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Error estimate of a finite element method using stress intensity factor

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ABSTRACT

An algorithm on computing accurate finite element approximation to the Poisson equation on a polygonal domain with corner singularities was studied in Kim and Lee (2016, 2017) numerically. The algorithm requires several iterations depending on singularities of the solution. Each iteration requires a solution of the standard finite element approximation to the Poisson equation with possible different Dirichlet data and the corresponding stress intensity factors. This paper provides an error estimate of the finite element approximation given by the algorithm, and, hence, determine the number of iterations needed to achieve full rates of convergence in both the energy and the L^2 norms.

1. Introduction

Let Ω be an open, bounded polygonal domain in R^2, and let Γ_D and Γ_N be a partition of the boundary of Ω such that $\partial\Omega = \overline{\Gamma}_D \cup \overline{\Gamma}_N$ and $\Gamma_D \cap \Gamma_N = \emptyset$. For simplicity, assume $\Gamma_D \neq \emptyset$. As a model problem, consider the following Poisson equation with homogeneous mixed boundary conditions:

$$\begin{cases} -\Delta u &= f, \quad \text{in } \Omega, \\ u &= 0, \quad \text{on } \Gamma_D, \\ \dfrac{\partial u}{\partial \nu} &= 0, \quad \text{on } \Gamma_N, \end{cases} \tag{1.1}$$

where $f \in L^2(\Omega)$, Δ stands for the Laplacian operator, and ν denotes the outward unit vector normal to the boundary.

When the solution of (1.1) is sufficiently smooth, e.g., $\Gamma_N = \emptyset$ and the domain is convex or smooth, the conforming linear finite element approximation is quasi-optimal. This is no longer true if the domain is non-convex. Let ω be the largest internal angle of reentrant corners of the domain Ω satisfying $\pi < \omega < 2\pi$. Then the solution u of (1.1) is only in $H^{1+r}(\Omega)$ with $r < 1 + \frac{\pi}{\omega}$, but not in $H^2(\Omega)$. Such lack of regularity affects the accuracy of the finite element approximation and, hence, the approximation to the stress intensity factor. There were several approaches in the literatures for overcoming this difficulty by making use of the following singular function representation of the solution:

$$u = w + \lambda \eta s,$$

where $w \in H^2(\Omega) \cap H^1_D(\Omega)$, η is a smooth cut-off function, and s is a singular function; the coefficient λ of the singular function is the so-called stress intensity factor and can be computed by an extraction formula. One approach is the so-called

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singular function method (SFM) by augmenting the approximation space with the singular functions (see, e.g., [1,2]). Another one is the dual singular function method (DSFM) where extraction formulas are used (see, e.g., [3–5]). Its multigrid version is given in [6]. To avoid iteration, a method was studied in [7,8] to first find the regular part of the solution and then the stress intensity factors and the solution. This approach results in a non-symmetric equation for the regular part of the solution due to the correction term (see [7]).

Recently, we [9] showed that the regular part of the solution satisfies the original Poisson equation with an updated Dirichlet data using the intensity factors (see (3.2)), where the intensity factors depend on the solution of the original problem. This leads to an accurate method which solves the original equation first to compute the intensity factors and then the Poisson equation for the regular part in (3.2). This was confirmed by numerical results for the Poisson problem with Dirichlet boundary condition in [9]. Unfortunately, this procedure does not produce quasi-optimal approximation if the underlying problem has a strong singularity, e.g., in the case of mixed boundary conditions with a reasonable large inner angle. In [10], we found numerically that several repetitions of the procedure in [9] does produce quasi-optimal approximation. The number of iterations depends on the singularity of the underlying problem. The purpose of this paper is to analyze this procedure by establishing an a priori error estimate and to quantify the number of iterations needed to achieve quasi-optimal approximation.

The paper is organized as follows. Section 2 introduces the singular and the dual singular functions together with the extraction formula for computing the stress intensity factors. An accurate numerical method is described in Section 3, and we establish a priori error estimates in Section 4.

We will use the standard notation and definitions for the Sobolev spaces $H^t(\Omega)$ for $t \geq 0$; the standard associated inner products are denoted by $(\cdot,\cdot)_{t,\Omega}$, and their respective norms and seminorms are denoted by $\|\cdot\|_{t,\Omega}$ and $|\cdot|_{t,\Omega}$. The space $L^2(\Omega)$ is interpreted as $H^0(\Omega)$, in which case the inner product and norm will be denoted by $(\cdot,\cdot)_\Omega$ and $\|\cdot\|_\Omega$, respectively. We will omit the subscript Ω when there is no ambiguity. Set $H^1_D(\Omega) = \{u \in H^1(\Omega) : u = 0 \text{ on } \Gamma_D\}$.

2. Singular function representation

This section describes singular and dual singular functions and the singular function representation of the solution of (1.1).

To this end, first consider the case that boundary condition does not change its type at vertices of reentrant corners with inner angle $\omega \in (\pi, 2\pi]$. The singular function is given by

$$s_2 = s_2(r,\theta) = \begin{cases} r^{\frac{\pi}{\omega}} \sin\dfrac{\pi\theta}{\omega}, & \text{Dirichlet boundary condition (D/D)}, \\ r^{\frac{\pi}{\omega}} \cos\dfrac{\pi\theta}{\omega}, & \text{Neumann boundary condition (N/N)} \end{cases} \tag{2.1}$$

for all $\theta \in [0, \omega]$.

In the case that boundary condition does change its type, the singular function depends on both the angle and the orientation. Denote by D/N and N/D the changes of boundary conditions passing through the singular point in the counterclockwise orientation, where D and N mean the respective Dirichlet and Neumann boundary conditions. For $\omega \in (\frac{\pi}{2}, \frac{3\pi}{2}]$, there is only one singular function of the form

$$s_1 = s_1(r,\theta) = \begin{cases} r^{\frac{\pi}{2\omega}} \sin\dfrac{\pi\theta}{2\omega}, & \text{D/N}, \\ r^{\frac{\pi}{2\omega}} \cos\dfrac{\pi\theta}{2\omega}, & \text{N/D}, \end{cases} \quad \forall\ \theta \in [0, \omega]. \tag{2.2}$$

For $\omega \in (\frac{3\pi}{2}, 2\pi]$, there are two singular functions: $s_1(r,\theta)$ defined in (2.2) and $s_3(r,\theta)$ of the form

$$s_3 = s_3(r,\theta) = \begin{cases} r^{\frac{3\pi}{2\omega}} \sin\dfrac{3\pi\theta}{2\omega}, & \text{D/N}, \\ r^{\frac{3\pi}{2\omega}} \cos\dfrac{3\pi\theta}{2\omega}, & \text{N/D} \end{cases} \quad \forall\ \theta \in [0, \omega]. \tag{2.3}$$

For convenience, denote the index set of singular functions by $L = \{2\}$ for (2.1), $\{1\}$ for (2.2), and $\{1, 3\}$ for the third case.

To state the singular function representation of the solution, one needs to introduce a cut-off function for isolating the singular behavior of the solution. To this end, set

$$B(r_1; r_2) = \{(r,\theta) : r_1 < r < r_2 \text{ and } 0 < \theta < \omega\} \cap \Omega \quad \text{and} \quad B(r_1) = B(0; r_1).$$

Then the cut-off function of r is given as follows:

$$\eta(r) = \begin{cases} 1, & \text{in } B(\frac{1}{2}\rho), \\ \dfrac{15}{16}\left\{\dfrac{8}{15} - p(r) + \dfrac{2}{3}p(r)^3 - \dfrac{1}{5}p(r)^5\right\}, & \text{in } \bar{B}(\frac{1}{2}\rho;\ \rho), \\ 0, & \text{in } \Omega \setminus \bar{B}(\rho), \end{cases} \tag{2.4}$$

where $p(r) = 4r/\rho - 3$ and ρ is a fixed constant such that the support of $\eta(r)\, s_j(r,\ \theta)$ is contained inside of Ω for $j = 1,\ 2,\ 3$. It is easy to see that $\eta(r)$ is $C^2(\Omega)$.

With the cut-off function defined above, it is well known that the solution of problem (1.1) has the following singular function representation [7,8]:

$$u = w_0 + \sum_{j\in L} \lambda_j\, \eta\, s_j, \tag{2.5}$$

with $w_0 \in H^2(\Omega) \cap H^1_D(\Omega)$. Moreover, the following regularity estimate holds:

$$\|w_0\|_{2,\Omega} + \sum_{j\in L} |\lambda_j| \le C_R \|f\|_{0,\Omega}, \tag{2.6}$$

where C_R is a positive constant depending on the domain and the diameter of the support of η.

To compute the intensity factors λ_j, one needs to introduce the dual singular functions. For singular functions $s_j = s_j(r,\theta) = r^{\frac{j\pi}{2\omega}} \phi\left(\frac{j\pi\theta}{2\omega}\right)$ with $\phi(\theta) = \sin\theta$ or $\cos\theta$ and $j \in L$, the corresponding dual singular functions have the form

$$s_{-j} = s_{-j}(r,\theta) = r^{-\frac{j\pi}{2\omega}} \phi\left(\frac{j\pi\theta}{2\omega}\right). \tag{2.7}$$

It is easy to verify that both the singular and the dual singular functions are harmonic in Ω and that

$$s_j \in H^{1+\frac{j\pi}{2\omega}-\epsilon}(\Omega)$$

for any $\epsilon > 0$. By some elementary computations, we have

Lemma 2.1. *There exists a positive constant C such that*

$$|s_j|_{1,\Omega} \le C \quad \forall\ j \in L. \tag{2.8}$$

With the dual functions defined above, the stress intensity factors λ_j can be computed by the following extraction formula (see, e.g., [8,11,12]):

$$\lambda_j(u) := \frac{2}{j\pi} \int_\Omega f \eta s_{-j} dx + \frac{2}{j\pi} \int_\Omega u \Delta(\eta s_{-j}) dx. \tag{2.9}$$

Let u_h^0 (see (3.6)) be the standard conforming linear finite element approximation to problem (1.1) and let $\lambda_j(u_h^0)$ be an approximation to the intensity factor computed by the extraction formula in (2.9). Then it is well known [6] that the following error estimate holds:

$$|\lambda_j(u) - \lambda_j(u_h^0)| \le C\, \|u - u_h^0\|_{0,\Omega} \le \begin{cases} C\, h^{\frac{2\pi}{\omega}-\epsilon}\, \|f\|_{0,\Omega}, & \text{for} \quad (D/D) \text{ or } (N/N), \\ C\, h^{\frac{\pi}{\omega}-\epsilon}\, \|f\|_{0,\Omega}, & \text{for} \quad (D/N) \text{ or } (N/D). \end{cases} \tag{2.10}$$

This indicates that the accuracy of the stress intensity factor approximation $\lambda_j(u_h^0)$ depends on the accuracy of the finite element approximation which, in turn, depends on the regularity of u (see, e.g., [6]).

3. Finite element approximation

For simplicity of presentation, we assume that there is only one singular point with an inner angle $\omega \in (\frac{\pi}{2},\ 2\pi)$ and that the type of boundary conditions around the singular point is D/N in the counterclockwise orientation. The algorithm and the analysis presented in this paper may be easily carried over to the case of many singular points with various types of boundary conditions.

For the case considered in this paper, there are either one or two singular functions depending on the inner angle. Let u be the solution of (1.1), and let

$$\xi = \begin{cases} 0, & \omega \in (\frac{\pi}{2},\ \frac{3\pi}{2}], \\ 1, & \omega \in (\frac{3\pi}{2},\ 2\pi], \end{cases}$$

then the singular part of the solution may be expressed as

$$g(u) = \lambda_1(u) s_1 + \xi\, \lambda_3(u) s_3, \quad \forall\ \omega \in (\frac{\pi}{2},\ 2\pi]. \tag{3.1}$$

with the singular functions given by

$$s_1(r,\ \theta) = r^{\frac{\pi}{2\omega}} \sin\frac{\pi\theta}{2\omega} \quad \text{and} \quad s_3(r,\ \theta) = r^{\frac{3\pi}{2\omega}} \sin\frac{3\pi\theta}{2\omega},$$

where $\lambda_j(u)$ is the stress intensity factor given in (2.9). Let w be the solution of the Poisson equation with modified Dirichlet boundary condition:

$$\begin{cases} -\Delta w &= & f, & \text{in } \Omega, \\ w &= & -g(u), & \text{on } \Gamma_D, \\ \dfrac{\partial w}{\partial \nu} &= & 0, & \text{on } \Gamma_N, \end{cases} \tag{3.2}$$

It was proved in [9,10] that (3.2) has a unique solution $w \in H^2(\Omega)$ satisfying

$$\|w\|_{2,\Omega} \le C_R' \|f\|_{0,\Omega} \tag{3.3}$$

and that we have the following singular function representation of u:

$$u = w + g(u) \tag{3.4}$$

Based on the representation in (3.4) and the extraction formula in (2.9), we proposed and numerically tested an accurate numerical method to solve Poisson equation (1.1) in [9,10]. To describe this approach, let $\mathcal{T}_h$ be a partition of the domain Ω into triangular elements; i.e., $\Omega = \cup_{K\in\mathcal{T}_h} K$ with $h = \max\{\text{diam}K : K \in \mathcal{T}_h\}$. Let V_h and $V_{h,D}$ be continuous piecewise linear finite element spaces defined by

$$V_h = \{\phi_h \in C^0(\Omega) : \phi_h|_K \in P_1(K)\ \ \forall K \in \mathcal{T}_h\} \subset H^1(\Omega),$$

and by

$$V_{h,D} = \{\phi_h \in V_h : \phi_h = 0 \text{ on } \Gamma_D\} \subset H^1_D(\Omega),$$

respectively, where $P_1(K)$ is the space of polynomials of degree less than or equal to one on K. It is well-known that

$$\inf_{\phi_h\in V_h, \phi_h|_{\Gamma_D} = I_{h,D}\phi} \{\|\phi - \phi_h\|_{0,\Omega} + h|\phi - \phi_h|_{1,\Omega}\} \le C_A h^{1+t} \|\phi\|_{1+t,\Omega} \tag{3.5}$$

for any $\phi \in H^1_D(\Omega) \cap H^{1+t}(\Omega)$, where $I_{h,D}\phi$ is the piecewise linear interpolant of $\phi|_{\Gamma_D}$.

Now, we are ready to introduce the numerical method as follows.

Algorithm 3.1. Let $u_h^0 \in V_{h,D}$ be the standard finite element approximation to problem (1.1), i.e.,

$$(\nabla u_h^0,\ \nabla v) = (f,\ v), \quad \forall\ v \in V_{h,D}. \tag{3.6}$$

For $i = 1,\ 2,\ \ldots,\ m$,

Step 1. compute $w_h^i \in V_h$ such that $w_h^i\big|_{\Gamma_D} = -I_{h,D}\left(g(u_h^{i-1})|_{\Gamma_D}\right)$ and that

$$(\nabla w_h^i,\ \nabla v) = (f,\ v), \quad \forall\ v \in V_{h,D}. \tag{3.7}$$

Step 2. set

$$u_h^i = w_h^i + g(u_h^{i-1}).$$

The number of iterations m in the above algorithm depends on singularity behavior of the underlying problem. It is shown in Theorem 4.1 that $m = 1$ and 2 are enough for $\omega \in (\frac{\pi}{2},\ \pi)$ and $\omega \in [\pi,\ 2\pi)$, respectively.

4. Error estimate

Let u_h^m be the approximation resulted from Algorithm 3.1 defined in the previous section. As an approximation to the solution of problem (1.1), we establish its a priori error bounds in both the energy and the L^2 norms. In particular, we show that $m = 1$ or 2 is enough to obtain a quasi-optimal approximation.

To this end, we need the following lemma on approximations to the solution w of problem (3.2) by replacing the unknown u by the current approximation u_h^{i-1}.

Lemma 4.1. *Let w^i be the solution of the following problem:*

$$\begin{cases} -\Delta w^i &= & f, & \text{in } \Omega, \\ w^i &= & -g(u_h^{i-1}), & \text{on } \Gamma_D, \\ \dfrac{\partial w^i}{\partial \nu} &= & 0, & \text{on } \Gamma_N. \end{cases} \tag{4.1}$$

then $u = w^i + g(u_h^{i-1})$ is the unique solution of (1.1). Moreover, we have

$$w^i = w + g(u) - g(u_h^{i-1}). \tag{4.2}$$

Proof. The first part of the lemma may be proven in the same fashion as that of Theorem 3.2 in [10]. Equality (4.2) is then a direct consequence of (3.4). ■

To derive an error representation, note first that $w_h^i \in V_h$ is the finite element approximation to problem (4.1) with $w_h^i|_{\Gamma_D} = -I_{h,D}g(u_h^{i-1})$. Hence, by Lemma 4.1 and Algorithm 3.1, we have the error representation:

$$u - u_h^i = \left(w^i + g(u_h^{i-1})\right) - \left(w_h^i + g(u_h^{i-1})\right) = w^i - w_h^i \tag{4.3}$$

Lemma 4.2. *Let w be the solution of problem (3.2) and $w_I \in V_h$ be the linear interpolant of w. Assume that $\left|\lambda_j(u) - \lambda_j(u_h^{i-1})\right| \le C\,h^{\alpha_i-\epsilon}\|f\|_{0,\Omega}$ for $i = 1, \ldots, m$. Then*

$$\inf_{v^i \in \hat{V}_{h,D}^i} |w_I - v^i|_{1,\Omega} \le C\,h^{\alpha_i-\epsilon}\|f\|_{0,\Omega} \tag{4.4}$$

where $\hat{V}_{h,D}^i = \{v \in V_h \,:\, v = -I_{h,D}g(u_h^{i-1}) \text{ on } \Gamma_D\}$.

Proof. Choose $v^i \in \hat{V}_{h,D}^i$ such that $w_I - v^i = 0$ for all interior and Neumann boundary vertices, and denote by $\mathcal{T}_{h,D} = \{K \in \mathcal{T}_h \,:\, \bar{K} \cap \Gamma_D \neq \emptyset\}$ the collection of elements who have edges belonging to Γ_D. Set

$$e_I^i = w_I - v^i.$$

It follows from integration by parts, the fact that $e_I^i \in V_h$, and the inverse inequality that

$$\begin{aligned} |e_I^i|_{1,\Omega}^2 &= \sum_{K\in\mathcal{T}_{h,D}} |e_I^i|_{1,K}^2 = \sum_{K\in\mathcal{T}_{h,D}} \int_{\partial K} e_I^i \frac{\partial e_I^i}{\partial \nu}\,ds \le \sum_{K\in\mathcal{T}_{h,D}} |e_I^i|_{1/2,\partial K} \left|\frac{\partial e_I^i}{\partial \nu}\right|_{-1/2,\partial K} \\ &\le C \sum_{K\in\mathcal{T}_{h,D}} h_K^{-1/2} |e_I^i|_{0,\partial K} \|e_I^i\|_{0,K}. \end{aligned} \tag{4.5}$$

For any $K \in \mathcal{T}_{h,D}$, denote its vertices by $\mathbf{a}_{l,K}$ for $l = 1, 2, 3$. Then

$$e_I^i(\mathbf{a}_{l,K}) = \begin{cases} \displaystyle\sum_{j\in\{1,3\}} \left(\lambda_j(u_h^{i-1}) - \lambda_j(u)\right) s_j(\mathbf{a}_{l,K}), & \mathbf{a}_{l,K} \in \Gamma_D, \\ 0, & \text{otherwise}, \end{cases}$$

which, together with the triangle inequality and the assumption, implies

$$\left(\sum_{l=1}^{3} \left(e_I^i\right)^2(\mathbf{a}_{l,K})\right)^{1/2} \le \sum_{l=1}^{3} \left|e_I^i(\mathbf{a}_{l,K})\right| \le C\,h^{\alpha_i-\epsilon}\|f\|_{0,\Omega}. \tag{4.6}$$

For any $K \in \mathcal{T}_{h,D}$, since $e_I^i\big|_K \in P_1(K)$, the scaling argument combining with (4.6) gives that

$$|e_I^i|_{0,\partial K} \le C\,h_K^{1/2} \left(\sum_{l=1}^{3} \left(e_I^i\right)^2(\mathbf{a}_{l,K})\right)^{1/2} \le C\,h^{1/2+\alpha_i-\epsilon}\|f\|_{0,\Omega}$$

and that

$$\|e_I^i\|_{0,K} \le C\,h_K \left(\sum_{l=1}^{3} \left(e_I^i\right)^2(\mathbf{a}_{l,K})\right)^{1/2} \le C\,h^{1+\alpha_i-\epsilon}\|f\|_{0,\Omega},$$

which, together with (4.5) and the fact that the number of elements in $\mathcal{T}_{h,D}$ is $O(h^{-1})$, implies the validity of (4.4). This completes the proof of the lemma. ■

Lemma 4.3. *Let w^i and w_h^i be the solutions of problems (4.1) and (3.7), respectively. Under the assumption of Lemma 4.2, we have*

$$|w^i - w_h^i|_{1,\Omega} \le C\,\left(h + h^{\alpha_i-\epsilon}\right)\|f\|_{0,\Omega}. \tag{4.7}$$

Proof. For any $v^i \in V_h$ satisfying $v^i = -I_{h,D}g(u_h^{i-1})$ on Γ_D, by the fact that $w_h^i - v^i \in V_{h,D}$ and the error equation, we have

$$\begin{aligned} |w^i - w_h^i|_{1,\Omega}^2 &= \left(\nabla(w^i - w_h^i),\, \nabla(w^i - w_h^i)\right) = \left(\nabla(w^i - w_h^i),\, \nabla\left(w^i - v^i - (w_h^i - v^i)\right)\right) \\ &= \left(\nabla(w^i - w_h^i),\, \nabla(w^i - v^i)\right), \end{aligned}$$

which, together with the Cauchy–Schwarz inequality, implies

$$|w^i - w_h^i|_{1,\Omega} \le |w^i - v^i|_{1,\Omega}.$$

Let $w \in H^2(\Omega)$ be the solution of (3.2), and denote by $w_I \in V_h$ the interpolant of w. It follows from (4.2), the triangle inequality, the approximation property, and (3.3) that

$$\begin{aligned}|w^i - v^i|_{1,\Omega} &= |w - v^i + g(u) - g(u_h^{i-1})|_{1,\Omega}\\ &\le |w - w_I|_{1,\Omega} + |w_I - v^i|_{1,\Omega} + |g(u) - g(u_h^{i-1})|_{1,\Omega}\\ &\le C\,h\,\|w\|_{2,\Omega} + C\,h^{\alpha_i-\epsilon}\,\|f\|_{0,\Omega} + \sum_{j\in\{1,3\}} |\lambda_j(u) - \lambda_j(u_h^{i-1})|\,|s_j|_{1,\Omega},\end{aligned}$$

which, together with (2.8) and the assumption, implies the validity of (4.7). This completes the proof of the lemma. ■

We shall use the standard duality argument to estimate the error in the L^2-norm. To this end, for $i = 1, \ldots, m$, consider the following adjoint problem:

$$\begin{cases} -\Delta z^i = w^i - w_h^i & \text{in } \Omega,\\ z^i = 0 & \text{on } \Gamma_D,\\ \dfrac{\partial z^i}{\partial \nu} = 0 & \text{on } \Gamma_N. \end{cases} \tag{4.8}$$

The regular part of z^i is the solution of the following problem:

$$\begin{cases} -\Delta w_z^i = w^i - w_h^i & \text{in } \Omega,\\ w_z^i = -\lambda_{1,z}^i s_1 - \lambda_{3,z}^i s_3 & \text{on } \Gamma_D,\\ \dfrac{\partial w_z^i}{\partial \nu} = 0 & \text{on } \Gamma_N, \end{cases} \tag{4.9}$$

where $\lambda_{1,z}^i$ and $\lambda_{3,z}^i$ are the stress intensity factors of z^i. A similar argument for the solutions of (1.1) and (3.2), we have that

$$w_z^i := z^i - \lambda_{1,z}^i s_1 - \lambda_{3,z}^i s_3 \in H^2(\Omega) \tag{4.10}$$

and that

$$\|w_z^i\|_{2,\Omega} \le C_R''\,\|w^i - w_h^i\|_{0,\Omega}. \tag{4.11}$$

Lemma 4.4. *Let w^i and w_h^i be the solutions of problems (4.1) and (3.7), respectively. Then*

$$\|w^i - w_h^i\|_{0,\Omega} \le C\,h\,|w^i - w_h^i|_{1,\Omega}. \tag{4.12}$$

Proof. The lemma follows from the standard duality argument

$$\begin{aligned}\|w^i - w_h^i\|_{0,\Omega}^2 &= (\nabla(w^i - w_h^i),\ \nabla w_z^i) = (\nabla(w^i - w_h^i),\ \nabla(w_z^i - I w_z^i))\\ &\le |w^i - w_h^i|_{1,\Omega}\cdot C_A\,h\,\|w_z^i\|_{2,\Omega} \le |w^i - w_h^i|_{1,\Omega}\cdot C_A\,C_R''\,h\,\|w^i - w_h^i\|_{0,\Omega},\end{aligned}$$

which implies (4.12) and, hence, the lemma. ■

Theorem 4.1. *Let u be the solution of (1.1) and u_h^i be given in Algorithm 3.1, then we have the following a priori error estimates:*

$$|u - u_h^i|_{1,\Omega} \le C\,h\,\|f\|_{0,\Omega} \quad \text{and} \quad \|u - u_h^i\|_{0,\Omega} \le C\,h^2\,\|f\|_{0,\Omega} \tag{4.13}$$

with $i = 1$ for $\omega \in (\frac{\pi}{2}, \pi)$ and $i = 2$ for $\omega \in [\pi, 2\pi)$.

Proof. For $\omega \in (\frac{\pi}{2}, \pi)$, (2.10) for the D/N case implies that $\alpha_1 = \pi/\omega \in (1, 2)$. Now, for $i = 1$, the first inequality in (4.13) is a direct consequence of (4.3) and Lemma 4.3. By Lemma 4.4, we may establish the second inequality in (4.13):

$$\begin{aligned}\|u - u_h^1\|_{0,\Omega} &\le \|w^1 - w_h^1\|_{0,\Omega} \le C\,h\,|w^1 - w_h^1|_{1,\Omega}\\ &\le C\,h\,\left(h + h^{\frac{\pi}{\omega}-\epsilon}\right)\,\|f\|_{0,\Omega} \le C\,h^2\|f\|_{0,\Omega}\end{aligned} \tag{4.14}$$

in a similar fashion.

For $\omega \in [\pi, 2\pi)$, then $\pi/\omega \in (1/2, 1]$. The inequality in (4.14) with (2.10) implies

$$|\lambda_j(u) - \lambda_j(u_h^1)| \le C\,h^{3/2}\,\|f\|_{0,\Omega} \qquad (j = 1, 3). \tag{4.15}$$

That is, $\alpha_2 = 3/2$. Hence, the error estimate for $i = 2$ may be established in a similar fashion. ■

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