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Least-squares Finite Element Approximations for the Reissner–Mindlin Plate

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Based on the Helmholtz decomposition of the transverse shear strain, Brezzi and Fortin in [7] introduced a three-stage algorithm for approximating the Reissner–Mindlin plate model with clamped boundary conditions and established uniform error estimates in the plate thickness. The first and third stages involve approximating two simple Poisson equations and the second stage approximates a perturbed Stokes equation. Instead of using the mixed finite element method which is subject to the 'infsup' condition, we consider a least-squares finite element approximation to such a perturbed Stokes equation. By introducing a new independent vector variable and associated div equation, we are able to establish the ellipticity and continuity of the homogeneous least-squares functional in an H^1 product norm appropriately weighted by the thickness. This immediately yields optimal discretization error estimates for finite element spaces in this norm which are uniform in the thickness. We show that the resulting algebraic equations can be uniformly well preconditioned by well-known techniques in the thickness. The Reissner–Mindlin model with pure traction boundary condition is also studied. Finally, we consider an alternative least-squares formulation for the perturbed Stokes equation by introducing an independent scalar variable.

KEY WORDS Reissner–Mindlin plate; least-square functional finite element; multigrid

1. Introduction

The Reissner–Mindlin plate model is frequently used by engineers in connection with plate and shell problems of small to moderate thickness. It is known that standard finite

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element approximations grossly underestimate the displacement of very thin plates. Such a phenomenon is referred to as *locking* of the numerical solution. There have been many studies to develop alternative approaches (see [1,6–9]) that are robust in the zero limit of the *plate thickness* t. But these alternatives are usually based on either *mixed* formulations that require finite element approximation spaces satisfying the so-called infsup condition or *stabilized mixed* formulations (see [9] and references therein). Both formulations lead to symmetric but indefinite algebraic systems that are difficult to solve (see [2,5]). Indeed, little attention seems to have been paid to the development of robust solution strategies for the resulting algebraic equations.

Among those *locking-free* discretization schemes, the fundamental work of Brezzi and Fortin in [7] must be noted. Based on the Helmholtz decomposition of the *transverse shear strain*, they introduced a three-stage algorithm for approximating the Reissner–Mindlin model with clamped boundary conditions and established the uniform convergence of discretization schemes in the thickness. The first and third stages solve simple Poisson equations for the respective *irrotational part* of the shear strain and *transverse displacement*. The second stage uses mixed finite element methods to solve a Stokes equation perturbed by a Laplacian term for the *rotation* of the fibers and the *solenoidal part* of the shear strain. As usual, mixed finite element methods are subject to the infsup condition and the resulting algebraic equations are difficult to solve.

Recently, there has been substantial interest in the use of least-squares principles for numerical approximations of elliptic partial differential equations and systems. See [11–13] for linear elasticity which is a parameter dependent problem. Its advantages over the usual mixed finite element discretizations include: (a) the choice of finite element spaces is not subject to the infsup condition, (b) the resulting algebraic equations can be efficiently solved by standard multigrid methods or preconditioned by well-known techniques, and (c) the value of the least-squares functional provides a good error indicator which can be used efficiently in a refinement process.

The purpose of this paper is to extend this methodology to the perturbed Stokes equation. By introducing a new independent vector variable, the vector curl of the solenoidal part of the shear strain and associated div equation, we reformulate such a perturbed Stokes equation as an equivalent first- and second-order system. We first apply a least-squares principle to this system using L^2- and H^{-1}-norms weighted appropriately by the thickness. We then show that the homogeneous part of the resulting functional is uniformly elliptic and continuous with respect to the thickness in a weighted H^1 product norm. The H^{-1}-norm and second-order differential operators in the functional are further replaced by the respective discrete H^{-1}-norm and discrete second-order differential operators to make the computation feasible (see [3,10,11]). Such a discrete functional is shown to be uniformly equivalent to the same weighted H^1 product norm. This property enables us to show that standard finite element discretization error estimates are optimal and uniform in the thickness. Moreover, the resulting discrete algebraic equations can be preconditioned by multigrid uniformly well with respect to the thickness, the mesh size and the number of levels.

The Reissner–Mindlin model is introduced in Section 2, along with some notation. We describe a least-squares formulation for the perturbed Stokes equation and establish its ellipticity and continuity in Section 3. Its discrete counterpart, the corresponding finite element approximation and an efficient preconditioner are discussed in Section 4. Section 5 studies the pure traction problem. This least-squares formulation involves the weighted H^1-norm of new independent variable, the curl of the solenoidal part of the shear strain, and hence, it has one extra regularity requirement in establishing standard error estimation (see

Theorem 4.2). As an alternative, we consider another least-squares formulation in Section 6 by introducing the curl of the shear strain that is the quantity representing a different scale. The latter has the same numerical properties as those of the former, but no extra regularity requirement.

2. The Reissner–Mindlin model

Let $\mathcal{D}(\Omega)$ be the linear space of infinitely differentiable functions with compact support on Ω. We use standard notation and definition for the Sobolev spaces $H^s(\Omega)^2$ for $s \geq 0$, the associated inner products are denoted by $(\cdot,\ \cdot)_{s,\,\Omega}$, and their norms by $\|\cdot\|_{s,\,\Omega}$. (We will omit Ω from the inner product and norm designation when there is no risk of confusion.) For $s = 0$, $H^s(\Omega)^2$ coincides with $L^2(\Omega)^2$. In this case, the norm and inner product will be denoted by $\|\cdot\|$ and $(\cdot,\ \cdot)$, respectively. As usual, $H_0^1(\Omega)$ is the closure of $\mathcal{D}(\Omega)$ with respect to the norm $\|\cdot\|_1$. Let $H_0^{-1}(\Omega)$ and $H^{-1}(\Omega)$ be duals of $H_0^1(\Omega)$ and $H^1(\Omega)$, correspondingly, with norms defined by

$$\|\psi\|_{-1,0} = \sup_{0\neq\phi\in H_0^1(\Omega)} \frac{(\psi,\phi)}{\|\phi\|_1} \quad \text{and} \quad \|\psi\|_{-1} = \sup_{0\neq\phi\in H^1(\Omega)} \frac{(\psi,\phi)}{\|\phi\|_1}$$

respectively. Define the product spaces $H_0^1(\Omega)^2 = \prod_{i=1}^2 H_0^1(\Omega)$ with standard product norms and let $L_0^2(\Omega)$ denote the subspace of $L^2(\Omega)$ consisting of all such functions in $L^2(\Omega)$ having mean value zero.

Let Ω be the region occupied by the plate and assume that Ω is a convex polygon in $\mathcal{R}^2$. Let ω and $\boldsymbol{\phi} = (\phi_1, \phi_2)^t$ denote the transverse displacement of Ω and the rotation of the fibers normal to Ω, respectively. The variational form of the Reissner–Mindlin plate model is to find $(\omega, \boldsymbol{\phi}) \in H_0^1(\Omega) \times H_0^1(\Omega)^2$ such that

$$a(\boldsymbol{\phi}, \boldsymbol{\psi}) + \lambda t^{-2}(\boldsymbol{\phi} - \nabla\omega, \boldsymbol{\psi} - \nabla v) = (g, v), \quad \forall (v, \boldsymbol{\psi}) \in H_0^1(\Omega) \times H_0^1(\Omega)^2 \tag{2.1}$$

where the symbol ∇ stands for the gradient operator; $t > 0$ is the plate thickness; $\lambda = Ek/2(1+\nu)$ is the shear modulus with E the Young's modulus, $\nu \in (-1, \frac{1}{2})$ the Poisson ratio and k the shear correction factor; and g is the given scaled transverse loading function (scaling by a constant multiple of the square of the thickness [7]). The bilinear form $a(\cdot,\cdot)$ is defined by

$$a(\boldsymbol{\phi}, \boldsymbol{\psi}) = \frac{E}{12(1-\nu^2)} \int_\Omega \Big[(\partial_1\phi_1 + \nu\partial_2\phi_2)\,\partial_1\psi_1 + (\nu\partial_1\phi_1 + \partial_2\phi_2)\,\partial_2\psi_2 + \frac{1-\nu}{2}(\partial_2\phi_1 + \partial_1\phi_2)(\partial_2\psi_1 + \partial_1\psi_2) \Big]$$

where $\partial_i\phi_j = \frac{\partial\phi_j}{\partial x_i}$ and $\partial_i\psi_j = \frac{\partial\psi_j}{\partial x_i}$ for $i, j = 1, 2$.

We will use standard curl notation for two dimensions by identifying $\mathcal{R}^2$ with the (x_1, x_2)-plane in $\mathcal{R}^3$. Thus, the curl of $\boldsymbol{\phi}$ means the scalar function

$$\nabla\times\boldsymbol{\phi} = \partial_1\phi_2 - \partial_2\phi_1$$

and $\nabla^\perp$ denotes its formal adjoint:

$$\nabla^\perp p = \begin{pmatrix} \partial_2 p \\ -\partial_1 p \end{pmatrix}$$

The gradient operator is extended to two-vectors componentwise:

$$\nabla \boldsymbol{\phi} = \begin{pmatrix} \nabla \phi_1 \\ \nabla \phi_2 \end{pmatrix}$$

In order to avoid the 'locking' phenomenon, Brezzi and Fortin [7] proposed a *three-stage* finite element method through the Helmholtz decomposition of the transverse shear strain vector. Such an approach converges uniformly in the thickness t. To this end, consider the following Helmholtz decomposition

$$\lambda t^{-2}(\nabla \omega - \boldsymbol{\phi}) = \nabla r - \nabla^\perp p \tag{2.2}$$

with $(r, p) \in H_0^1(\Omega) \times (H^1(\Omega)/\mathcal{R})$. For simplicity, assume that $\lambda = 1$. Then the equivalent formulation of (2.1) suggested by Brezzi and Fortin [7] is as follows:

(1) Find $r \in H_0^1(\Omega)$ such that

$$(\nabla r, \nabla \mu) = (g, \mu), \quad \forall\ \mu \in H_0^1(\Omega) \tag{2.3}$$

(2) Find $(\boldsymbol{\phi}, p) \in H_0^1(\Omega)^2 \times (H^1(\Omega)/\mathcal{R})$ such that

$$\begin{cases} a(\boldsymbol{\phi}, \boldsymbol{\psi}) + (\nabla^\perp p, \boldsymbol{\psi}) = (\nabla r, \boldsymbol{\psi}), & \forall\ \boldsymbol{\psi} \in H_0^1(\Omega)^2 \\ (\boldsymbol{\phi}, \nabla^\perp q) - t^2(\nabla^\perp p, \nabla^\perp q) = 0, & \forall\ q \in H^1(\Omega)/\mathcal{R} \end{cases} \tag{2.4}$$

(3) Find $\omega \in H_0^1(\Omega)$ such that

$$(\nabla \omega, \nabla s) = (\boldsymbol{\phi} + t^2 \nabla r, \nabla s), \quad \forall\ s \in H_0^1(\Omega) \tag{2.5}$$

The following a priori estimate was established by Brezzi and Fortin in [7].

Theorem 2.1. *Let Ω be a bounded convex polygon or have $C^{1,1}$ boundary in $\mathcal{R}^2$. For any $0 < t \le C$ and $g \in H^{-1}(\Omega)$, there exists a unique solution $(r, \boldsymbol{\phi}, p, \omega) \in H_0^1(\Omega) \times H_0^1(\Omega)^2 \times \left(H^1(\Omega)/\mathcal{R}\right) \times H_0^1(\Omega)$ of problem (2.3–2.5). Moreover, $\boldsymbol{\phi} \in H^2(\Omega)^2$ and there exists a positive constant C independent of t and g such that*

$$\|r\|_1 + \|\boldsymbol{\phi}\|_2 + \|p\|_1 + t\|p\|_2 + \|w\|_1 \le C\,\|g\|_{-1,0} \tag{2.6}$$

Note that problems (2.3) and (2.5) are the Poisson equations whose numerical computations are well understood. Hence, we only consider least-squares finite element approximations to problem (2.4). Note also that problem (2.4) is a perturbed Stokes equation with the following strong form:

$$\begin{cases} -\alpha_1 \Delta \boldsymbol{\phi} - \alpha_2 \nabla(\nabla \cdot \boldsymbol{\phi}) + \nabla^\perp p = \mathbf{f}, & \text{in } \Omega \\ \nabla \times \boldsymbol{\phi} + t^2 \Delta p = f_3, & \text{in } \Omega \end{cases} \tag{2.7}$$

where $\mathbf{f} = \nabla r$ and $f_3 = 0$, with boundary conditions

$$\boldsymbol{\phi} = \mathbf{0} \quad \text{and} \quad \nabla p \cdot \mathbf{n} = 0 \quad \text{on} \quad \partial\Omega \tag{2.8}$$

Here $\alpha_1 = \frac{E}{24(1+\nu)} > 0$, $\alpha_2 = \frac{E}{24(1-\nu)} > 0$, the symbol $\nabla\cdot$ stands for the divergence operator, and $\mathbf{n} = (n_1, n_2)^t$ is the outward unit vector normal to the boundary $\partial\Omega$.

Remark 1
It is easy to establish the following a priori estimate for the solution of problem (2.7–2.8):

$$\|\boldsymbol{\phi}\|_1 + \|p\| + \|t\nabla p\| + \|t^2\Delta p\| \le C\left(\|\mathbf{f}\|_{-1,0} + \|f_3\|\right)$$

where C is a positive constant independent of t. Based on this estimate, it is possible to develop a least-squares formulation, but such formulation involves $t^2\Delta p$ which complicates discretization and solution processes (cf. [4]).

Let

$$H(\text{div};\Omega) = \{\mathbf{v} \in L^2(\Omega)^2 : \ \nabla\cdot\mathbf{v} \in L^2(\Omega)\}$$

and

$$H(\text{curl};\Omega) = \{\mathbf{v} \in L^2(\Omega)^2 : \ \nabla\times\mathbf{v} \in L^2(\Omega)\}$$

which are Hilbert spaces under the respective norms:

$$\|\mathbf{v}\|_{H(\text{div};\Omega)} = \left(\|\mathbf{v}\|^2 + \|\nabla\cdot\mathbf{v}\|^2\right)^{\frac{1}{2}} \quad \text{and} \quad \|\mathbf{v}\|_{H(\text{curl};\Omega)} = \left(\|\mathbf{v}\|^2 + \|\nabla\times\mathbf{v}\|^2\right)^{\frac{1}{2}}$$

Define their respective subspaces:

$$H_0(\text{div};\Omega) = \{\mathbf{v} \in H(\text{div};\Omega) : \ \mathbf{n}\cdot\mathbf{v} = 0 \text{ on } \partial\Omega\}$$

and

$$H_0(\text{curl};\Omega) = \{\mathbf{v} \in H(\text{curl};\Omega) : \ \mathbf{n}\times\mathbf{v} = 0 \text{ on } \partial\Omega\}$$

Set

$$U = H_0(\text{div};\Omega) \cap H(\text{curl};\Omega) \quad \text{and} \quad W = H(\text{div};\Omega) \cap H_0(\text{curl};\Omega)$$

We will also make use of the following results (see [15]).

Theorem 2.2. *Assume that the domain Ω is a bounded convex polygon or has $C^{1,1}$ boundary. Then for any vector function $\mathbf{v}$ in either U or W, we have*

$$\|\mathbf{v}\|_1^2 \le C\left(\|\mathbf{v}\|^2 + \|\nabla\cdot\,\mathbf{v}\|^2 + \|\nabla\times\mathbf{v}\|^2\right) \tag{2.9}$$

If, in addition, the domain is simply connected, then

$$\|\mathbf{v}\|_1^2 \le C\left(\|\nabla\cdot\,\mathbf{v}\|^2 + \|\nabla\times\mathbf{v}\|^2\right) \tag{2.10}$$

3. A least-squares formulation

In this section, we first introduce a new independent vector variable for the treatment of the term $t^2\Delta p$ in (2.7) and write problem (2.7)–(2.8) as a first- and second-order system. We then apply the least-squares principle to such system. Our least-squares functional is defined by the weighted sum of the L^2- and H^{-1}-norms of the residual equations of the system. The ellipticity and continuity of the homogeneous functional are established in Theorem 3.1. This will in turn imply the well-posedness of the least-squares formulation and its equivalence to (2.4).

Introducing an independent vector variable

$$\mathbf{u} = \nabla^{\perp} p$$

and using the homogeneous Neumann boundary condition of p, we have the following properties

$$\nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega \quad \text{and} \quad \mathbf{u} \cdot \boldsymbol{\tau} = 0 \quad \text{on } \partial\Omega \tag{3.1}$$

where $\boldsymbol{\tau} = (-n_2, n_1)^t$ is the unit counter-clockwise vector tangent to the boundary $\partial\Omega$. Then problem (2.7)–(2.8) may be rewritten as following first- and second-order system:

$$\begin{cases} -\alpha_1 \Delta \boldsymbol{\phi} - \alpha_2 \nabla(\nabla \cdot \boldsymbol{\phi}) + \nabla^{\perp} p = \nabla r, & \text{in } \Omega \\ \nabla \times \boldsymbol{\phi} - t^2 \nabla \times \mathbf{u} = 0, & \text{in } \Omega \\ \mathbf{u} - \nabla^{\perp} p = \mathbf{0}, & \text{in } \Omega \\ \nabla \cdot \mathbf{u} = 0, & \text{in } \Omega \end{cases} \tag{3.2}$$

with boundary conditions

$$\begin{cases} \boldsymbol{\phi} = \mathbf{0}, & \text{on } \partial\Omega \\ \nabla p \cdot \mathbf{n} = 0, & \text{on } \partial\Omega \\ \mathbf{u} \cdot \boldsymbol{\tau} = 0, & \text{on } \partial\Omega \end{cases} \tag{3.3}$$

We define the least-squares functional in terms of appropriate weights and norms of the residuals for the above system:

$$\begin{aligned} G(\boldsymbol{\phi},\ p,\ \mathbf{u};\ r) = {} & \|\nabla r + \alpha_1 \Delta \boldsymbol{\phi} + \alpha_2 \nabla(\nabla \cdot \boldsymbol{\phi}) - \nabla^{\perp} p\|_{-1,0}^2 + \|\nabla \times \boldsymbol{\phi} - t^2 \nabla \times \mathbf{u}\|^2 \\ & + \|t(\mathbf{u} - \nabla^{\perp} p)\|^2 + \|t^2 \nabla \cdot \mathbf{u}\|^2 \end{aligned} \tag{3.4}$$

Let

$$\mathcal{V} = H_0^1(\Omega)^2 \times (H^1(\Omega)/\mathcal{R}) \times W$$

and denote the weighted norm over $\mathcal{V}$ by

$$|||(\boldsymbol{\phi},\ p,\ \mathbf{u})|||_{\mathcal{V}} \equiv \left(\|\boldsymbol{\phi}\|_1^2 + \|p\|^2 + \|t\nabla p\|^2 + \|t\mathbf{u}\|^2 + \|t^2 \nabla \mathbf{u}\|^2 \right)^{\frac{1}{2}}$$

The least-squares formulation for problem (2.7) with boundary condition (2.8) is to minimize the quadratic functional $G(\boldsymbol{\phi}, p, \mathbf{u};\ r)$ with given r over $\mathcal{V}$: find $(\boldsymbol{\phi}, p, \mathbf{u}) \in \mathcal{V}$ such that

$$G(\boldsymbol{\phi},\ p,\ \mathbf{u};\ r) = \inf_{(\boldsymbol{\psi}, p, \mathbf{u}) \in \mathcal{V}} G(\boldsymbol{\psi},\ s,\ \mathbf{v};\ r) \tag{3.5}$$

Below, we will use C with or without subscripts to denote a generic positive constant, possibly different at different occurrences, which is independent of the plate thickness t and the mesh size h, introduced in the subsequent section, but may depend on the domain Ω and α_i $(i = 1, 2)$.

Theorem 3.1. *The homogeneous functional $G(\boldsymbol{\phi}, p, \mathbf{u};\ 0)$ is elliptic and continuous in $\mathcal{V}$; i.e., there exists a positive constant C independent of the thickness t such that*

$$\frac{1}{C}\, |||(\boldsymbol{\phi},\ p,\ \mathbf{u})|||^2_{\mathcal{V}} \le G(\boldsymbol{\phi},\ p,\ \mathbf{u};\ 0) \tag{3.6}$$

and that

$$G(\boldsymbol{\phi},\ p,\ \mathbf{u};\ 0) \le C\, |||(\boldsymbol{\phi},\ p,\ \mathbf{u})|||^2_{\mathcal{V}} \tag{3.7}$$

for any $(\boldsymbol{\phi}, p, \mathbf{u})$ in $\mathcal{V}$.

Proof

The upper bound in (3.7) follows immediately from the triangle inequality and the easily established bounds

$$\|\Delta\boldsymbol{\phi}\|_{-1,0} \le \|\nabla\boldsymbol{\phi}\| \le \|\boldsymbol{\phi}\|_1, \quad \|\nabla(\nabla\cdot\boldsymbol{\phi})\|_{-1,0} \le \|\boldsymbol{\phi}\|_1, \quad \|\nabla^{\perp} p\|_{-1,0} \le \|p\| \tag{3.8}$$

We proceed to show the validity of the lower bound in (3.6) for $(\boldsymbol{\phi}, p, \mathbf{u}) \in \mathcal{V}$ satisfying that $\boldsymbol{\phi} \in H^2(\Omega)^2$. Then (3.6) will follow for $(\boldsymbol{\phi}, p, \mathbf{u}) \in \mathcal{V}$ by continuity. For any $p \in H^1(\Omega)/\mathfrak{R}$, note first that by using the well-known inequality (see, e.g., [15]), $\|p\| \le C\|\nabla p\|_{-1,0}$, and the change of variable, $(\hat{x}_1, \hat{x}_2) = (-x_2, x_1)$, we have that

$$\|p\| = \|\hat{p}\| \le C\|\hat{\nabla}\hat{p}\|_{-1,0} = C\|\nabla^{\perp} p\|_{-1,0} \tag{3.9}$$

It then follows from the triangle inequality and the first two inequalities in (3.8) that

$$\|p\| \le C\left(\|\nabla^{\perp} p - \alpha_1\Delta\boldsymbol{\phi} - \alpha_2\nabla(\nabla\cdot\boldsymbol{\phi})\|_{-1,0} + \|\boldsymbol{\phi}\|_1\right) \tag{3.10}$$

Using the Korn inequality, the integration by parts, and the Cauchy–Schwarz inequality, we have that

$$\begin{aligned} C_0\|\boldsymbol{\phi}\|_1^2 &\le \alpha_1\,(\nabla\boldsymbol{\phi},\ \nabla\boldsymbol{\phi}) + \alpha_2\,(\nabla\cdot\boldsymbol{\phi},\ \nabla\cdot\boldsymbol{\phi}) \\ &= (-\alpha_1\Delta\boldsymbol{\phi} - \alpha_2\nabla(\nabla\cdot\boldsymbol{\phi}),\ \boldsymbol{\phi}) \\ &= \left(-\alpha_1\Delta\boldsymbol{\phi} - \alpha_2\nabla(\nabla\cdot\boldsymbol{\phi}) + \nabla^{\perp} p,\ \boldsymbol{\phi}\right) - \left(\nabla^{\perp} p,\ \boldsymbol{\phi}\right) \\ &\le G^{\frac{1}{2}}(\boldsymbol{\phi}, p, \mathbf{u};\ 0)\|\boldsymbol{\phi}\|_1 - (p, \nabla\times\boldsymbol{\phi}) \end{aligned} \tag{3.11}$$

and that

$$\begin{aligned} -(p,\ \nabla\times\boldsymbol{\phi}) &= (p,\ -\nabla\times\boldsymbol{\phi} + t^2\nabla\times\mathbf{u}) - (p,\ t^2\nabla\times\mathbf{u}) \\ &= (p,\ -\nabla\times\boldsymbol{\phi} + t^2\nabla\times\mathbf{u}) - (t\nabla^{\perp} p,\ t\mathbf{u}) \\ &= (p,\ -\nabla\times\boldsymbol{\phi} + t^2\nabla\times\mathbf{u}) - (t(\nabla^{\perp} p - \mathbf{u}),\ t\mathbf{u}) - \|t\mathbf{u}\|^2 \\ &\le (\|p\| + \|t\mathbf{u}\|)\, G^{\frac{1}{2}}(\boldsymbol{\phi}, p, \mathbf{u};\ 0) - \|t\mathbf{u}\|^2 \end{aligned} \tag{3.12}$$

where $G^{\frac{1}{2}}(\boldsymbol{\phi}, p, \mathbf{u}; 0) \equiv \sqrt{G(\boldsymbol{\phi}, p, \mathbf{u}; 0)}$. Combining the above two inequalities implies that

$$\begin{aligned} C_0\|\boldsymbol{\phi}\|_1^2 + \|t\mathbf{u}\|^2 &\le C\, G^{\frac{1}{2}}(\boldsymbol{\phi}, p, \mathbf{u}; 0)\,(\|\boldsymbol{\phi}\|_1 + \|p\| + \|t\mathbf{u}\|) \\ &\le C\, G(\boldsymbol{\phi}, p, \mathbf{u}; 0) + C\, G^{\frac{1}{2}}(\boldsymbol{\phi}, p, \mathbf{u}; 0)\,(\|\boldsymbol{\phi}\|_1 + \|t\mathbf{u}\|) \end{aligned}$$

The last inequality used bound (3.10). It then follows from the ϵ-inequality that

$$\|\boldsymbol{\phi}\|_1^2 + \|t\mathbf{u}\|^2 \le C\, G(\boldsymbol{\phi}, p, \mathbf{u}; 0)$$

Now, upper bounds in (3.6) for the terms, $\|p\|^2$, $\|t\nabla p\|^2$, and $\|t^2\nabla\mathbf{u}\|^2$, are immediate consequences of (3.10), the triangle inequality, and Theorem 2.2. This completes the proof of the validity of (3.6) and, hence, the theorem. ■

4. Finite element approximations

This section presents a discrete H^{-1} least-squares finite element approximation for the perturbed Stokes problem based on (3.5). We first discuss the well-posedness of the discrete problem and then establish optimal order error estimates in the weighted H^1-norms for each variable.

We approximate the minimum of the least-squares functional $G(\boldsymbol{\phi}, p, \mathbf{u}; r)$ defined in (3.4) using a Rayleigh–Ritz type finite element method. Let $\mathcal{T}_h$ be a partition of the Ω into finite elements; i.e., $\Omega = \cup_{K\in\mathcal{T}_h} K$ with $h = \max\{h_K = \mathrm{diam}(K) : K \in \mathcal{T}_h\}$. Assume that the triangulation $\mathcal{T}_h$ is quasi-uniform; i.e., it is regular and satisfies the inverse assumption (see [14]). Let $\mathcal{V}^h = \boldsymbol{\Phi}^h \times P^h \times \mathbf{U}^h$ be a finite-dimensional subspace of $\mathcal{V}$ such that for any $(\boldsymbol{\psi}, q, \mathbf{v}) \in (H^{\gamma+1}(\Omega)^2 \times H^{\gamma+1}(\Omega) \times H^{\gamma+1}(\Omega)^2) \cap \mathcal{V}$, there exists an interpolant of $(\boldsymbol{\psi}, q, \mathbf{v})$, denoted by $(\boldsymbol{\psi}^I, q^I, \mathbf{v}^I)$, in $\mathcal{V}^h$ satisfying

$$\|\boldsymbol{\psi} - \boldsymbol{\psi}^I\| + h\|\boldsymbol{\psi} - \boldsymbol{\psi}^I\|_1 \le Ch^{\gamma+1}\|\boldsymbol{\psi}\|_{\gamma+1} \tag{4.1}$$

$$\sum_{K\in\mathcal{T}^h} h_K(\|\Delta(\boldsymbol{\psi} - \boldsymbol{\psi}^I)\|_{0,K} + \|\nabla(\nabla\cdot(\boldsymbol{\psi} - \boldsymbol{\psi}^I))\|_{0,K}) \le Ch^{\gamma}\|\boldsymbol{\psi}\|_{\gamma+1} \tag{4.2}$$

$$\|q - q^I\| + h\|q - q^I\|_1 \le Ch^{\gamma+1}\|q\|_{\gamma+1} \tag{4.3}$$

$$\sum_{K\in\mathcal{T}^h} h_K\|\nabla(q - q^I)\|_{0,K} \le Ch^{\gamma}\|q\|_{\gamma} \tag{4.4}$$

$$\|\mathbf{v} - \mathbf{v}^I\| + h\|\mathbf{v} - \mathbf{v}^I\|_1 \le Ch^{\gamma+1}\|\mathbf{v}\|_{\gamma+1} \tag{4.5}$$

where $\gamma \ge 0$ for (4.1), (4.3) and (4.5) and $\gamma \ge 1$ for (4.2) and (4.4), and $(\cdot,\cdot)_{0,K}$ and $\|\cdot\|_{0,K}$ indicate the respective inner product and norm in $L^2(K)$. It is well known that (4.1)–(4.5) hold for typical finite element spaces consisting of continuous piecewise polynomials with respect to quasi-uniform triangulations (cf. [14]).

Note that the functional $G(\boldsymbol{\phi}, p, \mathbf{u}; r)$ defined in (3.4) involves the H^{-1}-norm, which requires solution of a boundary value problem for its evaluation, and second-order differential operators. Hence, we need to replace the H^{-1}-norm in (3.4) by a computationally feasible discrete H^{-1}-norm that ensures the equivalence on $\mathcal{V}^h$ between the standard norm in $\mathcal{V}$ and that induced by the discrete homogeneous functional. (A discrete H^{-1} approach was introduced in [3] for scalar elliptic equations and was extended to the Stokes problem in [11] and linear elasticity in [13] in the context of least-squares methods and was used for the Stokes problem in [10] in the context of stabilized finite element methods.) Also, we need to replace the Laplacian and gradient operators in the first term of the functional by the corresponding 'discrete' operators so that we can use C^0 finite element approximations.

To this end, define the operator A: $H_0^{-1}(\Omega)^2 \to H_0^1(\Omega)^2$ as the solution operator for the Poisson problem

$$\begin{cases} -\Delta\boldsymbol{\psi} + \boldsymbol{\psi} = \mathbf{v} & \text{in} \quad \Omega \\ \boldsymbol{\psi} = \mathbf{0} & \text{on} \ \partial\Omega \end{cases} \tag{4.6}$$

i.e., $A\mathbf{v} = \boldsymbol{\psi}$ for a given $\mathbf{v} \in H_0^{-1}(\Omega)^2$ is the solution of (4.6). It is well known that $(A\cdot, \cdot)^{\frac{1}{2}}$ defines a norm that is equivalent to the H_0^{-1}-norm. Let A_h: $H_0^{-1}(\Omega)^2 \to \mathbf{\Phi}^h$ be the discrete solution operator $\boldsymbol{\psi} = A_h\mathbf{v} \in \mathbf{\Phi}^h$ for the Poisson problem (4.6) defined by

$$\int_\Omega (\nabla\boldsymbol{\psi} \cdot \nabla\mathbf{w} + \boldsymbol{\psi} \cdot \mathbf{w}) = (\mathbf{v},\ \mathbf{w}), \quad \mathbf{w} \in \mathbf{\Phi}^h$$

It is easy to check that $(A_h\cdot,\ \cdot)^{\frac{1}{2}}$ defines a semi-norm on $H_0^{-1}(\Omega)^2$ which is equivalent to the discrete H_0^{-1} semi-norm

$$\|\cdot\|_{-1,h} \equiv \sup_{\boldsymbol{\psi}\in\mathbf{\Phi}^h} \frac{(\cdot,\ \boldsymbol{\psi})}{\|\boldsymbol{\psi}\|_1}$$

Assume that there is a preconditioner $B_h : H_0^{-1}(\Omega)^2 \to \mathbf{\Phi}^h$ that is symmetric with respect to the $L^2(\Omega)^2$-inner product and spectrally equivalent to A_h; i.e., there exists a positive constant C, independent of the mesh size h such that

$$\frac{1}{C}(A_h\mathbf{v},\ \mathbf{v}) \le (B_h\mathbf{v},\ \mathbf{v}) \le C(A_h\mathbf{v},\ \mathbf{v}), \quad \mathbf{v} \in \mathbf{\Phi}^h \tag{4.7}$$

Remark 1

(1) By introducing the L^2-orthogonal projection operator $Q_h : L^2(\Omega)^2 \to \mathbf{\Phi}^h$ and noting the relations that $A_h = A_hQ_h$ and $B_h = B_hQ_h$, it is easy to check that the spectral equivalence in (4.7) holds for every $\mathbf{v} \in L^2(\Omega)^2$.

(2) The spectral equivalence in (4.7) implies that

$$|\cdot|_{-1,h} \equiv (B_h\cdot,\ \cdot)^{\frac{1}{2}}$$

defines a semi-norm on $H_0^{-1}(\Omega)^2$, which is equivalent to $\|\cdot\|_{-1,h}$.

Finally, we introduce 'discrete' Laplacian and gradient operators: the 'discrete' Laplacian operator, Δ_h: $H_0^1(\Omega)^2 \to \mathbf{\Phi}^h$, for given $\mathbf{v} \in H_0^1(\Omega)^2$ is defined by $\boldsymbol{\psi} = \Delta_h\mathbf{v} \in \mathbf{\Phi}^h$ satisfying

$$(\boldsymbol{\psi},\ \mathbf{w}) = -(\nabla\mathbf{v},\ \nabla\mathbf{w}), \quad \forall\ \mathbf{w} \in \mathbf{\Phi}^h$$

and the 'discrete' gradient operator, ∇_h: $L^2(\Omega) \to \mathbf{\Phi}^h$, for given $q \in L^2(\Omega)$ is defined by $\mathbf{v} = \nabla_h q \in \mathbf{\Phi}^h$ satisfying

$$(\mathbf{v}, \mathbf{w}) = -(q, \nabla\cdot\mathbf{w}), \quad \forall\mathbf{w} \in \mathbf{\Phi}^h$$

Now, we are ready to define the discrete counterparts of the least-squares functional G as follows:

$$\begin{aligned} G_h(\boldsymbol{\phi}, p, \mathbf{u}; r) = {} & |\nabla r + \alpha_1 \Delta_h \boldsymbol{\phi} + \alpha_2 \nabla_h(\nabla \cdot \boldsymbol{\phi}) - \nabla^\perp p|^2_{-1,h} \\ & + \sum_{K \in \mathcal{T}_h} h_K^2 \|\nabla r + \alpha_1 \Delta \boldsymbol{\phi} + \alpha_2 \nabla(\nabla \cdot \boldsymbol{\phi}) - \nabla^\perp p\|^2_{0,K} \\ & + \|\nabla \times \boldsymbol{\phi} - t^2 \nabla \times \mathbf{u}\|^2 + \|t(\mathbf{u} - \nabla^\perp p)\|^2 + \|t^2 \nabla \cdot \mathbf{u}\|^2 \end{aligned} \tag{4.8}$$

Then the least-squares finite element approximation to (3.5) is to find $(\boldsymbol{\phi}^h, p^h, \mathbf{u}^h) \in \mathcal{V}^h$ such that

$$G_h(\boldsymbol{\phi}^h,\ p^h,\ \mathbf{u}^h;\ r) = \inf_{(\boldsymbol{\psi}, q, \mathbf{v}) \in \mathcal{V}^h} G_h(\boldsymbol{\psi},\ q,\ \mathbf{v};\ r)$$

The corresponding variational problem is to find $(\boldsymbol{\phi}^h, p^h, \mathbf{u}^h) \in \mathcal{V}^h$ such that

$$b_h(\boldsymbol{\phi}^h, p^h, \mathbf{u}^h;\ \boldsymbol{\psi}, q, \mathbf{v}) = r(\boldsymbol{\psi}, q, \mathbf{v}), \quad \forall\ (\boldsymbol{\psi}, q, \mathbf{v}) \in \mathcal{V}^h \tag{4.9}$$

where the bilinear form $b_h(\cdot;\cdot)$ is induced by the quadratic form $G_h(\cdot;0)$ and the linear form $r(\cdot)$ is given by

$$\begin{aligned} r(\boldsymbol{\psi}, q, \mathbf{v}) = {} & \left(B_h \nabla r, -\alpha_1 \Delta_h \boldsymbol{\psi} - \alpha_2 \nabla_h(\nabla \cdot \boldsymbol{\psi}) + \nabla^\perp q\right) \\ & + \sum_{K \in \mathcal{T}_h} h_K^2 \left(\nabla r, -\alpha_1 \Delta \boldsymbol{\psi} - \alpha_2 \nabla(\nabla \cdot \boldsymbol{\psi}) + \nabla^\perp q\right)_{0,K} \end{aligned}$$

Theorem 4.1. *The homogeneous functional $G_h(\boldsymbol{\phi}, p, \mathbf{u};\ 0)$ is elliptic and continuous in $\mathcal{V}^h$; i.e., there exists a positive constant C independent of h and t such that*

$$\frac{1}{C} |||(\boldsymbol{\phi},\ p,\ \mathbf{u})|||^2_{\mathcal{V}} \le G_h(\boldsymbol{\phi}, p, \mathbf{u};\ 0) \tag{4.10}$$

and that

$$G_h(\boldsymbol{\phi}, p, \mathbf{u};\ 0) \le C\, |||(\boldsymbol{\phi},\ p,\ \mathbf{u})|||^2_{\mathcal{V}} \tag{4.11}$$

for any $(\boldsymbol{\phi}, p, \mathbf{u}) \in \mathcal{V}^h$.

Proof

By the definitions of the 'discrete' Laplacian and gradient operators and the Cauchy–Schwarz inequality, we have that for $\boldsymbol{\phi} \in \boldsymbol{\Phi}^h$,

$$\|\Delta_h \boldsymbol{\phi}\|_{-1,h} = \sup_{\boldsymbol{\psi} \in \boldsymbol{\Phi}^h} \frac{|(\Delta_h \boldsymbol{\phi}, \boldsymbol{\psi})|}{\|\boldsymbol{\psi}\|_1} = \sup_{\boldsymbol{\psi} \in \boldsymbol{\Phi}^h} \frac{|(\nabla \boldsymbol{\phi}, \nabla \boldsymbol{\psi})|}{\|\boldsymbol{\psi}\|_1} \le \|\nabla \boldsymbol{\phi}\| \tag{4.12}$$

and

$$\|\nabla_h(\nabla \cdot \boldsymbol{\phi})\|_{-1,h} = \sup_{\boldsymbol{\psi} \in \boldsymbol{\Phi}^h} \frac{|(\nabla_h(\nabla \cdot \boldsymbol{\phi}),\ \boldsymbol{\psi})|}{\|\boldsymbol{\psi}\|_1} = \sup_{\boldsymbol{\psi} \in \boldsymbol{\Phi}^h} \frac{|(\nabla \cdot \boldsymbol{\phi},\ \nabla \cdot \boldsymbol{\psi})|}{\|\boldsymbol{\psi}\|_1} \le \|\nabla \boldsymbol{\phi}\| \tag{4.13}$$

The upper bound in (4.11) is then a straightforward consequence of Remark 1, the triangle and inverse inequalities, and the bound that

$$|\nabla^{\perp} p|_{-1,h} \le C\|\nabla^{\perp} p\|_{-1,h} \le C\|\nabla^{\perp} p\|_{-1,0} \le C\|p\|$$

Next, we show the validity of (4.10). Note first that (4.1) with $\gamma = 0$ implies that, for any $\boldsymbol{\psi} \in H_0^1(\Omega)^2$,

$$\|\boldsymbol{\psi} - Q_h \boldsymbol{\psi}\| \le Ch\|\boldsymbol{\psi}\|_1 \quad \text{and} \quad \|Q_h \boldsymbol{\psi}\|_1 \le C\|\boldsymbol{\psi}\|_1$$

For any $\boldsymbol{\psi} \in L^2(\Omega)^2$, a standard duality argument implies that

$$\|\boldsymbol{\psi} - Q_h \boldsymbol{\psi}\|_{-1,0} \le Ch\|\boldsymbol{\psi}\| \le C\left(\sum_{K\in\mathcal{T}_h} h_K^2 \|\boldsymbol{\psi}\|_{0,K}^2\right)^{\frac{1}{2}}$$

and

$$\|Q_h \boldsymbol{\psi}\|_{-1,0} = \sup_{\mathbf{w}\in H_0^1(\Omega)^2} \frac{(Q_h \boldsymbol{\psi},\ \mathbf{w})}{\|\mathbf{w}\|_1} \le \sup_{\mathbf{w}\in H_0^1(\Omega)^2} \frac{(\boldsymbol{\psi},\ Q_h \mathbf{w})}{\|Q_h \mathbf{w}\|_1} \le C\|\boldsymbol{\psi}\|_{-1,h}$$

Hence,

$$\|\boldsymbol{\psi}\|_{-1,0}^2 \le C\left(\sum_{K\in\mathcal{T}_h} h_K^2 \|\boldsymbol{\psi}\|_{0,K}^2 + \|\boldsymbol{\psi}\|_{-1,h}^2\right)$$

which, with the choice $\boldsymbol{\psi} = \nabla^{\perp} p$ and the inequality (3.9), gives

$$\|p\|^2 \le C\left(\sum_{K\in\mathcal{T}_h} h_K^2 \|\nabla^{\perp} p\|_{0,K}^2 + \|\nabla^{\perp} p\|_{-1,h}^2\right), \quad p \in P^h$$

It then follows from the triangle and inverse inequalities, (4.12), (4.13), and Remark 1 that

$$\begin{aligned}\|p\|^2 \le{}& C\left(\sum_{K\in\mathcal{T}_h} h_K^2 (\|\alpha_1 \Delta \boldsymbol{\phi} + \alpha_2 \nabla(\nabla\cdot\boldsymbol{\phi}) - \nabla^{\perp} p\|_{0,K}^2 + \|\Delta\boldsymbol{\phi}\|_{0,K}^2 + \|\nabla(\nabla\cdot\boldsymbol{\phi})\|_{0,K}^2)\right)\\ &+C\left(\|\alpha_1\Delta_h\boldsymbol{\phi} + \alpha_2\nabla_h(\nabla\cdot\boldsymbol{\phi}) - \nabla^{\perp} p\|_{-1,h}^2 + \|\Delta_h\boldsymbol{\phi}\|_{-1,h}^2 + \|\nabla_h(\nabla\cdot\boldsymbol{\phi})\|_{-1,h}^2\right)\\ &\le C\, G_h(\boldsymbol{\phi}, p, \mathbf{u};\ 0) + C\|\boldsymbol{\phi}\|_1^2 \end{aligned} \tag{4.14}$$

By the Korn inequality, the definitions of the 'discrete' Laplacian and gradient operators, and Remark 1, we have that for any $\boldsymbol{\phi} \in \boldsymbol{\Phi}^h$

$$\begin{aligned} C_0\|\boldsymbol{\phi}\|_1 &\le \alpha_1 (\nabla\boldsymbol{\phi},\ \nabla\boldsymbol{\phi}) + \alpha_2 (\nabla\cdot\boldsymbol{\phi},\ \ \nabla\cdot\boldsymbol{\phi})\\ &= (-\alpha_1\Delta_h\boldsymbol{\phi} - \alpha_2\nabla_h(\nabla\cdot\boldsymbol{\phi}),\ \ \boldsymbol{\phi})\\ &= \left(-\alpha_1\Delta_h\boldsymbol{\phi} - \alpha_2\nabla_h(\nabla\cdot\boldsymbol{\phi}) + \nabla^{\perp} p,\ \boldsymbol{\phi}\right) - \left(\nabla^{\perp} p,\ \boldsymbol{\phi}\right)\\ &\le C\,|-\alpha_1\Delta_h\boldsymbol{\phi} - \alpha_2\nabla_h(\nabla\cdot\boldsymbol{\phi}) + \nabla^{\perp} p|_{-1,h}\|\boldsymbol{\phi}\|_1 - (p, \nabla\times\boldsymbol{\phi})\\ &\le C\, G_h^{\frac{1}{2}}(\boldsymbol{\phi}, p, \mathbf{u};\ 0)\|\boldsymbol{\phi}\|_1 + (\|p\| + \|t\mathbf{u}\|)\, G_h^{\frac{1}{2}}(\boldsymbol{\phi}, p, \mathbf{u};\ 0) - \|t\mathbf{u}\|^2 \end{aligned}$$

The last inequality follows similarly to the proof of (3.12). We then have by the ϵ-inequality and (4.14) that

$$\|\boldsymbol{\phi}\|_1^2 + \|t\mathbf{u}\|^2 \le C\, G_h(\boldsymbol{\phi}, p, \mathbf{u};\ 0)$$

Upper bounds in (4.10) for the terms, $\|p\|^2$, $\|t\nabla p\|^2$, and $\|t^2\nabla\mathbf{u}\|^2$, can be established by an argument similar to that in the proof of Theorem 3.1. This completes the proof of the theorem. ■

Theorem 4.2. *Let $(\boldsymbol{\phi}, p, \mathbf{u})$ and $(\boldsymbol{\phi}^h, p^h, \mathbf{u}^h) \in \mathcal{V}^h$ be the solutions of (3.5) and (4.9), respectively. Assume that $(\boldsymbol{\phi}, p)$ is in $H^{\gamma+1}(\Omega)^2 \times H^{\gamma+2}(\Omega)$ with $\gamma \ge 1$. Then there exists a positive constant C independent of the thickness t and the mesh size h such that*

$$\begin{aligned}\|\boldsymbol{\phi} - \boldsymbol{\phi}^h\|_1 + \|p - p^h\| + \|t\nabla(p - p^h)\| + \|t(\mathbf{u} - \mathbf{u}^h)\| + \|t^2\nabla(\mathbf{u} - \mathbf{u}^h)\| \\ \le Ch^\gamma \left(\|\boldsymbol{\phi}\|_{\gamma+1} + \|p\|_\gamma + \|tp\|_{\gamma+1} + \|t\mathbf{u}\|_\gamma + \|t^2\mathbf{u}\|_{\gamma+1}\right)\end{aligned} \tag{4.15}$$

Moreover, if the domain Ω is bounded and sufficiently smooth, then we have that for $\gamma = 1$,

$$\|\boldsymbol{\phi}-\boldsymbol{\phi}^h\|_1+\|p-p^h\|+\|t\nabla(p-p^h)\|+\|t(\mathbf{u}-\mathbf{u}^h)\|+\|t^2\nabla(\mathbf{u}-\mathbf{u}^h)\| \le Ch\|g\|_{-1,0} \tag{4.16}$$

Proof
It is easy to see that the approximation error, $(\boldsymbol{\phi} - \boldsymbol{\phi}^h, p - p^h, \mathbf{u} - \mathbf{u}^h)$, satisfies the error equation

$$b_h(\boldsymbol{\phi} - \boldsymbol{\phi}^h, p - p^h, \mathbf{u} - \mathbf{u}^h;\ \boldsymbol{\psi}, q, \mathbf{v}) = 0, \quad \forall\ (\boldsymbol{\psi}, q, \mathbf{v}) \in \mathcal{V}^h$$

Let $(\boldsymbol{\phi}^I, p^I, \mathbf{u}^I)$ be an interpolant of $(\boldsymbol{\phi}, p, \mathbf{u})$ satisfying (4.1)–(4.5). To show the validity of (4.15), it suffices to prove that

$$\begin{aligned}\|\boldsymbol{\phi}^I - \boldsymbol{\phi}^h\|_1 + \|p^I - p^h\| + \|t\nabla(p^I - p^h)\| + \|t(\mathbf{u}^I - \mathbf{u}^h)\| + \|t^2\nabla(\mathbf{u}^I - \mathbf{u}^h)\| \\ \le Ch^\gamma \left(\|\boldsymbol{\phi}\|_{\gamma+1} + \|p\|_\gamma + \|tp\|_{\gamma+1} + \|t\mathbf{u}\|_\gamma + \|t^2\mathbf{u}\|_{\gamma+1}\right)\end{aligned}$$

This follows from Theorem 4.1, the above error equation, the Cauchy–Schwarz triangle and inverse inequalities, and approximation properties (4.1), (4.3) and (4.5). If the domain Ω is sufficiently smooth, then the second equation in (2.7) implies that

$$\|t^2 p\|_3 \le C\|\nabla \times \boldsymbol{\phi}\|_1$$

Now, (4.16) is then a direct consequence of (4.15), Theorem 2.1 and the relation $\mathbf{u} = \nabla^\perp p$. This completes the proof of the theorem. ■

Theorem 4.1 indicates that the discrete H^{-1} least-squares functional $G_h(\boldsymbol{\phi}, p, \mathbf{u}; 0)$ defined in (4.8) can be preconditioned by the functional

$$\|\boldsymbol{\phi}\|_1^2 + \|p\|^2 + \|t\nabla p\|^2 + \|\mathbf{u}\|^2 + \|t^2\nabla\mathbf{u}\|^2$$

that decouples $\boldsymbol{\phi}$, p and $\mathbf{u}$ unkonwns, because they are spectrally equivalent uniformly in the thickness t and the mesh size h. This means that the discrete system (4.9) can be uniformly preconditioned by a block diagonal preconditioner

$$\begin{pmatrix} A_{h,\boldsymbol{\phi}} & 0 & 0 \\ 0 & A_{h,p} & 0 \\ 0 & 0 & A_{h,\mathbf{u}} \end{pmatrix}$$

Here the blocks correspond to the ordering of unknowns of $\boldsymbol{\phi}$, p and $\mathbf{u}$, and $A_{h,\boldsymbol{\phi}}$, $A_{h,p}$ and $A_{h,\mathbf{u}}$ are the respective discrete solution operators of the Poisson problems:

$$\begin{array}{llll} \text{find} & \boldsymbol{\phi} \in \boldsymbol{\Phi}^h & \text{such that} & (\nabla \boldsymbol{\phi}, \nabla \boldsymbol{\psi}) + (\boldsymbol{\phi}, \boldsymbol{\psi}) = (\boldsymbol{\phi}_0, \boldsymbol{\psi}), \quad \forall\ \boldsymbol{\psi} \in \boldsymbol{\Phi}^h \\ \text{find} & p \in P^h & \text{such that} & t^2(\nabla p, \nabla q) + (p, q) = (p_0, q), \quad \forall\ q \in P^h \\ \text{find} & \mathbf{u} \in \mathbf{U}^h & \text{such that} & t^4(\nabla \mathbf{u}, \nabla \mathbf{v}) + t^2(\mathbf{u}, \mathbf{v}) = (\mathbf{u}_0, \mathbf{v}), \quad \forall\ \mathbf{v} \in \mathbf{U}^h \end{array}$$

We can then replace $A_{h,i}$ for $i = \boldsymbol{\phi},\ p,\ \mathbf{u}$ by any effective elliptic preconditioners including those of multigrid type. Note that the Poisson problems for p and $\mathbf{u}$ are weighted by t in such a way that the condition numbers of the discrete operators are $O\left((\frac{t}{h})^2 + 1\right)$. Therefore, we can replace $A_{h,p}$ and $A_{h,\mathbf{u}}$ by simple preconditioners including those of diagonal matrix type when the thickness t is relatively small compared with the mesh size h. With preconditioners mentioned above, the preconditioned system has h-independent condition number but we have no estimate of its actual size.

5. Pure traction boundary conditions

The strong form of the Reissner–Mindlin plate model with pure traction boundary conditions and $\lambda = 1$ is given by

$$\begin{cases} -\alpha_1 \Delta \boldsymbol{\phi} - \alpha_2 \nabla(\nabla \cdot \boldsymbol{\phi}) + t^{-2}(\boldsymbol{\phi} - \nabla \omega) = \mathbf{0}, & \text{in } \Omega \\ t^{-2} \nabla \cdot (\boldsymbol{\phi} - \nabla \omega) = g, & \text{in } \Omega \end{cases} \tag{5.1}$$

which satisfies the following boundary conditions

$$\sigma(\boldsymbol{\phi})\mathbf{n} \equiv \frac{E}{12(1-\nu^2)}\left((1-\nu)\mathscr{E}(\boldsymbol{\phi}) + \nu \nabla \cdot \boldsymbol{\phi} \mathscr{I}\right)\mathbf{n} = \mathbf{0} \quad \text{and} \quad \boldsymbol{\phi}\cdot\mathbf{n} - \nabla\omega\cdot\mathbf{n} = 0 \quad \text{on } \partial\Omega \tag{5.2}$$

where $\mathscr{I}$ is the 2×2 identity matrix and $\mathscr{E}(\boldsymbol{\phi}) = \left(\frac{1}{2}(\partial_i \phi_j + \partial_j \phi_i)\right)_{2\times 2}$ the deformation. Let $\mathscr{L}$ be the space of linear functions on Ω; i.e.,

$$\mathscr{L} = \{a + bx_1 + cx_2 \ :\ a, b, c, \in R\}$$

and $\mathscr{L}^{\perp}$ its orthogonal complement in $L^2(\Omega)$. Then (the weak form of) boundary value problem (5.1)–(5.2) has a unique solution, $(\boldsymbol{\phi}, \omega) \in H^1(\Omega)^2 \times H^1(\Omega)$, up to additive functions in $\left(\nabla\mathscr{L}\right) \times \mathscr{L}$ for any $g \in H^{-1}(\Omega) \cap \mathscr{L}^{\perp}$.

In this section, we extend our previous discussions to the pure traction problem (5.1)–(5.2). We also consider imposing some boundary conditions on the functional rather than the solution space. Because the development of computable finite element approximations and the corresponding efficient iterative solvers or preconditioners, based on the least-squares functional involving the H^{-1}-norm, becomes standard (see, for example, Section 4), we are, therefore, focusing on establishing ellipticity here and in the subsequent section.

To this end, define $r \in H^1(\Omega)$ to be the solution of

$$\begin{cases} -\Delta r = g, & \text{in } \Omega, \\ \nabla r \cdot \mathbf{n} = 0, & \text{on } \partial\Omega \end{cases} \tag{5.3}$$

Then we again have the Helmholtz decomposition (2.2) for the transverse shear strain, but the boundary condition for p is now that $\nabla p \cdot \boldsymbol{\tau} = 0$ on $\partial\Omega$. We may normalize p so that $p = 0$ on $\partial\Omega$. Substituting (2.2) into the first equation in (5.1) and the second boundary condition in (5.2) and applying $\nabla\times$ to (2.2) give the perturbed Stokes equation (2.7) but with the following boundary conditions

$$\sigma(\boldsymbol{\phi})\mathbf{n} = \mathbf{0} \quad \text{and} \quad p = 0 \quad \text{on } \partial\Omega \tag{5.4}$$

The second equation and the second boundary condition in the respective (5.1) and (5.2) lead to

$$\begin{cases} -\Delta\omega = t^2 g - \nabla \cdot \boldsymbol{\phi}, & \text{in } \Omega \\ \nabla\omega \cdot \mathbf{n} = \quad \boldsymbol{\phi} \cdot \mathbf{n}, & \text{on } \partial\Omega \end{cases} \tag{5.5}$$

Let

$$\mathcal{V}_P \equiv \mathbf{\Phi} \times H_0^1(\Omega) \times U \quad \text{and} \quad \mathbf{\Phi} \equiv \{\boldsymbol{\psi} \in (H^1(\Omega)/R)^2 \,:\, \sigma(\boldsymbol{\phi})\mathbf{n} = \mathbf{0} \text{ on } \partial\Omega\}$$

Then the similar proof as that of Theorem 3.1 gives:

Theorem 5.1. *For the functional G defined in* (3.4)*, there exists a positive constant C independent of the thickness t such that*

$$\frac{1}{C}|||(\boldsymbol{\phi},\ p,\ \mathbf{u})|||_{\mathcal{V}}^2 \le G(\boldsymbol{\phi},\ p,\ \mathbf{u};\ 0) \le C\,|||(\boldsymbol{\phi},\ p,\ \mathbf{u})|||_{\mathcal{V}}^2 \tag{5.6}$$

for any $(\boldsymbol{\phi},\ p,\ \mathbf{u}) \in \mathcal{V}_P$.

Instead of imposing traction boundary conditions of $\boldsymbol{\phi}$ on the solution space, we may enforce them weakly in the least-squares functional. More specifically, we modify the functional G as follows:

$$\hat{G}(\boldsymbol{\phi},\ p,\ \mathbf{u};\ r) = G(\boldsymbol{\phi},\ p,\ \mathbf{u};\ r) + \|\sigma(\boldsymbol{\phi})\mathbf{n}\|_{\partial\Omega,-\frac{1}{2}}^2$$

where $\|\cdot\|_{\partial\Omega,-\frac{1}{2}}$ is $H^{-\frac{1}{2}}(\partial\Omega)$ norm (see [16] for the discrete counterpart of the boundary integral term). Let

$$\hat{\mathcal{V}}_P \equiv (H^1(\Omega)/\mathcal{R})^2 \times H_0^1(\Omega) \times U$$

and denote the weighted norm on $\hat{\mathcal{V}}_P$ by

$$|||(\boldsymbol{\phi},\ p,\ \mathbf{u})|||_{\hat{\mathcal{V}}_P} \equiv \left(|||(\boldsymbol{\phi},\ p,\ \mathbf{u})|||_{\mathcal{V}}^2 + \|\sigma(\boldsymbol{\phi})\mathbf{n}\|_{\partial\Omega,-\frac{1}{2}}^2\right)^{\frac{1}{2}}$$

Theorem 5.2. *There exists a positive constant C independent of the thickness t such that*

$$\frac{1}{C}|||(\boldsymbol{\phi},\ p,\ \mathbf{u})|||_{\hat{\mathcal{V}}_P}^2 \le \hat{G}(\boldsymbol{\phi},\ p,\ \mathbf{u};\ 0) \le C\,|||(\boldsymbol{\phi},\ p,\ \mathbf{u})|||_{\hat{\mathcal{V}}_P}^2 \tag{5.7}$$

for any $(\boldsymbol{\phi}, p, \mathbf{u}) \in \hat{\mathcal{V}}_P$.

Proof
Upper bound (3.7) for G immediately implies the upper bound in (5.7). Note that

$$\alpha_1 (\nabla\boldsymbol{\phi}, \nabla\boldsymbol{\phi}) + \alpha_2 (\nabla\cdot\boldsymbol{\phi}, \nabla\cdot\boldsymbol{\phi}) = (-\nabla\cdot\sigma(\boldsymbol{\phi}), \boldsymbol{\phi}) + \int_{\partial\Omega} \boldsymbol{\phi}\cdot(\sigma(\boldsymbol{\phi})\mathbf{n})\mathrm{d}s$$

and that

$$\left|\int_{\partial\Omega} \boldsymbol{\phi}\cdot(\sigma(\boldsymbol{\phi})\mathbf{n})\mathrm{d}s\right| \le \|\sigma(\boldsymbol{\phi})\mathbf{n}\|_{\partial\Omega,-\frac{1}{2}}\|\boldsymbol{\phi}\|_{\partial\Omega,\frac{1}{2}} \le C\|\sigma(\boldsymbol{\phi})\mathbf{n}\|^2_{\partial\Omega,-\frac{1}{2}}\|\boldsymbol{\phi}\|_1$$

for any $\boldsymbol{\phi} \in \left(H^1(\Omega)/R\right)^2$. Now the proof of the lower bound in (5.7) follows in a similar fashion as that of lower bound (3.6). ■

6. Another least-squares formulation

The least-squares formulation introduced in Section 3 involves the weighted H^1-norm of $\mathbf{u}$ and hence, error estimates in Theorem 4.2 require that $p \in H^{\gamma+2}(\Omega)$ while $\boldsymbol{\phi} \in H^{\gamma+1}(\Omega)^2$. As an alternative, this section presents another least-squares formulation by introducing the curl of the shear strain that is the quantity representing a different scale. The latter has the same numerical properties as those of the former, but no extra regularity requirement. As in the previous section, we are focusing on establishing ellipticity here.

To this end, let $w = \nabla\times\boldsymbol{\phi}$, then the perturbed Stokes equation may be rewritten as

$$\begin{cases} -\alpha_1\Delta\boldsymbol{\phi} - \alpha_2\nabla(\nabla\cdot\boldsymbol{\phi}) + \nabla^{\perp}p = \nabla r, & \text{in } \Omega \\ w + t^2\Delta p = 0, & \text{in } \Omega \\ w - \nabla\times\boldsymbol{\phi} = 0, & \text{in } \Omega \end{cases} \tag{6.1}$$

with boundary conditions

$$\boldsymbol{\phi} = \mathbf{0} \quad \text{and} \quad \nabla p\cdot\mathbf{n} = 0 \quad \text{on } \partial\Omega$$

and the compatibility condition

$$\int_{\Omega} w\,\mathrm{d}x = 0$$

Let

$$\mathcal{V}_1 \equiv H_0^1(\Omega)^2 \times \mathcal{P} \times L_0^2(\Omega) \quad \text{and} \quad \mathcal{P} = \{q \in H^1(\Omega)/R \,:\, \nabla q\cdot\mathbf{n} = 0 \text{ an } \partial\Omega\}$$

and denote the weighted norm over $\mathcal{V}_1$ by

$$|||(\boldsymbol{\phi}, p, w)|||_{\mathcal{V}_1} \equiv \left(\|\boldsymbol{\phi}\|_1^2 + \|p\|^2 + \|t\nabla^{\perp}p\|^2 + \|\frac{1}{t}w\|_{-1}^2 + \|w\|^2\right)^{\frac{1}{2}}$$

We then consider the following least-squares functional

$$G_1(\boldsymbol{\phi},\ p,\ w;\ r) = \|\nabla r + \alpha_1 \Delta \boldsymbol{\phi} + \alpha_2 \nabla(\nabla \cdot \boldsymbol{\phi}) - \nabla^{\perp} p\|^2_{-1,0} + \|\tfrac{1}{t}(w + t^2 \Delta p)\|^2_{-1} + \|w - \nabla \times \boldsymbol{\phi}\|^2 \tag{6.2}$$

for any $(\boldsymbol{\phi}, p, w) \in \mathcal{V}_1$.

Theorem 6.1. *There exists a positive constant C independent of the thickness t such that*

$$\frac{1}{C} |||(\boldsymbol{\phi},\ p,\ w)|||^2_{\mathcal{V}_1} \le G_1(\boldsymbol{\phi},\ p,\ w; 0) \le C\,|||(\boldsymbol{\phi},\ p,\ w)|||^2_{\mathcal{V}_1} \tag{6.3}$$

for any $(\boldsymbol{\phi}, p, w) \in \mathcal{V}_1$.

Proof

The upper bound in (6.3) is a straightforward consequence of the triangle inequalities, (3.8), and the easily established bound

$$\|\Delta p\|_{-1} = \sup_{v \in H^1(\Omega)} \frac{(\Delta p, v)}{\|v\|_1} = \sup_{v \in H^1(\Omega)} \frac{(\nabla p, \nabla v)}{\|v\|_1} \le \|\nabla^{\perp} p\| \tag{6.4}$$

It follows from the integration by parts, the Cauchy–Schwarz and Poincaré–Friedrichs inequalities, and the definition of the H^{-1}-norm that

$$\begin{aligned} -(p, \nabla \times \boldsymbol{\phi}) &= (p,\ w - \nabla \times \boldsymbol{\phi}) - (p, w + t^2 \Delta p) + (p,\ t^2 \Delta p) \\ &= (p,\ w - \nabla \times \boldsymbol{\phi}) - (tp,\ \frac{1}{t}(w + t^2 \Delta p)) - \|t \nabla^{\perp} p\|^2 \\ &\le \|p\|\,\|w - \nabla \times \boldsymbol{\phi}\| + C\|t \nabla^{\perp} p\|\,\|\frac{1}{t}(w + t^2 \Delta p)\|_{-1} - \|t \nabla^{\perp} p\|^2 \end{aligned} \tag{6.5}$$

Combining with (3.11), and using (3.10) and the ϵ-inequality imply that

$$C_0 \|\boldsymbol{\phi}\|_1^2 + \|t \nabla^{\perp} p\|^2 \le C G_1(\boldsymbol{\phi}, p, w; 0)$$

and, hence, that

$$\|p\|^2 \le C G_1(\boldsymbol{\phi}, p, w; 0)$$

Lower bounds in (6.3) for the terms $\|\frac{1}{t} w\|^2_{-1}$ and $\|w\|^2$ are immediate consequences of the triangle inequality and (6.4). This completes the proof of the validity of the lower bound in (6.3) and, hence, the theorem. ■

As in the previous section, we may enforce weakly the boundary condition of p in the least-squares functional by modifying the functional G_1 as follows:

$$\hat{G}_1(\boldsymbol{\phi}, p, w; r) = G_1(\boldsymbol{\phi}, p, w; r) + t^2 \|\frac{\partial p}{\partial \mathbf{n}}\|^2_{\partial\Omega, -\frac{1}{2}}$$

for any $(\boldsymbol{\phi}, p, w)$ in

$$\hat{\mathcal{V}}_1 \equiv H^1_0(\Omega)^2 \times (H^1(\Omega)/R) \times L^2_0(\Omega)$$

By noting that the inequality (6.4) becomes

$$\|\Delta p\|_{-1} = \sup_{v \in H^1(\Omega)} \frac{-(\nabla p, \nabla v) + \int_{\partial\Omega} v \frac{\partial p}{\partial \mathbf{n}} \mathrm{d}s}{\|v\|_1} \le \|\nabla^{\perp} p\| + C \|\frac{\partial p}{\partial \mathbf{n}}\|^2_{\partial\Omega, -\frac{1}{2}}$$

and that (6.5) becomes

$$\begin{aligned} -(p, \nabla \times \boldsymbol{\phi}) &= (p,\ w - \nabla \times \boldsymbol{\phi}) - (tp,\ \frac{1}{t}(w + t^2 \Delta p)) - \|t \nabla^{\perp} p\|^2 + t^2 \int_{\partial\Omega} p \frac{\partial p}{\partial \mathbf{n}} \mathrm{d}s \\ &\le \|p\|\, \|w - \nabla \times \boldsymbol{\phi}\| + C \|t \nabla^{\perp} p\|\, \|\frac{1}{t}(w + t^2 \Delta p)\|_{-1} - \|t \nabla^{\perp} p\|^2 \\ &\quad + C \|t \nabla^{\perp} p\|\, \|t \frac{\partial p}{\partial \mathbf{n}}\|^2_{\partial\Omega, -\frac{1}{2}} \end{aligned}$$

Then the same argument as that in the proof of Theorem 6.1 gives

Theorem 6.2. *There exists a positive constant C independent of the thickness t such that*

$$\frac{1}{C} |||(\boldsymbol{\phi},\ p,\ w)|||^2_{\hat{\mathcal{V}}_1} \le \hat{G}_1(\boldsymbol{\phi},\ p,\ w; 0) \le C\, |||(\boldsymbol{\phi},\ p,\ w)|||^2_{\hat{\mathcal{V}}_1} \qquad (6.6)$$

for any $(\boldsymbol{\phi}, p, w) \in \hat{\mathcal{V}}_1$ with

$$|||(\boldsymbol{\phi},\ p,\ w)|||_{\hat{\mathcal{V}}_1} \equiv \left(|||(\boldsymbol{\phi},\ p,\ w)|||^2_{\mathcal{V}_1} + \|\frac{\partial p}{\partial \mathbf{n}}\|^2_{\partial\Omega, -\frac{1}{2}} \right)^{\frac{1}{2}}$$

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