and the Lax–Milgram lemma. □

THEOREM 7.2. Let $(\boldsymbol{u}, p)$ and $(\boldsymbol{u}^h, p^h)$ be the solutions of (2.1) and (7.2), respectively. Assume that $(\boldsymbol{u}, p) \in H^{r+1}(\Omega)^d \times H^r(\Omega)$ with $r \geq 1$. Let α_0 be chosen so that the conclusions of Theorem 7.1 hold. Then,

$$\|\boldsymbol{u} - \boldsymbol{u}^h\|_1 + \|p - p^h\| \leq Ch^r(\|\boldsymbol{u}\|_{r+1} + \|p\|_r). \tag{7.3}$$

If, in addition, $(\boldsymbol{u}, p)$ is H^2-regular, then

$$\|\boldsymbol{u} - \boldsymbol{u}^h\| \leq Ch^{r+1}(\|\boldsymbol{u}\|_{r+1} + \|p\|_r). \tag{7.4}$$

PROOF. The proof is similar to that for Theorem 4.2. □

As (4.10), (7.2) is nonsymmetric. It is easy to see by straightforward manipulations that the symmetric part of $b_t^h(\cdot\,;\cdot)$ has the form

$$b_{t,s}^h(\boldsymbol{u}, p; \boldsymbol{v}, q) = b_s^h(\boldsymbol{u}, p; \boldsymbol{v}, q) + \frac{\alpha(1-t)}{2}(a_t^h(-\Delta_h \boldsymbol{u} + \nabla p, \Delta_h \boldsymbol{v}) + a_t^h(\Delta_h \boldsymbol{u}, -\Delta_h \boldsymbol{v} + \nabla q)) \tag{7.5}$$

and the skew-symmetric part the form

$$b_{t,s}^h(\boldsymbol{u}, p; \boldsymbol{v}, q) = b_a(\boldsymbol{u}, p; \boldsymbol{v}, q) + \frac{\alpha(1-t)}{2}(a_t^h(-\Delta_h \boldsymbol{u} + \nabla p, \Delta_h \boldsymbol{v}) - a_t^h(\Delta_h \boldsymbol{u}, -\Delta_h \boldsymbol{v} + \nabla q)) \tag{7.6}$$

where, with $\alpha_1 = \alpha_2 = \alpha$, the bilinear forms $b_s^h(\cdot\,;\cdot)$ and $b_a(\cdot\,;\cdot)$ are defined in (4.8) and (3.3). Let $B_{t,s}$ and $B_{t,a}$ denote operators associated with the bilinear forms $b_{t,s}^h(\cdot\,;\cdot)$ and $b_{t,a}^h(\cdot\,;\cdot)$. Then, the symmetrization procedures of Section 5 applied to (7.2) give the symmetric but indefinite problem (5.5) and the symmetric positive-definite problem (5.12) with $B_s = B_{t,s}$, $B_a = B_{t,a}$, and $B_0 = B_{t,0}$, where we assume that $B_{t,0}$ is spectrally equivalent to $B_{t,s}$.

THEOREM 7.3. Under the assumptions of Theorem 7.1, $B_{t,s}$ and $B_{t,s} + B_{t,a}^* B_{t,s}^{-1} B_{t,a}$ are elliptic and continuous on $\mathcal{V}^h$. Moreover, $(B_{t,s} + B_{t,a})^* B_{t,0}^{-1}(B_{t,s} + B_{t,a})$ is spectrally equivalent to $B_{t,s} + B_{t,a}^* B_{t,s}^{-1} B_{t,a}$.

PROOF. The ellipticity and continuity of $B_{t,s}$ are immediate consequences of Theorem 7.1 and the fact that

$$(B_{t,s}(\boldsymbol{v}, q), (\boldsymbol{v}, q)) = b_t^h(\boldsymbol{v}, q; \boldsymbol{v}, q).$$

It is now easy to see that $B_{t,s} + B_{t,a}^* B_{t,s}^{-1} B_{t,a}$ is elliptic. The proof of its continuity parallels that given in the proof of Theorem 5.1. The spectral equivalence is proved in an analogous fashion to that employed in Theorem 5.2. □

8. The time-dependent Stokes problem

Consider the time-dependent Stokes problem

$$\begin{cases} \dfrac{\partial \boldsymbol{u}}{\partial t} - \Delta \boldsymbol{u} + \nabla p = \boldsymbol{f} & \text{in } \Omega \times J, \\ \nabla \cdot \boldsymbol{u} = 0 & \text{in } \Omega \times J, \\ \boldsymbol{u} = \boldsymbol{0} & \text{on } \partial\Omega \times J, \\ \boldsymbol{u}(0, x) = \boldsymbol{u}_0(x) & \text{in } \Omega, \end{cases} \tag{8.1}$$

where $J = (0, T]$ and $\boldsymbol{u}_0(x)$ is the given initial condition. Let $t^n = n\,\Delta t \in J$; then implicit backward differencing gives

$$\begin{cases} -\Delta \boldsymbol{u}^n + \dfrac{1}{\Delta t}\boldsymbol{u}^n + \nabla p^n = \boldsymbol{f}^n + \dfrac{\boldsymbol{u}^{n-1}}{\Delta t} & \text{in } \Omega, \\ \nabla \cdot \boldsymbol{u}^n = 0 & \text{in } \Omega, \\ \boldsymbol{u}^n = \boldsymbol{0} & \text{on } \partial\Omega, \\ \boldsymbol{u}^0 = \boldsymbol{u}_0(x) & \text{in } \Omega. \end{cases} \tag{8.2}$$

At each time level t^n, (8.2) represents an elliptic system for $\boldsymbol{u}^n$ which is similar to and has better numerical properties than that of the Stokes problem (2.1) because of the term $\boldsymbol{u}^n/\Delta t$ in the first equation. When the stabilized finite element method and the preconditioned iterations (with initial guess obtained by extrapolation from the previous time levels) described in the respective Sections 4 and 5 are employed, only a fixed number (usually, a small number) of iterations is required for each time step.

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