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Adaptive Finite Element Method for Dirichlet Boundary Control of Elliptic Partial Differential Equations

Shaohong Du[1] · **Zhiqiang Cai**[2]

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Abstract
In this paper, we consider the Dirichlet boundary control problem of elliptic partial differential equations, and get a coupling system of the state and adjoint state by cancelling the control variable in terms of the control rule, and prove that this coupling system is equivalent to the known Karush–Kuhn–Tucker (KKT) system. For corresponding finite element approximation, we find a measure of the numerical errors by employing harmonic extension, based on this measure, we develop residual-based a posteriori error analytical technique for the Dirichlet boundary control problem. The derived estimators for the coupling system and the KKT system are proved to be reliable and efficient over adaptive mesh. Numerical examples are presented to validate our theory.

Keywords Dirichlet boundary control problem · A coupling system of the state and adjoint state · The KKT system · Equivalence · A posteriori error estimates · Reliability and efficiency

Mathematics Subject Classification 65N06 · 65N12 · 65N15 · 65N30 · 65J15

1 Introduction

Let $\Omega \subset \mathbb{R}^d, d \geq 2$, be a bounded polygonal or polyhedral convex domain with Lipschitz boundary $\Gamma = \partial\Omega$. Consider the following Dirichlet boundary control problem of elliptic partial differential equations (PDEs):

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✉ Shaohong Du
duzheyan.student@sina.com

Zhiqiang Cai
caiz@purdue.edu

1 School of Mathematics and Statistics, Chongqing Jiaotong University, Chongqing 400074, China

2 Department of Mathematics, Purdue University, 150 N. University Street, West Lafayette, IN 47907-2067, USA

$$\min\ J(u),\quad J(u)=\frac{1}{2}||y-y_d||^2_{L^2(\Omega)}+\frac{\gamma}{2}||u||^2_{L^2(\Gamma)}, \tag{1.1}$$

where $\gamma > 0$ and y is the solution of the Poisson equation with nonhomogeneous Dirichlet boundary conditions

$$-\triangle y = f \ \text{ in } \Omega, \tag{1.2}$$

$$y = u \ \text{ on } \Gamma. \tag{1.3}$$

There have been some efforts on the error estimates for finite element approximation to the distributed control problems governed by PDEs, since Falk and Geveci made the pioneering works in the literature [1,2]. For semilinear elliptic control problem, the error estimates on the control were derived by Arada et al. [3,4] in the L^∞ and L^2 norms; For some important flow control problems, their error estimates of finite element approximation were studied in [5,6], and the study of the Neumann boundary control problem was carried out by Casas et al. [7].

It is well known that the Dirichlet boundary control plays an important role in many applications such as flow control problems and has been a hot topic for decades. However, the Dirichlet boundary control problems are extremely difficult to solve from both the theoretical and the numerical points of view, because the Dirichlet boundary data cannot be directly involved in a standard variational setting for the PDEs. On the one hand, the traditional finite element (FE) method such as [8–12] deals with the state variable in a very weak sense; on the other hand, the attempt of the first order optimality condition involves the normal derivative of the adjoint state on the boundary of the domain. Therefore, it is crucial to obtain this normal derivative numerically by using additional information. But in doing so the problem becomes complicated in both theoretical analysis and numerical practice.

To overcome the difficulty mentioned above, two remedies to deal with the control variable were presented in [13–15]. One of them was to replace the L^2 norm in the cost functional with the $H^{1/2}$ norm because the $H^{1/2}(\Gamma)$-setting yields smoother solution which is more favorable in practice. In fact, the $L^2(\Gamma)$-setting is more popular because it is easier to derive the first order optimality condition. Note that both imposed the Dirichlet boundary condition as essential boundary condition. The other was to approximate the nonhomogeneous Dirichlet boundary condition with a Robin boundary condition or weak boundary penalization. However, the former changes the problem and the latter pays expensive cost to deal with penalization. Note that techniques similar to [13] have been applied in [14–16].

Recently, the mixed FE method for the Dirichlet boundary control problem was presented by Gong et al. [17], where the optimal control and the adjoint state were involved in variational form in a natural sense. This approach makes the theoretical analysis straightforward, but the corresponding fluxes of the two states are involved. It is pointed out that the mixed FE method obtained the same rate of convergence as the order of regularity of the control on boundary. Very recently, a hybridizable discontinuous Galerkin method was analyzed by Hu et al. [18], where they obtained optimal a priori error estimates for the control by only solving the trace of the primal variables on the inter-element boundaries which comes from the hybridization of the scheme.

Based on both the fact that the control u is equal to the restriction of the state y to the boundary [see the original Eq. (1.3)], and the fact that the restriction of an approximation of the state to the boundary is naturally an approximation of the control, we realise that the restriction of the numerical errors for the state to the boundary can be used to measure the numerical error of the control in $L^2(\Gamma)$-norm. In particular, the state y and its adjoint state z will be coupled with the original Eq. (1.3) and an extra Eq. (2.6) as well as the right-hand side term y of the Eq. (2.4), i.e., the control and the normal derivative of the adjoint state along the boundary can be cancelled. This idea is different from the one in literatures [8–16], where the original Eq. (1.3) and an extra Eq. (2.6) have been taken into account in variational formulation. Note that for the constrained case, since the variational inequality can be equivalently written as a projection of the adjoint state, we can formally eliminate the control, at this moment, we are not sure if our approach is applicable to the control constrained case.

Owing to this observation above, Du et al. developed a variational setting in [19], and established its well-posedness (unique solvability and stability), and analyzed finite element method based on this variational formulation. The derived estimate is optimal for the control variable, but it is not for the state and adjoint variables. In this paper, we firstly prove that the coupling system of the state and adjoint state is equivalent to the known KKT system (the corresponding discrete forms are the same), and improve the L^2 estimate for the state and adjoint state with the help of the a priori estimates for finite element approximation to the KKT system ([11]). The numerical experiments in [19] indicated that adaptive mesh based on a posteriori estimator is in urgent need for the Dirichlet boundary control problems, since the solution of this type of problems is of strong singularity over the polygonal domain.

When the PDEs for optimal control problem were involved in many problems of practical interest, such as interface singularities, discontinuities in the form of shock-like front, and of interior or boundary layers, adaptive finite element method (AFEM), proposed since the pioneer work of Babuška and Rheinboldt [20], has become a popular approach in the community of engineering and scientific computing. It is well known that a posteriori error estimation is an essential ingredient of adaptivity, and that error estimators in literature can be categorized into three classes: residual based, gradient recovery based, and hierarchical bases based, and different types of estimators have been developed in the last decades for different types of problems and for different approximation methods [21–27], we refer to [28] for an overview.

AFEM has been successfully applied to optimal control problems governed by PDEs, starting from Becker et al. [29] and Liu et al. [30]. In [29] a dual-weighted goal-oriented adaptivity has been proposed for optimal control problem; in [30] the authors have derived residual-based a posteriori error estimators for convex distributed optimal control problem. About a posteriori estimates for optimal control problems governed by different PDEs, we refer to literatures [31–33]. Recently, Kohls et al. [34,35] have developed a unifying framework for the a posteriori error analysis for control constrained optimal control problem by using either variational discretization or full control discretization. Very recently, Schneider et al. [36] have complimented the framework of [34,35]. But both these approaches exploit the first-order optimality conditions to derive a posteriori error estimates.

For Dirichlet boundary control problem, reliable and efficient a posteriori error estimates of residual-type have been derived in energy space in [15,37]. However, the a posteriori error estimates of the $L^2(\Gamma)$-setting have been not proposed, its main difficulties lie in the following facts: One is that the primal control problem is concerned in optimizing the control variable in $L^2(\Gamma)$-norm, this results in that the energy norm to measure the numerical error for the state variable seems slightly strong when the state equation is regarded as non-homogeneous Dirichlet boundary value problem. Another is that the numerical error for the control variable involves $L^2(\Gamma)$-norm, owing to the control equation, it may be understood as $H^{1/2}$-norm of the numerical error of the state variable. A natural question is what indicator it will be controlled by. Unfortunately, such a problem has not been studied in literatures. The third one is that the control (the restriction of the state to the boundary) is taken accounted into the variational system as a unknown function, this is an essential difference from general non-homogeneous Dirichlet boundary value problem (see [38]), since the restriction of the discrete state to the boundary is not an interpolation or a projection of the control. Owing to these observations, we employ harmonic extension to give a measure of numerical errors, and develop corresponding technique of a posteriori analysis based on the standard tools. The estimators are based on this measure, and are derived for both the coupling system of the state and adjoint state and the KKT system, and are proved to reliable and efficient.

It is pointed out that here we indeed give a way to develop residual-based a posteriori error estimation for finite element approximation to the KKT system. Its idea is that the coupling system of the state and adjoint state variables is used as a bridge, through which the residual functional for the control variable can be defined, and a measure for the numerical errors can be found, this is owed to their equivalence. However, these two goals are not easily achieved directly through the KKT system, since the discrete KKT system was not obtained by first-optimise-then-discretze (the first approach), but derived by using so-called first-discretize-then-optimise (the second approach). Note that these two approaches are not always equivalent, especially when the governing state equation is not self-adjoint, and that the second approach is more favorable because it preserves the structure of the optimization problems. Furthermore, the three equations of the discrete KKT system are coupled, this challenges a posteriori error estimation for the KKT system.

This paper is organized as follows. In Sect. 2, we introduce some notations and a variational setting. In Sect. 3, we prove the equivalence between the coupling system of the state and adjoint state and the KKT system, including their corresponding discrete formulations, and give an improved estimate for the state and adjoint state variables in L^2-norm, and contain a preliminary result. In Sect. 4, we employ the harmonic extension to obtain a measure of the numerical errors, and develop a technique for residual-based a posteriori error analysis based on this measure. Estimators derived for the coupling system and the KKT system are proved to be reliable. An efficient lower bound is provided under a reasonable assumption in Sect. 5. Finally numerical tests are provided in Sect. 6 to support the theoretical results.

2 Notations and a Variational Setting

For any subdomain ω of Ω with a Lipschitz boundary ϑ, denote by $(\cdot,\cdot)_\omega$ $((\cdot,\cdot)_\vartheta)$ the L^2 inner-product on ω (ϑ), and by $<\cdot,\cdot>_\omega$ the L^2 inner-product of the duality pairings between $H^1(\omega)$ and $H^1(\omega)'$. Moreover, denote $L^2(\vartheta)$ and $H^m(\omega)$ the standard Lebesgue and Sobolev spaces equipped with standard norms $\|\cdot\|_{0,\vartheta} = \|\cdot\|_{L^2(\vartheta)}$ and $\|\cdot\|_{m,\omega} = \|\cdot\|_{H^m(\omega)}$, $m \in \mathbb{N}$. Note that $H^0(\omega) = L^2(\omega)$. We denote $|\cdot|_{m,\omega}$ the semi-norm in $H^m(\omega)$. We shall omit the symbol Ω in the notations above if $\omega = \Omega$. In particular, for $1 \le p < \infty$ and $0 < s < 1$, the norm of the fractional Sobolev space $W^{s,p}(\omega)$ is defined as

$$||v||_{W^{s,p}(\omega)} := \left\{ ||v||^p_{L^p(\omega)} + \int_\omega \int_\omega \frac{|v(x) - v(y)|^p}{|x-y|^{d+ps}} dxdy \right\}^{1/p} \quad \text{for } v \in W^{s,p}(\omega).$$

When $p = 2$, we write $H^s(\omega)$ for $W^{s,2}(\omega)$.

We introduce finite element spaces. To this end, let $\mathcal{T}_h$ be a shape regular partition of Ω into triangles (tetrahedra for $d = 3$) or parallelograms (parallelepiped for $d = 3$) satisfying the angle condition [39], i.e., there exists a constant C_0 such that

$$C_0^{-1} h_K^d \le |K| \le C_0 h_K^d \qquad \forall\, K \in \mathcal{T}_h, \tag{2.1}$$

where $h_K := \text{diam}(K)$, and define h as the mesh-size function (piecewise constant function). Denote $P_k(K)$ be the space of polynomials of total degree at most k if K is a simplex, or the space of polynomials with degree at most k for each variable if K is a parallelogram/parallelepiped. Define the finite element spaces V_h and V_h^0 by

$$V_h := \{v_h \in C(\overline{\Omega}) : v_h|_K \in P_k(K),\ \forall K \in \mathcal{T}_h\}$$
$$\text{and } V_h^0 := \{v_h \in V_h : v_h|_\Gamma = 0\},\ \text{respectively.}$$

We introduce some notations that will be used below. Denote $\mathcal{E}_h^0$ the set of interior sides (if $d = 2$) or faces (if $d = 3$) in $\mathcal{T}_h$, $\mathcal{E}_h^\partial$ the set of boundary sides/faces in $\mathcal{T}_h$, $\mathcal{E}_K$ the set of sides or faces of $K \in \mathcal{T}_h$. For a side or face E in $\mathcal{E}_h$, which is the set of element sides or faces in $\mathcal{T}_h$, let h_E be the diameter of E, and ω_E be the union of all elements in $\mathcal{T}_h$ sharing E. For a function v in the "broken Sobolev space" $H^1(\bigcup \mathcal{T}_h)$, we define $[v]|_E := (v|_{K_+})|_E - (v|_{K_-})|_E$ as the jump of v across an interior side or face E, where K_+ and K_- are the two neighboring elements such that $E = K_+ \cap K_-$.

Throughout of this paper, we denote by C a constant independent of mesh size with different context in different occurrence, and also use the notation $A \lesssim F$ to represent $A \leq CF$ with a generic constant $C > 0$ independent of mesh size. In addition, $A \approx F$ abbreviates $A \lesssim F \lesssim A$.

It is well known that the Dirichlet boundary control problem (1.1)–(1.3) is equivalent to the optimality system

$$-\triangle y = f \quad \text{in} \quad \Omega, \tag{2.2}$$
$$y = u \quad \text{on} \quad \Gamma, \tag{2.3}$$
$$-\triangle z = y - y_d \quad \text{in} \quad \Omega, \tag{2.4}$$
$$z = 0 \quad \text{on} \quad \Gamma, \tag{2.5}$$
$$u = \frac{1}{\gamma}\frac{\partial z}{\partial \mathbf{n}} \quad \text{on} \quad \Gamma, \tag{2.6}$$

where $\mathbf{n}$ is the unit outer normal to Γ. Note that these equations must be understood in a very weak sense. For a 2D convex polygonal domain, we recall a regularity result of May et al. in [11] below, which gives conditions on the domain and data to guarantee the regularity of the solution. To this end, let $\omega_{\max}$ be the maximum interior angle of the polygonal domain Ω, and denote p_*^Ω by

$$p_*^\Omega = 2\omega_{\max}/(2\omega_{\max} - \pi), \tag{2.7}$$

including the special case $p_*^\Omega = \infty$ for $\omega_{\max} = \pi/2$. For a higher dimensional convex polygonal domain, we do not attempt to provide condition on the regularity of the solution, because it is not an emphasis in this paper. Of course, the regularity theory is more complicated in three-dimensional case.

Lemma 2.1 ([11] Lemma 2.9). *Suppose that $f \in L^2(\Omega)$ and $y_d \in L^{p_*^d}(\Omega)$, $p_*^d > 2$, and that $\Omega \subset \mathbb{R}^2$ is a bounded convex domain with polygonal boundary Γ. Let $p_*^\Omega \geq 2$ be defined by* (2.7) *and $p_* := \min(p_*^d, p_*^\Omega)$. Then, the solution (y, u) of the optimization problem* (1.1)–(1.3) *and the associated adjoint state determined by* (2.4) *have the regularity properties*

$$(y, u, z) \in H^{3/2-1/p}(\Omega) \times H^{1-1/p}(\Gamma) \times (H_0^1(\Omega) \cap W_p^2(\Omega)),\ \ 2 \leq p < p_*.$$

The Dirichlet boundary condition (2.3) indicates that the control u is equal to the restriction of the state y to the boundary Γ. Therefore, we simultaneously obtain the control u if the state y is got. The Eq. (2.6) is an additional equation with respect to the adjoint state z. Here, we don't regard (2.6) as an additional equation, but understand it as a boundary condition, through which the state y and its adjoint state z will be coupled on the boundary. So the control u can be cancelled in form, but it can be reflected by the state y in essence. It is pointed out that the right hand term of (2.4) includes the state variable y, through which the adjoint state z is coupled over the whole domain. Based on this observation and the regularity of the solutions in Lemma 2.1, we present the following variational formulation: Find $(y, z) \in H^1(\Omega) \times H_0^1(\Omega)$ such that

$$(\nabla y, \nabla \psi) = (f, \psi) \quad \forall \psi \in H_0^1(\Omega), \tag{2.8}$$

$$(\nabla z, \nabla \phi) - (\gamma y, \phi)_\Gamma - (y, \phi) = -(y_d, \phi) \quad \forall \phi \in H^1(\Omega). \tag{2.9}$$

In [19], well-posedness (unique solvability and stability) for the coupling system of the state and adjoint state in (2.8)–(2.9) has been analyzed, and a corresponding finite element approximation of order k has been developed, and the a priori error estimates have been proven for the control, state and adjoint state, in $L^2(\Gamma)$-norm, L^2-norm, and semi-norm, respectively. The estimate is optimal for the control, however, it is not for the state and adjoint state.

For any $q \in H^{1/2}(\Gamma)$ there exists the harmonic extension $Bq \in H^1(\Omega)$ as the unique solution of the nonhomogeneous Dirichlet boundary value problem

$$-\triangle Bq = 0 \quad \text{in} \ \ \Omega, \ \ Bq = q \ \ \text{on} \ \Gamma.$$

We recall the following a priori bounds for the harmonic extension Bq as Lemma 2.2.

Lemma 2.2 ([11] Lemma 2.2) *Suppose that $\Omega \subset \mathbb{R}^d$ is a bounded convex polygonal or polyhedral domain with boundary Γ. For $0 \le s \le 1$ the harmonic extension is continuously defined from $H^s(\Gamma)$ into $H^{s+1/2}(\Omega)$ and satisfies*

$$||Bq||_{H^{s+1/2}(\Omega)} \le c\|q\|_{H^s(\Gamma)}. \tag{2.10}$$

Note that Lemma 2.2 has been given for $d = 2$ in [11], but the proof for $d = 3$ is similar.

To avoid the use of very weak solutions, and to remove the nonhomogeneous boundary conditions, the regularity of the solution triplet allows for the following KKT system: Find the triplet $\{\tilde{y}, u, z\} \in H_0^1(\Omega) \times H^{1/2}(\Gamma) \times H_0^1(\Omega)$ such that

$$\begin{cases} (\nabla \tilde{y}, \nabla \phi) = (f, \phi), \ \ \forall \phi \in H_0^1(\Omega) \\ (\gamma u, \chi)_\Gamma + (\tilde{y} + Bu, B\chi) = (y_d, B\chi), \ \ \forall \chi \in H^{1/2}(\Gamma) \\ (\nabla z, \nabla \psi) - (\tilde{y} + Bu, \psi) = -(y_d, \psi), \ \ \forall \psi \in H_0^1(\Omega). \end{cases} \tag{2.11}$$

Since the regularity of the solution triplet is essentially determined by that of the adjoint state, May et al. have analyzed the regularity of the solution triplet in [11], and derived the above formula (2.11) based on this regularity.

3 An Equivalence and an Improved Estimate

In this section, we prove the equivalence between the coupling system in (2.8)–(2.9) and the KKT system in (2.11), including their corresponding discrete formulations, and obtain an improved estimate of the state and adjoint in L^2-norm with the help of the a priori estimate for the KKT system.

Theorem 3.1 *The coupling system in (2.8)–(2.9) is equivalent to the KKT system in (2.11).*

Proof We first prove that the solution pair for the coupling system in (2.8)–(2.9) satisfies the KKT system in (2.11). To this end, let $(y, z) \in H^1(\Omega) \times H_0^1(\Omega)$ solve the coupling system in (2.8)–(2.9), and denote $u = y|_\Gamma$, and let Bu be the harmonic extension of u, i.e., Bu satisfies $Bu \in H^1(\Omega)$ and solves the following problem

$$(\nabla Bu, \nabla \phi) = 0, \quad \forall \phi \in H_0^1(\Omega), \quad Bu|_\Gamma = u. \tag{3.1}$$

Setting $y - Bu = \tilde{y}$ indicates $\tilde{y} \in H_0^1(\Omega)$. Inserting $y = \tilde{y} + Bu$ into (2.8), and owing to (3.1), we have

$$(\nabla \tilde{y} + \nabla Bu, \nabla \phi) = (\nabla \tilde{y}, \nabla \phi) = (f, \phi), \quad \forall \phi \in H_0^1(\Omega). \tag{3.2}$$

Arbitrary $\psi \in H_0^1(\Omega)$, we obtain from (2.9)

$$(\nabla z, \nabla \psi) - (\tilde{y} + Bu, \psi) = -(y_d, \psi). \tag{3.3}$$

Given $\chi \in H^{1/2}(\Gamma)$, denote $B\chi$ the harmonic extension of χ. Obviously, $-B\chi \in H^1(\Omega)$. Inserting $\psi = -B\chi$ into (2.9) leads to

$$(\nabla z, -\nabla B\chi) - (\gamma(\tilde{y} + Bu), -B\chi)_\Gamma - (\tilde{y} + Bu, -B\chi) = -(y_d, -B\chi),$$

which results in

$$(\gamma u, \chi)_\Gamma + (\tilde{y} + Bu, B\chi) = (y_d, B\chi). \tag{3.4}$$

In the last step above, we employ $(\nabla z, -\nabla B\chi) = 0$, because of $z \in H_0^1(\Omega)$. Therefore, $(\tilde{y}, u, z) \in H_0^1(\Omega) \times H^{1/2}(\Gamma) \times H_0^1(\Omega)$, and (3.2)–(3.4) show that $(\tilde{y}, u, z)$ solves the KKT system in (2.11).

In what follows, we prove that the solution triplet for the KKT system in (2.11) satisfies the coupling system in (2.8)–(2.9). To this end, denote $\{\tilde{y}, u, z\}$ the solution triplet for the KKT system in (2.11), and set $\tilde{y} + Bu = y \in H^1(\Omega)$. Owing to $\tilde{y}|_\Gamma = 0$ and $Bu|_\Gamma = u$, we obtain $u = y$ on Γ. Inserting $\tilde{y} = y - Bu$ into the first equation in (2.11), and using the definition (3.1) of the harmonic extension, yield to

$$(\nabla \tilde{y}, \nabla \phi) = (\nabla y - \nabla Bu, \nabla \phi) = (\nabla y, \nabla \phi) = (f, \phi), \quad \forall \phi \in H_0^1(\Omega). \tag{3.5}$$

Arbitrary given $\psi \in H^1(\Omega)$, denote q the restriction of ψ to the boundary, i.e., $q = \psi|_\Gamma$, and let Bq be the harmonic extension of q. Obviously, $\psi - Bq \in H_0^1(\Omega)$. Inserting $\psi - Bq$ into the third equation in (2.11), and noticing $\tilde{y} + Bu = y$, we have

$$(\nabla z, \nabla \psi - \nabla Bq) - (y, \psi - Bq) = -(y_d, \psi - Bq),$$

which results in

$$(\nabla z, \nabla \psi) - (y, \psi) + (y, Bq) = -(y_d, \psi) + (y_d, Bq). \tag{3.6}$$

In the last step above, we employ $(\nabla z, -\nabla Bq) = -(\nabla z, \nabla Bq) = 0$, since $z \in H_0^1(\Omega)$.

Noticing $q \in H^{1/2}(\Gamma)$, we attain from the second equation in (2.11)

$$(\gamma u, q)_\Gamma + (y, Bq) = (y_d, Bq),$$

which results in

$$(y, Bq) = (y_d, Bq) - (\gamma y, \psi)_\Gamma. \tag{3.7}$$

In the last step above, we employ $u = y$ and $q = \psi$ on Γ. Inserting (3.7) into (3.6), we get

$$(\nabla z, \nabla \psi) - (\gamma y, \psi)_\Gamma - (y, \psi) = -(y_d, \psi). \tag{3.8}$$

(3.5) and (3.8) show that the pair $(y = \tilde{y} + Bu, z)$ solves the coupling system in (2.8)–(2.9). □

We are now the place where the discrete form of the coupling system in (2.8)–(2.9) should be rephrased: Find $(y_h, z_h) \in V_h \times V_h^0$ such that

$$(\nabla y_h, \nabla \psi_h) = (f, \psi_h) \quad \forall\, \psi_h \in V_h^0, \tag{3.9}$$

$$(\nabla z_h, \nabla \phi_h) - (\gamma y_h, \phi_h)_\Gamma - (y_h, \phi_h) = -(y_d, \phi_h) \quad \forall\, \phi_h \in V_h. \tag{3.10}$$

Denote V_h^∂ the trace space corresponding to V_h, and for $q_h \in V_h^\partial$, $B_h q_h$ the "discrete harmonic extension" defined by

$$(\nabla B_h q_h, \nabla \varphi_h) = 0 \ \ \forall \varphi_h \in V_h^0, \quad B_h q_h|_\Gamma = q_h.$$

Note that May et al. [11] didn't directly discretize the continuous KKT systen in (2.11), but derived the discrete KKT system by using the discrete optimal control problem based on the Euler-Lagrange principle, because of $Bq_h \neq B_h q_h$ for $q_h \in V_h^\partial$. Their discrete formulation (see [11]) reads: Find $\{\tilde{y}_h, u_h, z_h\} \in V_h^0 \times V_h^\partial \times V_h^0$ such that

$$\begin{cases} (\nabla \tilde{y}_h, \nabla \phi_h) = (f, \phi_h), \ \ \forall \phi_h \in V_h^0 \\ (\gamma u_h, \chi_h)_\Gamma + (\tilde{y}_h + B_h u_h, B_h \chi_h) = (y_d, B_h \chi_h), \ \ \forall \chi_h \in V_h^\partial \\ (\nabla z_h, \nabla \psi_h) - (\tilde{y}_h + B_h u_h, \psi_h) = -(y_d, \psi_h), \ \ \forall \psi_h \in V_h^0. \end{cases} \tag{3.11}$$

Next, we show the equivalence between the corresponding discrete formulations of the coupling system in (2.8)–(2.9) and the KKT system in (2.11).

Theorem 3.2 *The discrete coupling system in (3.9)–(3.10) is equivalent to the discrete KKT system in (3.11).*

Proof Repeating the proof of Theorem 3.1, we obtain the desired result. □

In [11], May et al. have developed the a priori error estimates for the state and adjoint state variables in L^2-norm.

Lemma 3.3 ([11] Corollaries 5.3-5.4) *Let* $\{\tilde{y}, u, z\} \in H_0^1(\Omega) \times H^{1/2}(\Gamma) \times H_0^1(\Omega)$ *and* $\{\tilde{y}_h, u_h, z_h\} \in V_h^0 \times V_h^\partial \times V_h^0$ *($k = 1$) solve the continuous KKT system in (2.11) and the discrete KKT system in (3.11), respectively. For the numerical errors on the primal state variable and the adjoint state variable, there holds for the lowest-order finite element approximation in the case of two dimensions ($d = 2$)*

$$||y - y_h|| + ||z - z_h|| \leq Ch \left(\|u\|_{H^{1/2}(\Gamma)} + ||f|| + ||z||_2 \right). \tag{3.12}$$

Note that Lemma 3.3 is a combination of the results in Corollaries 5.3-5.4, and is also a special case of $p = r = 2$ in Corollaries 5.3-5.4 in [11].

Theorem 3.4 *Let* $(y, z) \in H^1(\Omega) \times H_0^1(\Omega)$ *and* $(y_h, z_h) \in V_h \times V_h^0$ *($k = 1$) be the solution pair for the coupling system in (2.8)–(2.9) and the discrete coupling system in (3.9)–(3.10), respectively. For the numerical errors on the state variable and the adjoint state variable, there holds for the lowest-order finite element approximation in the case of two dimensions ($d = 2$)*

$$||y - y_h|| + ||z - z_h|| \leq Ch \left(\|u\|_{H^{1/2}(\Gamma)} + ||f|| + ||z||_2 \right). \tag{3.13}$$

Proof We obtain the desired estimate (3.13) from a combination of Theorems 3.1–3.2, and Lemma 3.3. □

Remark 3.1 Note that the stability of the KKT system in (2.11) was included in Lemma 2.9 of [11], as a special case of the regularity result.

4 A posteriori Error Analysis

In this section, we shall develop a posteriori estimates for numerical errors between the exact solution pair of (2.8)–(2.9) and an approximation solution pair of (3.9)–(3.10). Owing to the equivalence between the coupling system and the KKT system, simultaneously, we shall obtain a posteriori error estimation for the KKT system.

To give a measure of numerical errors, we recall the proof of Theorems 3.1–3.2, and get a decomposition of the continuous and discrete state variables

$$y = \tilde{y} + Bu, \quad y_h = \tilde{y}_h + B_h u_h,$$

where $\tilde{y} \in H_0^1(\Omega)$ and $\tilde{y}_h \in V_h^0$ are the solutions of the first equation of (2.11) and (3.11), respectively, and Bu and $B_h u_h$ are the continuous and discrete harmonic extension of u and u_h ($u = y|_\Gamma, u_h = y_h|_\Gamma$), respectively. Let $P_h : L^2(\Gamma) \to V_h^\partial$ be the L^2 projection operator defined by

$$(u - P_h u, \chi_h) = 0, \quad \forall \chi_h \in V_h^\partial. \tag{4.1}$$

According to the definition of the discrete harmonic extension, $B_h P_h u$ is a finite element approximation to Bu in V_h in the sense that the discrete Dirichlet data is chosen as the projection of u onto V_h^∂.

Define a measure of the numerical errors as following

$$\begin{aligned}\mathcal{E} := \{||y - y_h||^2 + ||\gamma^{1/2}(u - u_h)||_{0,\Gamma}^2 + ||\nabla(\tilde{y} - \tilde{y}_h)||^2 \\ +\|\nabla(Bu - B_h P_h u)\|^2 + ||\nabla(z - z_h)||^2 + ||\nabla(B - B_h)u_h||^2\}^{1/2},\end{aligned} \tag{4.2}$$

and denote a series of indicators $\eta_i, i = 1, \cdots, 7$ by

$$\eta_1^2 := \sum_{K \in \mathcal{T}_h} \left(h_K^2 ||f + \triangle y_h||_K^2 + \frac{1}{2} \sum_{E \in \mathcal{E}_K \cap \mathcal{E}_h^0} h_E \left|\left| \left[\frac{\partial y_h}{\partial \mathbf{n}} \right] \right|\right|_E^2 \right), \tag{4.3}$$

$$\eta_2^2 := \sum_{K \in \mathcal{T}_h} \left(h_K^2 ||y_h - y_d + \triangle z_h||_K^2 + \frac{1}{2} \sum_{E \in \mathcal{E}_K \cap \mathcal{E}_h^0} h_E \left|\left| \left[\frac{\partial z_h}{\partial \mathbf{n}} \right] \right|\right|_E^2 \right), \tag{4.4}$$

$$\eta_3^2 := \sum_{K \in \mathcal{T}_h} \left(h_K^2 ||f + \triangle \tilde{y}_h||_K^2 + \frac{1}{2} \sum_{E \in \mathcal{E}_K \cap \mathcal{E}_h^0} h_E \left|\left| \left[\frac{\partial \tilde{y}_h}{\partial \mathbf{n}} \right] \right|\right|_E^2 \right), \tag{4.5}$$

$$\eta_4^2 := \sum_{K \in \mathcal{T}_h} \left(h_K^2 ||\triangle B_h u_h||_K^2 + \frac{1}{2} \sum_{E \in \mathcal{E}_K \cap \mathcal{E}_h^0} h_E \left|\left| \left[\frac{\partial B_h u_h}{\partial \mathbf{n}} \right] \right|\right|_E^2 \right), \tag{4.6}$$

$$\eta_5^2 := \eta_2^2 + \sum_{E \in \mathcal{E}_h^\partial} h_E \|\nabla z_h \cdot \mathbf{n} - \gamma u_h\|_E^2, \tag{4.7}$$

$$\eta_6^2 := \sum_{K\in\mathcal{T}_h}\left(h_K^2||\triangle B_h P_h u||_K^2 + \frac{1}{2}\sum_{E\in\mathcal{E}_K\cap\mathcal{E}_h^0} h_E\left|\left|\left[\frac{\partial B_h P_h u}{\partial \mathbf{n}}\right]\right|\right|_E^2\right), \tag{4.8}$$

$$\eta_7^2 := \left|\left|\gamma^{\frac{1}{2}}u_h - \gamma^{-\frac{1}{2}}\frac{\partial z_h}{\partial \mathbf{n}}\right|\right|_{0,\Gamma}^2. \tag{4.9}$$

Lemma 4.1 *Let $\tilde{y} \in H_0^1(\Omega)$ and $\tilde{y}_h \in V_h^0$ be the solutions to the first equation of (2.11) and (3.11), respectively. Then it holds*

$$||\nabla(\tilde{y} - \tilde{y}_h)|| \lesssim \eta_3. \tag{4.10}$$

Proof The desired estimate (4.10) follows from the standard analysis for the residual-type a posteriori estimator for the Laplace equation with homogeneous Dirichlet boundary condition. □

Lemma 4.2 *Denote Bu_h and $B_h u_h$ the continuous and discrete harmonic extensions of u_h, respectively. Then there holds*

$$||\nabla(B - B_h)u_h|| \lesssim \eta_4. \tag{4.11}$$

Proof Since $B_h u_h$ is just the finite element solution of Bu_h in V_h, and following the standard technique for the Laplace equation, we derive the desired estimate (4.11). □

For an edge or a side $E \in \mathcal{E}_h^\partial$ and a function $g|_E \in H^2(E)$ for all $E \in \mathcal{E}_h^\partial$, denote by $\partial_{\mathcal{E}}^2 g$ the edgewise second derivative of g along E (with respect to a proper Cartesian coordinate system along the flat $d-1$ dimensional manifold E). Define the "broken Sobolev space"

$$H^2\left(\bigcup \mathcal{E}_h^\partial\right) := \{\chi \in L^2(\Gamma) : \chi|_E \in H^2(E), \forall E \in \mathcal{E}_h^\partial\}.$$

Lemma 4.3 *Denote Bu the continuous harmonic extension of u, $P_h u$ the L^2 projection of u onto V_h^∂, and $B_h P_h u$ the discrete harmonic extensions of $P_h u$. If $u \in H^2(\cup\mathcal{E}_h^\partial)$, then there hold the following reliable a posteriori error estimates*

$$||\nabla(Bu - B_h P_h u)|| \lesssim \eta_6 + \|h_{\mathcal{E}}\|_{L^\infty(\Gamma)}^{1/2}\|h_{\mathcal{E}}\partial_{\mathcal{E}}^2 u\|_{0,\Gamma}. \tag{4.12}$$

Proof According to the definition of the discrete harmonic extension, $B_h P_h u$ is just a finite element approximation to Bu in V_h in the sense that the discrete Dirichlet data is chosen as the L^2 projection of u onto V_h^∂, (4.12) is a direct result of Theorem 6.2 in [38]. □

Note that the assumption of the regularity of u is a very weak requirement in Lemma 4.3, and this assumption is usually satisfied in practice. For convenience, denote the high order term by

$$\text{h.o.t} = \|h_{\mathcal{E}}\|_{L^\infty(\Gamma)}^{1/2}\|h_{\mathcal{E}}\partial_{\mathcal{E}}^2 u\|_{0,\Gamma}.$$

Theorem 4.4 *Let $(y, z) \in H^1(\Omega) \times H_0^1(\Omega)$ and $(y_h, z_h) \in V_h \times V_h^0$ be the solutions to (2.8)–(2.9) and (3.9)–(3.10), respectively, and denote an indicator by $\eta = \left(\sum_{i=1}^6 \eta_i^2\right)^{1/2}$. Assume that $u \in H^2(\cup\mathcal{E}_h^\partial)$. For the measure $\mathcal{E}$ of numerical errors defined in (4.2), there exists a positive constant C (independent on h and γ) satisfying*

$$\mathcal{E} \leq C(\eta + \text{h.o.t}). \tag{4.13}$$

Proof Recall $u = y|_\Gamma$, and denote $B((\cdot,\cdot),(\cdot,\cdot))$ a bilinear form on $H^1(\Omega)\times H_0^1(\Omega)$ by

$$B((y,z),(\psi,\phi)) = (\nabla y,\nabla\phi) + (\gamma u,\psi)_\Gamma + (y,\psi) - (\nabla z,\nabla\psi).$$

We associate with $\{u_h, y_h, z_h\}$ two residuals R_1 and R_2 by setting for every $\phi\in H_0^1(\Omega)$ and every $\psi\in H^1(\Omega)$

$$< R_1,\phi > = \int_\Omega f\phi - \int_\Omega \nabla y_h\cdot\nabla\phi,\quad \forall\phi\in H_0^1(\Omega), \tag{4.14}$$

$$< R_2,\psi > = \int_\Omega y_d\psi + \int_\Omega \nabla z_h\cdot\nabla\psi - \int_\Gamma \gamma u_h\psi - \int_\Omega y_h\psi,\quad \forall\psi\in H^1(\Omega), \tag{4.15}$$

where $u_h = y_h|_\Gamma$. Owing to $V_h^0\subset H_0^1(\Omega)$ and $V_h\subset H^1(\Omega)$, the residuals satisfy the Galerkin orthogonality

$$< R_1,\phi_h >= 0,\quad \forall\phi_h\in V_h^0,\quad < R_2,\psi_h >= 0,\quad \forall\psi_h\in V_h. \tag{4.16}$$

Notice that the residuals $R_i (i = 1, 2)$ are related to the error by

$$B((y-y_h, z-z_h),(\psi,\phi)) =< R_1,\phi > + < R_2,\psi > \quad \forall(\psi,\phi)\in H^1(\Omega)\times H_0^1(\Omega). \tag{4.17}$$

Let $I_h(I_h^0) : L^2(\Omega)\to V_h(V_h^0)$ be the Clément interpolation operator (cf. [40], [39, Exercise 3.2.3], [22,41]), we have

$$\|v - I_h v\|_K (||v - I_h^0 v||_K) \lesssim h_K||\nabla v||_{\tilde\omega_K},\quad \forall K\in\mathcal{T}_h,\ v\in H^1(\tilde\omega_K), \tag{4.18}$$

$$\|v - I_h v\|_E (||v - I_h^0 v||_E) \lesssim h_E^{1/2}||\nabla v||_{\omega_E},\quad \forall E\in\mathcal{E}_h,\ v\in H^1(\omega_E). \tag{4.19}$$

From the Galerkin orthogonality (4.16) and the properties (4.18)–(4.19), of the Clément interpolation operator, we have

$$\begin{aligned} < R_1,\phi > &= < R_1,\phi - I_h^0\phi > \\ &= \sum_{K\in\mathcal{T}_h}\left(\int_K (f+\triangle y_h)(\phi - I_h^0\phi) - \int_{\partial K}\frac{\partial y_h}{\partial\mathbf{n}}(\phi - I_h^0\phi)\right) \\ &\lesssim \eta_1||\nabla\phi||. \end{aligned} \tag{4.20}$$

For $\psi\in H_0^1(\Omega)$, we get from the Galerkin orthogonality (4.16), integration by parts, and the properties (4.18)–(4.19), of the Clément interpolation operator

$$\begin{aligned} < R_2,\psi > &= < R_2,\psi - I_h^0\psi > \\ &= \int_\Omega y_d(\psi - I_h^0\psi) + \int_\Omega \nabla z_h\cdot\nabla(\psi - I_h^0\psi) \\ &\quad - \int_\Gamma \gamma u_h(\psi - I_h^0\psi) - \int_\Omega y_h(\psi - I_h^0\psi) \\ &= \sum_{K\in\mathcal{T}_h}\int_K (y_d - y_h - \triangle z_h)(\psi - I_h^0\psi) + \sum_{E\in\mathcal{E}_h^0}\int_E\left[\frac{\partial z_h}{\partial\mathbf{n}}\right](\psi - I_h^0\psi) \\ &\lesssim \eta_2||\nabla\psi||. \end{aligned} \tag{4.21}$$

Similarly, for $\psi\in H^1(\Omega)$, we obtain

$$< R_2,\psi >=< R_2,\psi - I_h\psi >\lesssim \eta_5||\nabla\psi||. \tag{4.22}$$

Noticing $u - u_h = (y - y_h)|_\Gamma$, we attain from (4.17)

$$\begin{aligned}
||\gamma^{1/2}(u-u_h)||_{0,\Gamma}^2 + ||y-y_h||^2 &= ||\gamma^{1/2}(y-y_h)||_{0,\Gamma}^2 + ||y-y_h||^2 \\
&= B((y-y_h, z-z_h), (y-y_h, z-z_h)) \\
&= < R_1, z-z_h > + < R_2, y-y_h > . \qquad (4.23)
\end{aligned}$$

In what follows, we separately estimate $< R_1, z - z_h >$ and $< R_2, y - y_h >$. We obtain from (4.20)

$$< R_1, z - z_h > \lesssim \eta_1 ||\nabla(z - z_h)||. \qquad (4.24)$$

Since $z - z_h \in H_0^1(\Omega)$, we have

$$||\nabla(z - z_h)|| \leq \sup_{0 \neq w \in H_0^1(\Omega)} \frac{(\nabla(z - z_h), \nabla w)}{||\nabla w||}. \qquad (4.25)$$

Using integration by parts, we get

$$\begin{aligned}
&(\nabla(z - z_h), \nabla w) \\
&= (\nabla(z - z_h), \nabla(w - I_h^0 w)) + (\nabla(z - z_h), \nabla(I_h^0 w)) \\
&= \sum_{K \in \mathcal{T}_h} \int_K (-\triangle z + \triangle z_h)(w - I_h^0 w) + \int_{\partial K} \frac{\partial(z - z_h)}{\partial \mathbf{n}}(w - I_h^0 w) \\
&\quad +(\nabla(z - z_h), \nabla(I_h^0 w)). \qquad (4.26)
\end{aligned}$$

Noticing $I_h^0 w \in V_h^0$, we attain from (2.9) and (3.10)

$$(\nabla(z - z_h), \nabla(I_h^0 w)) = \left(\gamma(y - y_h), I_h^0 w\right)_\Gamma + (y - y_h, I_h^0 w) = (y - y_h, I_h^0 w). \qquad (4.27)$$

Combining (4.26) with (4.27), and noticing $-\triangle z = y - y_d$ in Ω, we get

$$\begin{aligned}
(\nabla(z - z_h), \nabla w) &= \sum_{K \in \mathcal{T}_h} \int_K (y_d - y_h - \triangle z_h)(w - I_h^0 w) \\
&\quad - \sum_{E \in \mathcal{E}_h^0} \int_E \left[\frac{\partial z_h}{\partial \mathbf{n}}\right](w - I_h^0 w) + (y - y_h, w). \qquad (4.28)
\end{aligned}$$

A Combination of (4.25) and (4.28), and the use of the properties (4.18)–(4.19), of the Clément interpolation operator, and the Poincaré inequality, yield

$$||\nabla(z - z_h)|| \lesssim \eta_2 + ||y - y_h||. \qquad (4.29)$$

Combining (4.24) with (4.29), and employing Young's inequality, we arrive at

$$< R_1, z - z_h > \leq C\left(\eta_1^2 + \eta_2^2\right) + \frac{1}{4}||y - y_h||^2. \qquad (4.30)$$

We now estimate $< R_2, y - y_h >$. Recalling the decomposition of y and y_h given at the beginning of this section, we have

$$< R_2, y - y_h > = < R_2, \tilde{y} - \tilde{y}_h > + < R_2, Bu - B_h u_h > . \qquad (4.31)$$

Since $\tilde{y} \in H_0^1(\Omega)$ and $\tilde{y}_h \in V_h^0$, repeating the proof of (4.21), we obtain

$$< R_2, \tilde{y} - \tilde{y}_h > \lesssim \eta_2 ||\nabla(\tilde{y} - \tilde{y}_h)||. \qquad (4.32)$$

Employing Young's inequality and (4.10) to (4.32), we reach

$$< R_2, \tilde{y} - \tilde{y}_h > \lesssim \eta_2^2 + \eta_3^2. \tag{4.33}$$

Recalling Bu_h the harmonic extension of u_h, we obtain by adding and subtracting Bu_h

$$< R_2, Bu - B_h u_h > = < R_2, B(u - u_h) > + < R_2, (B - B_h)u_h > . \tag{4.34}$$

Noticing $(B - B_h)u_h \in H_0^1(\Omega)$, and employing (4.21) and (4.11), we derive an estimate of the second term on the right side of (4.34)

$$< R_2, (B - B_h)u_h > \lesssim \eta_2 ||\nabla(B - B_h)u_h|| \lesssim \eta_2\eta_4 \lesssim \eta_2^2 + \eta_4^2. \tag{4.35}$$

We now estimate the first term on the right side of (4.34). For convenience, write $\chi = u - u_h$, denote $B\chi$ be the harmonic extension of χ. Recall the projection operator P_h defined in (4.1), let $B_h P_h \chi$ be the discrete harmonic extension of $P_h \chi$, set $\psi = B\chi - B_h P_h \chi \in H^1(\Omega)$. Noticing the Galerkin orthogonality in (4.16), recalling the Clemént interpolation operator I_h, and applying the estimate (4.22), we have

$$\begin{aligned} < R_2, B\chi > &= < R_2, B\chi - B_h P_h \chi > = < R_2, \psi > = < R_2, \psi - I_h \psi > \\ &\lesssim \eta_5 ||\nabla(B\chi - B_h P_h \chi) \\ &\le \eta_5 (\|\nabla(Bu - B_h P_h u)\| + \|\nabla(Bu_h - B_h u_h)\|). \end{aligned} \tag{4.36}$$

By making the use of (4.11), we obtain from (4.36)

$$< R_2, B(u - u_h) > \le C(\eta_4^2 + \eta_5^2) + \frac{1}{4}\|\nabla(Bu - B_h P_h u)\|^2 \tag{4.37}$$

Collecting (4.23), (4.30)–(4.31), (4.33)–(4.35), and (4.37), we obtain

$$||\gamma^{1/2}(u - u_h)||_{0,\Gamma}^2 + ||y - y_h||^2 \le C \sum_{i=1}^{5} \eta_i^2 + \frac{1}{4}\left(||\nabla(Bu - B_h P_h u)||^2 + ||y - y_h||^2\right). \tag{4.38}$$

Combining (4.10)–(4.12) with (4.38), we have

$$\begin{aligned} &\frac{3}{4}\left(||\gamma^{1/2}(u - u_h)||_{0,\Gamma}^2 + ||y - y_h||^2 + \|\nabla(\tilde{y} - \tilde{y}_h)\|^2 + \|\nabla(Bu - B_h P_h u)\|^2\right) \\ &\le ||\gamma^{1/2}(u - u_h)||_{0,\Gamma}^2 + \frac{3}{4}||y - y_h||^2 + \|\nabla(\tilde{y} - \tilde{y}_h)\|^2 + \frac{3}{4}\|\nabla(Bu - B_h P_h u)\|^2 \\ &\le C\left(\sum_{i=1}^{6} \eta_i^2 + (\text{h.o.t})^2\right), \end{aligned} \tag{4.39}$$

which results in

$$||\gamma^{1/2}(u - u_h)||_{0,\Gamma} + ||y - y_h|| + \|\nabla(\tilde{y} - \tilde{y}_h)\| + \|\nabla(Bu - B_h P_h u)\| \le C(\eta + \text{h.o.t}). \tag{4.40}$$

A combination of (4.29) and (4.40) yields

$$||\nabla(z - z_h)|| \le C(\eta + \text{h.o.t}). \tag{4.41}$$

Recalling the measure $\mathcal{E}$ of numerical errors defined in (4.2), and collecting (4.40)–(4.41), we get the desired estimate (4.13). □

Theorem 4.5 *Let* $\{\tilde{y}, u, z\} \in H_0^1(\Omega) \times H^{1/2}(\Gamma) \times H_0^1(\Omega)$ *and* $\{\tilde{y}_h, u_h, z_h\} \in V_h^0 \times V_h^\partial \times V_h^0$ *be the solution triplet to the continuous KKT sysytem in* (2.11) *and the discrete KKT system in* (3.11)*, respectively, and denote* $\tilde{\mathcal{E}}$ *a measure of numerical errors by*

$$\tilde{\mathcal{E}} = (||\nabla(\tilde{y} - \tilde{y}_h)||^2 + ||\gamma^{1/2}(u - u_h)||_{0,\Gamma}^2 + ||\nabla(z - z_h)||^2 \\ +||\nabla(B - B_h)u_h||^2 + \|\nabla(Bu - B_h P_h u)\|^2)^{1/2}.$$

If $u \in H^2(\cup\mathcal{E}_h^\partial)$*, then there holds the following reliable estimate*

$$\tilde{\mathcal{E}} \leq C(\eta + \text{h.o.t}). \tag{4.42}$$

Proof The equivalence between (2.11) and (2.8)–(2.9) ((3.11) and (3.9)–(3.10)) implies $\tilde{\mathcal{E}} \leq \mathcal{E}$, the assertion (4.42) follows from (4.13). □

Remark 4.1 Since the decomposed components $\tilde{y}_h$ and $B_h u_h$ of y_h are not derived from the discrete coupling system in (3.9)–(3.10), the indicators η_3 and η_4, which are used to control the numerical errors $||\nabla(\tilde{y} - \tilde{y}_h)||$ and $||\nabla(B - B_h)u_h||$, respectively, can not be directly computed in terms of the solution pair (y_h, z_h). To this end, let $\tilde{B}_h u_h \in V_h$ denote that extension which coincides with u_h at each nodal point $\mathbf{x}_i \in \Gamma$ but vanishes at each nodal point $\mathbf{x}_i \in \Omega$. The discrete harmonic extension $B_h u_h$ shows that $\tilde{B}_h u_h$ is indeed an approximation to $B_h u_h$, so $y_h - \tilde{B}_h u_h$ is a corresponding approximation to $\tilde{y}_h$. This suggests that one can substitute $B_h u_h$ and $\tilde{y}_h$ with $\tilde{B}_h u_h$ and $y_h - \tilde{B}_h u_h$, respectively, in order to compute the indicators η_4 and η_3. However, for the discrete KKT system in (3.11), the indicators η_3 and η_4 can be directly computed in terms of $\tilde{y}_h$ and $B_h u_h$ after $B_h u_h$ is replaced by $\tilde{B}_h u_h$ or $B_h u_h$ is solved from the second equation of (3.11). $B_h P_h u$ is a finite element solution of Bu, however, the indicator η_6 can not be evaluated, beacause the control variable u is unknown. Since $u_h = y_h|_\Gamma$ is an approximation to u, therefore, we can replace $P_h u$ by u_h, i.e., $B_h P_h u$ can be replaced by $B_h u_h$ in order to compute the indicator η_6.

Remark 4.2 It is known that the decomposed components $\tilde{y}$ and Bu of y are not easily obtained even if the exact solution triplet $\{y, u, z\}$ is known. Since $\tilde{y}_h$ and $B_h u_h$ are the standard finite element approximations to $\tilde{y}$ and Bu_h, respectively, the numerical errors $||\nabla(\tilde{y} - \tilde{y}_h)||$ and $||\nabla(B - B_h)u_h||$ are equivalent to the indicators η_3 and η_4, respectively. This suggests that the exact numerical errors $||\nabla(\tilde{y} - \tilde{y}_h)||$ and $||\nabla(B - B_h)u_h||$ may be replaced by the corresponding indicators for both the coupling system and the KKT system. Similarly, the exact error $\|\nabla(Bu - B_h P_h u)\|$ may be replaced by the indicator η_6.

Theorem 4.6 *Denote* η_7 *the edge/side residual indicator defined in (4.9). There holds*

$$\mathcal{E} + \left|\left|\gamma^{-1/2}\frac{\partial(z - z_h)}{\partial\mathbf{n}}\right|\right|_{0,\Gamma} \leq C(\eta + \eta_7 + \text{h.o.t}). \tag{4.43}$$

Proof Owing to the control rule $u = \frac{1}{\gamma}\frac{\partial z}{\partial \mathbf{n}}$, yields

$$\begin{aligned}\left|\left|\gamma^{-1/2}\frac{\partial(z - z_h)}{\partial\mathbf{n}}\right|\right|_{0,\Gamma}^2 &= \left(u - \frac{1}{\gamma}\frac{\partial z_h}{\partial\mathbf{n}}, \frac{\partial(z - z_h)}{\partial\mathbf{n}}\right)_\Gamma \\ &= \left(u - u_h, \frac{\partial(z - z_h)}{\partial\mathbf{n}}\right)_\Gamma + \left(u_h - \frac{1}{\gamma}\frac{\partial z_h}{\partial\mathbf{n}}, \frac{\partial(z - z_h)}{\partial\mathbf{n}}\right)_\Gamma \\ &\leq \left|\left|\gamma^{\frac{1}{2}}(u - u_h)\right|\right|_{0,\Gamma}\left|\left|\gamma^{-\frac{1}{2}}\frac{\partial(z - z_h)}{\partial\mathbf{n}}\right|\right|_{0,\Gamma} + \eta_7\left|\left|\gamma^{-\frac{1}{2}}\frac{\partial(z - z_h)}{\partial\mathbf{n}}\right|\right|_{0,\Gamma},\end{aligned}$$

which results in

$$\left\|\gamma^{-1/2}\frac{\partial(z-z_h)}{\partial \mathbf{n}}\right\|_{0,\Gamma} \leq \left\|\gamma^{\frac{1}{2}}(u-u_h)\right\|_{0,\Gamma} + \eta_7. \tag{4.44}$$

The estimate (4.43) follows from a combination of (4.40) and (4.44). □

5 Analysis of Efficiency

This section is devoted to the efficiency of the estimators developed in Sect. 5. To avoid the appearance of high order terms, we assume f and y_d are piecewise polynomials. For the simplicity of analysis, we consider only piecewise linear finite element approximation, i.e., $k = 1$, because the lowest order element is widely used in adaptivity. Since the indicators η_i $(i = 1, \cdots, 6)$ are composed of the element residuals and the side/face residuals, the efficient estimates (the lower bound) can be derived from the local efficiency of the element residuals and the side/face residuals.

Lemma 5.1 *There holds the following local efficiency for the element residual* $h_K||f + \triangle y_h||_K$

$$h_K||f + \triangle y_h||_K \lesssim ||\nabla(\tilde{y} - \tilde{y}_h)||_K. \tag{5.1}$$

Proof For convenience, write $v = f + \triangle y_h$. Let ψ_K be the bubble function on K. From the equivalence of norms $||\psi_K^{1/2} \cdot ||_K$ and $|| \cdot ||_K$ for polynomials, and $\psi_K v \in H_0^1(K)$ with the support K, we derive from integration by parts and the decomposition of y and y_h.

$$\begin{aligned} ||v||_K^2 &\approx (\psi_K v, f + \triangle y_h)_K = (-\triangle(y - y_h), \psi_K v)_K \\ &= (\nabla(y - y_h), \nabla(\psi_K v))_K \\ &= (\nabla(\tilde{y} - \tilde{y}_h), \nabla(\psi_K v))_K + (\nabla(Bu - B_h u_h), \nabla(\psi_K v))_K. \end{aligned} \tag{5.2}$$

Applying integration by parts, and noticing $B_h u_h \in V_h$ (piecewise linear finite element space), we have

$$\int_K \nabla(Bu - B_h u_h) \cdot \nabla(\psi_K v) = \int_K (-\triangle Bu + \triangle B_h u_h)\psi_K v + \int_{\partial K} \frac{\partial(Bu - B_h u_h)}{\partial \mathbf{n}} \psi_K v = 0. \tag{5.3}$$

We get from inverse estimate and the properties of the bubble function ψ_K

$$(\nabla(\tilde{y} - \tilde{y}_h), \nabla(\psi_K v))_K \lesssim h_K^{-1}||v||_K||\nabla(\tilde{y} - \tilde{y}_h)||_K. \tag{5.4}$$

Combining (5.2)–(5.3) with (5.4), we obtain the desired estimate (5.1) by multiplying the mesh-size function h_K and dividing by $||v||_K$. □

Lemma 5.2 *There holds the following local efficiency for the element residuals*

$$h_K||y_h - y_d + \triangle z_h||_K \lesssim ||\nabla(z - z_h)||_K + h_K||y - y_h||_K. \tag{5.5}$$

Proof Let $v = y_h - y_d + \triangle z_h$, and ψ_K be the bubble function introduced in Lemma 5.1. By repeating the proof of Lemma 5.1, we have

$$\begin{aligned} ||v||_K^2 &\approx (\psi_K v, v)_K = (y_h - y_d + \triangle z_h, \psi_K v)_K \\ &= (y - y_d + \triangle z_h, \psi_K v)_K - (y - y_h, \psi_K v)_K \end{aligned}$$

$$
\begin{aligned}
&= (-\triangle(z - z_h), \psi_K v)_K - (y - y_h, \psi_K v)_K \\
&= (\nabla(z - z_h), \nabla(\psi_K v))_K - (y - y_h, \psi_K v)_K \\
&\lesssim (h_K^{-1}||\nabla(z - z_h)||_K + ||y - y_h||_K)||v||_K,
\end{aligned}
$$

which results in the desired estimates (5.5) by dividing by $||v||_K$ and by multiplying by h_K. □

Lemma 5.3 *There holds the following local efficiency for the element residuals* $h_K||f + \triangle\tilde{y}_h||_K$

$$
h_K||f + \triangle\tilde{y}_h||_K \lesssim ||\nabla(\tilde{y} - \tilde{y}_h)||_K, \tag{5.6}
$$

Proof The estimate (5.6) follows from the standard analysis for Laplace equation. □

Lemma 5.4 *There holds the following local efficiency for the side/face residual* $h_E^{1/2}||[\partial y_h/\partial \mathbf{n}]||_E$, $E \in \mathcal{E}_h^0$

$$
h_E^{1/2}||[\partial y_h/\partial \mathbf{n}]||_E \lesssim ||\nabla(\tilde{y} - \tilde{y}_h)||_{\omega_E} + ||\nabla(B - B_h)u_h||_{\omega_E}. \tag{5.7}
$$

Proof Arbitrary $E \in \mathcal{E}_h^0$, let $\sigma = [\partial y_h/\partial \mathbf{n}]|_E$, and ψ_E be a bubble function of E. It is well known [42,43] that there exists an extension operator $P : C(E) \to C(\omega_E)$ such that

$$
P\sigma|_E = \sigma|_E; \quad ||\psi_E P\sigma||_{\omega_E} \lesssim h_E^{1/2}||\sigma||_E. \tag{5.8}
$$

Notice that

$$
([\partial y_h/\partial \mathbf{n}], \psi_E P\sigma)_E = -(\nabla(y - y_h), \nabla(\psi_E P\sigma))_{\omega_E} + \sum_{K\in\omega_E} (-\triangle(y - y_h), \psi_E P\sigma)_K,
$$

which results in, altogether with the equivalence of norms $||\psi_E^{1/2} \cdot ||_E$ and $|| \cdot ||_E$ for polynomials,

$$
\begin{aligned}
||\sigma||_E^2 &\lesssim ([\partial y_h/\partial \mathbf{n}], \psi_E P\sigma)_E \\
&= \sum_{K\in\omega_E} (f + \triangle y_h, \psi_E P\sigma)_K - (\nabla(y - y_h), \nabla(\psi_E P\sigma))_{\omega_E}.
\end{aligned} \tag{5.9}
$$

Owing to the decomposition of y and $\tilde{y}_h$, we have

$$
\begin{aligned}
&(\nabla(y - y_h), \nabla(\psi_E P\sigma))_{\omega_E} \\
&\quad = (\nabla(\tilde{y} - \tilde{y}_h), \nabla(\psi_E P\sigma))_{\omega_E} + (\nabla(Bu - B_h u_h), \nabla(\psi_E P\sigma))_{\omega_E}.
\end{aligned} \tag{5.10}
$$

By adding and subtracting Bu_h, employing integration by parts, and noticing the definition of the harmonic extension, we get

$$
\begin{aligned}
&(\nabla(Bu - B_h u_h), \nabla(\psi_E P\sigma))_{\omega_E} \\
&= (\nabla B(u - u_h), \nabla(\psi_E P\sigma))_{\omega_E} + (\nabla(Bu_h - B_h u_h), \nabla(\psi_E P\sigma))_{\omega_E} \\
&= \int_{\omega_E} -\triangle B(u - u_h)(\psi_E P\sigma) + \int_{\partial\omega_E} \frac{\partial B(u - u_h)}{\partial \mathbf{n}}(\psi_E P\sigma) \\
&\quad +(\nabla(Bu_h - B_h u_h), \nabla(\psi_E P\sigma))_{\omega_E} \\
&= (\nabla(B - B_h)u_h, \nabla(\psi_E P\sigma))_{\omega_E}.
\end{aligned} \tag{5.11}
$$

Combining (5.9)–(5.10) with (5.11), and applying (5.8) and inverse estimate, we arrive at

$$
\begin{aligned}
||\sigma||_E^2 &\lesssim \sum_{K\in\omega_E} ||f + \triangle y_h||_K ||\psi_E P\sigma||_K + (||\nabla(\tilde{y} - \tilde{y}_h)||_{\omega_E} \\
&\quad + ||\nabla(B - B_h)u_h||_{\omega_E})||\nabla(\psi_E P\sigma)||_{\omega_E} \\
&\lesssim \left(\sum_{K\in\omega_E} h_E ||f + \triangle y_h||_K + ||\nabla(\tilde{y} - \tilde{y}_h)||_{\omega_E} + ||\nabla(B - B_h)u_h||_{\omega_E}\right) h_E^{-\frac{1}{2}} ||\sigma||_E.
\end{aligned}
\tag{5.12}
$$

Multiplying (5.12) by $h_E^{1/2}$, dividing (5.12) by $||\sigma||_E$, and employing (5.1), we obtain the desired result (5.7). □

Lemma 5.5 *There holds the following local efficiency for internal side/face residuals of the adjoint state ($E \in \mathcal{E}_h^0$)*

$$
h_E^{\frac{1}{2}} \left|\left|[\frac{\partial z_h}{\partial \mathbf{n}}]\right|\right|_E \lesssim h_E \left|\left|y - y_h\right|\right|_{\omega_E} + \left|\left|\nabla(z - z_h)\right|\right|_{\omega_E}. \tag{5.13}
$$

Proof For $E \in \mathcal{E}_h^0$, let $\sigma_E = [\partial z_h/\partial \mathbf{n}]|_E$, $v_E = \psi_E P\sigma_E$, where the bubble function ψ_E and the extension operator P are the same as that in Lemma 5.4. Following the proof of (5.7), we have

$$
\begin{aligned}
||\sigma_E||_E^2 &\approx \big([\partial z_h/\partial \mathbf{n}], v_E\big)_E \\
&= \sum_{K\in\omega_E} (-\triangle(z - z_h), v_E)_K - (\nabla(z - z_h), \nabla v_E)_{\omega_E} \\
&= \sum_{K\in\omega_E} (y - y_d + \triangle z_h, v_E)_K - (\nabla(z - z_h), \nabla v_E)_{\omega_E} \\
&= (y - y_h, v_E)_{\omega_E} + \sum_{K\in\omega_E} (y_h - y_d + \triangle z_h, v_E)_K - (\nabla(z - z_h), \nabla v_E)_{\omega_E} \\
&\lesssim \Big\{\big(||y - y_h||_{\omega_E} + \sum_{K\in\omega_E} ||y_h - y_d + \triangle z_h||_K\big) h_E^{1/2} \\
&\quad + ||\nabla(z - z_h)||_{\omega_E} h_E^{-1/2}\Big\} ||\sigma_E||_E.
\end{aligned}
\tag{5.14}
$$

Dividing (5.14) by $||\sigma_E||_E$, and multiplying (5.14) by $h_E^{1/2}$, and employing (5.5), we obtain the estimate (5.13). □

Lemma 5.6 *There holds the following local efficiency for internal side/face residuals $h_E^{1/2}||[\partial\tilde{y}_h/\partial\mathbf{n}]||_E$, $h_E^{1/2}||[\partial B_h u_h/\partial\mathbf{n}]||_E$, and $h_E^{1/2}||[\partial B_h P_h u/\partial\mathbf{n}]||_E$ ($E \in \mathcal{E}_h^0$)*

$$
h_E^{1/2}||[\partial\tilde{y}_h/\partial\mathbf{n}]||_E \lesssim ||\nabla(\tilde{y} - \tilde{y}_h)||_{\omega_E}, \tag{5.15}
$$

$$
h_E^{1/2}||[\partial B_h u_h/\partial\mathbf{n}]||_E \lesssim ||\nabla(B - B_h)u_h||_{\omega_E}, \tag{5.16}
$$

$$
h_E^{1/2}||[\partial B_h P_h u/\partial\mathbf{n}]||_E \lesssim ||\nabla(Bu - B_h P_h u)||_{\omega_E}. \tag{5.17}
$$

Proof These three estimates (5.15), (5.16), and (5.17) are the standard results for Poisson equation. □

Lemma 5.7 *There holds the following local efficiency for boundary edge/side residuals*

$$h_E^{\frac{1}{2}}||\sigma||_E \lesssim h_K\big|\big|y-y_h\big|\big|_K + \gamma^{\frac{1}{2}}h_E^{\frac{1}{2}}\big|\big|\gamma^{\frac{1}{2}}(u-u_h)\big|\big|_{0,E} + \big|\big|\nabla(z-z_h)\big|\big|_K, \quad (5.18)$$

where $\sigma = \gamma u_h - \frac{\partial z_h}{\partial \mathbf{n}}$, $E \subset \partial K \cap \Gamma$.

Proof Let $v_E = \psi_E P\sigma$, and notice

$$\begin{aligned}\int_K -\triangle(z-z_h)v_E &= \int_K \nabla(z-z_h)\cdot\nabla v_E - \int_{\partial K}\frac{\partial(z-z_h)}{\partial\mathbf{n}}v_E\\ &= \int_K \nabla(z-z_h)\cdot\nabla v_E - \int_E \frac{\partial(z-z_h)}{\partial\mathbf{n}}v_E,\end{aligned}$$

which results in

$$\begin{aligned}||\sigma||_E^2 &\approx (\gamma u_h - \partial z_h/\partial\mathbf{n}, v_E)_E\\ &= \int_E \frac{\partial(z-z_h)}{\partial\mathbf{n}}v_E + \int_E\left(\gamma u_h - \frac{\partial z}{\partial\mathbf{n}}\right)v_E\\ &= \int_K \nabla(z-z_h)\cdot\nabla v_E - \int_K -\triangle(z-z_h)v_E + \int_E\left(\gamma u_h - \frac{\partial z}{\partial\mathbf{n}}\right)v_E\\ &= (\nabla(z-z_h),\nabla v_E)_K + (y_d - y - \triangle z_h, v_E)_K\\ &\quad +(\gamma u_h - \gamma y + \gamma y - \frac{\partial z}{\partial\mathbf{n}}, v_E)_E\\ &\le ||\nabla(z-z_h)||_K||\nabla v_E||_K + (||y_d - y_h - \triangle z_h||_K + ||y_h - y||_K)||v_E||_K\\ &\quad +||\gamma(u_h-u)||_E||v_E||_E\\ &\lesssim \left(h_E^{-1/2}||\nabla(z-z_h)||_K + h_E^{1/2}(||y_h - y_d + \triangle z_h||_K + ||y-y_h||_K)\right)||\sigma||_E\\ &\quad +\gamma^{1/2}||\gamma^{1/2}(u-u_h)||_{0,E}||\sigma||_E. \end{aligned}\quad (5.19)$$

In the fifth step above, we employ the boundary information $u = y|_\Gamma$ and the control rule $\gamma y = \partial z/\partial\mathbf{n}$ on Γ; In the sixth step above, we use $||v_E||_E \approx ||\sigma||_E$. Dividing (5.19) by $||\sigma||_E$, and multiplying (5.19) by $h_E^{1/2}$, and employing (5.5), we obtain the desired estimate (5.18). □

Theorem 5.8 *Let* $(y, z) \in H^1(\Omega)\times H_0^1(\Omega)$ *and* $(y_h, z_h) \in V_h \times V_h^0$ *be the solutions to* (2.8)–(2.9) *and* (3.9)–(3.10)*, respectively, and denote* η_i *(*$i = 1, \cdots, 6$*) the indictors defined in* (4.3)–(4.9)*. There hold the following global efficiency estimates for the indicators* η_i *(*$i = 1, \ldots, 6$*)*

$$\eta_1^2 \lesssim ||\nabla(\tilde{y}-\tilde{y}_h)||^2 + ||\nabla(B-B_h)u_h||^2, \quad (5.20)$$
$$\eta_2^2 \lesssim ||\nabla(z-z_h)||^2 + ||h(y-y_h)||^2, \quad (5.21)$$
$$\eta_3^2 \lesssim ||\nabla(\tilde{y}-\tilde{y}_h)||^2, \quad (5.22)$$
$$\eta_4^2 \lesssim ||\nabla(B-B_h)u_h||^2, \quad (5.23)$$
$$\eta_5^2 \lesssim ||\nabla(z-z_h)||^2 + ||h(y-y_h)||^2 + \gamma\|h^{1/2}\gamma^{1/2}(u-u_h)\|_{0,\Gamma}^2, \quad (5.24)$$
$$\eta_6^2 \lesssim \|\nabla(Bu - B_h P_h u)\|^2, \quad (5.25)$$
$$\eta_7^2 \lesssim \big|\big|\gamma^{1/2}(u-u_h)\big|\big|_{0,\Gamma}^2 + \left|\left|\gamma^{-1/2}\frac{\partial(z-z_h)}{\partial\mathbf{n}}\right|\right|_{0,\Gamma}^2. \quad (5.26)$$

For the total numerical error $\mathcal{E}$ defined in (4.2) and the a posteriori indicator η defined in Theorem 4.4, there exists a positive constant C_γ dependent on γ, such that

$$C_\gamma \eta \le \mathcal{E}. \tag{5.27}$$

Furthermore, for the boundary residual indicator η_7, there holds the following global efficiency estimate

$$C_\gamma (\eta + \eta_7) \le \mathcal{E} + \left\| \gamma^{-1/2} \frac{\partial (z - z_h)}{\partial \mathbf{n}} \right\|_{0,\Gamma}. \tag{5.28}$$

Proof Notice that (5.28) follows from a combination of (5.27) and (5.26), and that (5.27) is a direct result of (5.20)–(5.25). (5.26) is obtained by adding and subtracting $\gamma^{1/2}u$, and by using triangle inequality and the control rule $u = \frac{1}{\gamma}\frac{\partial z}{\partial \mathbf{n}}$.

Collecting Lemmas 5.1-5.7, and summing over all elements $K \in \mathcal{T}_h$ and all edges/sides $E \in \mathcal{E}_h$, we immediately get (5.20)–(5.25). □

Theorem 5.9 *Let $\{\tilde{y}, u, z\} \in H_0^1(\Omega) \times H^{1/2}(\Gamma) \times H_0^1(\Omega)$ and $\{\tilde{y}_h, u_h, z_h\} \in V_h^0 \times V_h^\partial \times V_h^0$ be the solution triplet to the continuous KKT system in (2.11) and the discrete KKT system in (3.11), respectively, and $\tilde{\mathcal{E}}$ be the measure of numerical errors defined in Theorem 4.5. There holds the following global efficiency estimate*

$$C_\gamma \eta \le \tilde{\mathcal{E}}. \tag{5.29}$$

Moreover, for the boundary residual indicator η_7, there holds the following global efficiency estimate

$$C_\gamma (\eta + \eta_7) \le \tilde{\mathcal{E}} + \left\| \gamma^{-1/2} \frac{\partial (z - z_h)}{\partial \mathbf{n}} \right\|_{0,\Gamma}. \tag{5.30}$$

Proof We have from the decomposition of y and y_h, Poincaré inequality, and Lemma 2.2

$$\begin{aligned} ||y - y_h|| &\le ||\tilde{y} - \tilde{y}_h|| + ||Bu - B_h u_h|| \\ &\le ||\tilde{y} - \tilde{y}_h|| + ||B(u - u_h)|| + ||(B - B_h)u_h|| \\ &\lesssim ||\nabla(\tilde{y} - \tilde{y}_h)|| + ||B(u - u_h)||_{H^{1/2}(\Omega)} + ||\nabla(B - B_h)u_h|| \\ &\lesssim ||\nabla(\tilde{y} - \tilde{y}_h)|| + ||u - u_h||_{0,\Gamma} + ||\nabla(B - B_h)u_h||, \end{aligned}$$

which results in $\mathcal{E} \lesssim \tilde{C}_\gamma \tilde{\mathcal{E}}$. This yields (5.29) with the help of (5.27). A combination of (5.29) and (5.26) gives (5.30). □

6 Numerical Experiments

In this section, we test the performance of the a posteriori estimates developed in this paper with two model problems. We are thus able to study the behaviour of the a posteriori estimators over adaptive mesh. Note that we shall employ piecewise linear element in both examples.

6.1 Example One

We consider the problem (1.1)–(1.2) over a unit square $\Omega = (0, 1) \times (0, 1)$ with

$$f = -\frac{4}{\gamma}, \; y_d = \left(2 + \frac{1}{\gamma}\right) \left(x_1^2 - x_1 + x_2^2 - x_2\right).$$

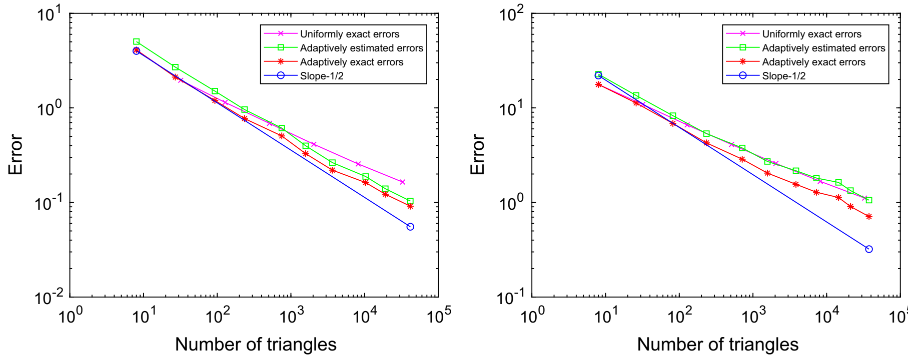

Fig. 1 Estimated and exact errors against the number of elements in uniformly/adaptively refined meshes for $\gamma = 1$ (left) and $\gamma = 0.1$ (right) for marking parameter $\theta = 0.7$

The exact solutions are given by

$$u = \frac{x_1^2 - x_1 + x_2^2 - x_2}{\gamma}, y = \frac{x_1^2 - x_1 + x_2^2 - x_2}{\gamma}, z = \left(x_1^2 - x_1\right)\left(x_2^2 - x_2\right).$$

It is easy to verify that the control u, state y, and adjoint state z satisfy

$$u = y|_\Gamma = \frac{1}{\gamma}\frac{\partial z}{\partial \mathbf{n}}\Big|_\Gamma.$$

Here, we consider different settings of regularization parameter γ.

In the adaptive algorithm, we employ the Dörfler marking strategy with different marking parameter θ and the "longest edge" refinement to obtain an admissible mesh, and start with an initial mesh consisting of 8 congruent right triangles.

Figure 1 reports the estimated error (η) and the exact error ($\mathcal{E}$) against the number of elements in uniformly/adaptively refined meshes for $\gamma = 1$ (left) and $\gamma = 0.1$ (right), including an optimal convergence line Slope-1/2. Note that the actual errors $||\nabla(\tilde{y} - \tilde{y}_h)||$, $||\nabla(B - B_h)u_h||$, and $\|\nabla(Bu - B_h P_h u)\|$ are substituted with their corresponding indicators η_3 and η_4, respectively (see Remark 4.2). Figure 1 shows that the estimated and actual errors are close over adaptively refined meshes, which suggests the estimator η is rather efficient, since the solutions for this model problem are the global polynomial functions, which are of good regularity.

Figure 2 shows the estimated values of the indicator η_7 against the number of elements in adaptively refined meshes for different regularization $\gamma = 1, 0.1$ (left) and $\gamma = 0.01, 0.001$ (right) in case of the marking parameter $\theta = 0.5$, including a convergence line Slope-1. It can be seen that the boundary indicator η_7 has convergence rate of order 1 in numbers of elements for boundary concentrated meshes.

6.2 Example Two

We consider a 2D example on a square domain $\Omega = (0, 1/4) \times (0, 1/4) \subset \mathbb{R}^2$. The data is chosen as

$$f = 0, \quad y_d = (x_1^2 + x_2^2)^s,$$

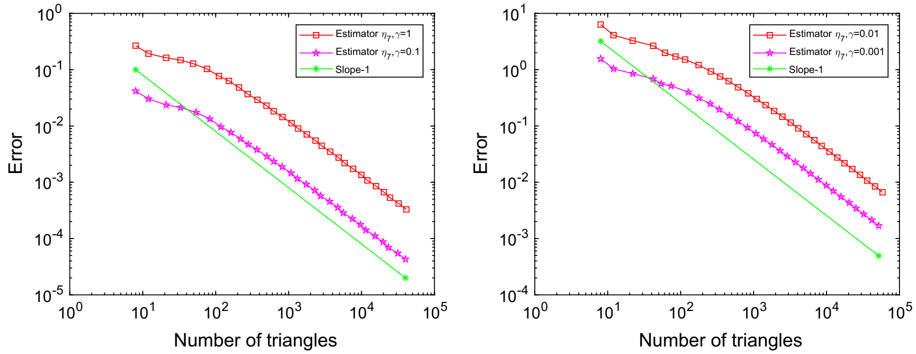

Fig. 2 Estimated value of the indicator η_7 against the number of elements in adaptively refined meshes for different regularization parameter $\gamma = 1, 0.1$ (left) and $\gamma = 0.01, 0.001$ (right) in case of the marking parameter $\theta = 0.5$

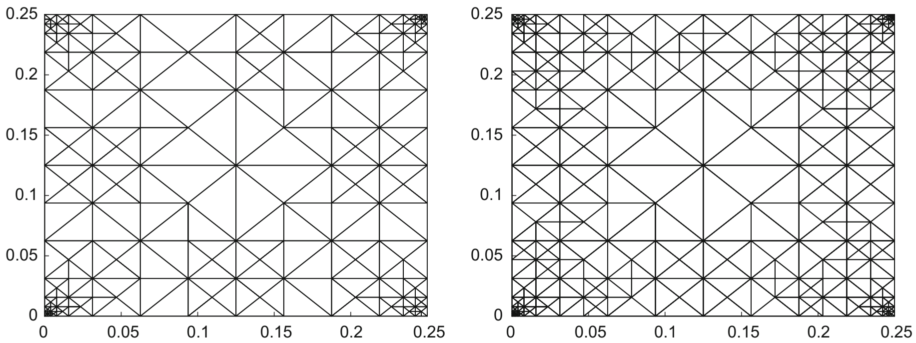

Fig. 3 A mesh with 313 triangles, iteration 8 (left) and a mesh with 510 triangles, iteration 9 (right) in case of $\gamma = 0.1$ and the marking parameter $\theta = 0.5$

where $s = 10^{-5}$. Since we do not have an explicit expression for the exact solution, and the solution has strong singularity at four corners of the boundary, adaptive finite element method based on a posteriori estimates is very applicable to this problem. In the adaptive algorithm, we begin with an initial mesh consisting of 8 congruent right triangles, and first solve the discrete system (3.9)–(3.10), then mark elements in terms of Dörfler marking, and finally use the "longest edge" refinement.

Figures 3 and 4 shows the meshes generated by the estimator η by refinement of iterations from 8 to 11 for $\gamma = 0.1$ and the marking parameter $\theta = 0.5$. It can be seen that the refinement mainly concentrates around the four corners of the boundary, which suggests the predicted error estimator η captures well the singularity of the solution. Figure 5 reports an approximation to the state variable y and the adjoint state variable z over mesh with 9830 elements in case of $\gamma = 0.1$.

The first picture of Fig. 6 shows an approximation solution to the control variable u over the mesh with 25812 elements in case of $\gamma = 0.1, \theta = 0.5$ (left). The second picture of Fig. 6 reports the global estimated error η over the adaptively refined meshes in case of $\gamma = 1, 0.1, \theta = 0.9$ (right), including an optimal convergence line Slope-1/2. In addition, it can be also seen that the estimated errors are close to the optimal convergence line after several iterations, since the total error $\mathcal{E}$ includes the energy norm errors of $\tilde{y} - \tilde{y}_h$, $Bu_h - B_h u_h$,

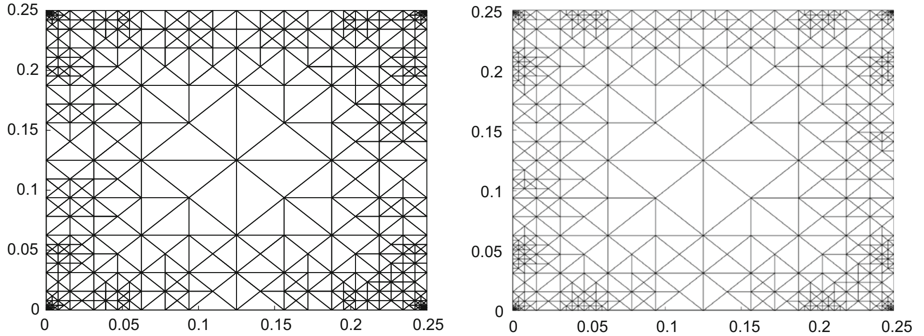

Fig. 4 A mesh with 790 triangles, iteration 10 (left) and a mesh with 1045 triangles, iteration 11 (right) in case of $\gamma = 0.1$ and the marking parameter $\theta = 0.5$

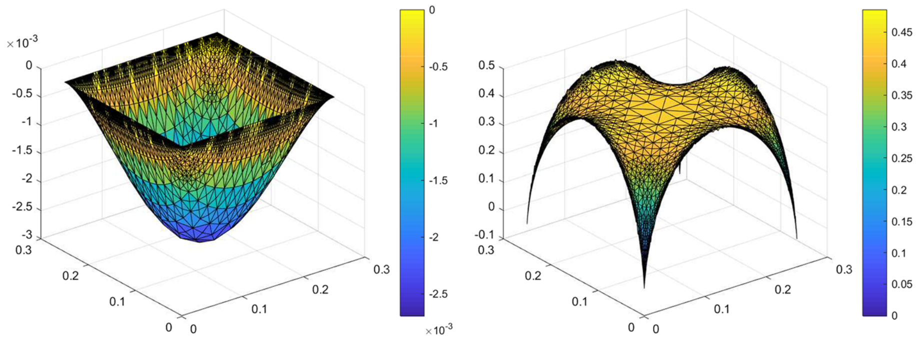

Fig. 5 An approximation solution to the adjoint state variable (left) and the state variable (right) over the mesh with 9765 triangles in case of $\gamma = 0.1$

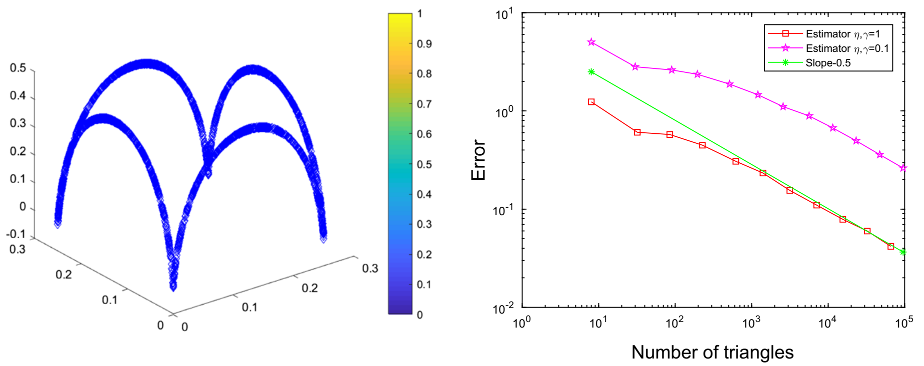

Fig. 6 An approximation solution to the control variable over the mesh with 25812 triangles for $\gamma = 0.1, \theta = 0.5$ (left) and the estimated errors in case of $\gamma = 1, 0.1, \theta = 0.9$ (right)

$Bu - B_h P_h u$, and $z - z_h$. (Note that the convergence order $1/2$ is optimal for the linear conforming element for Poisson equation in energy norm.)

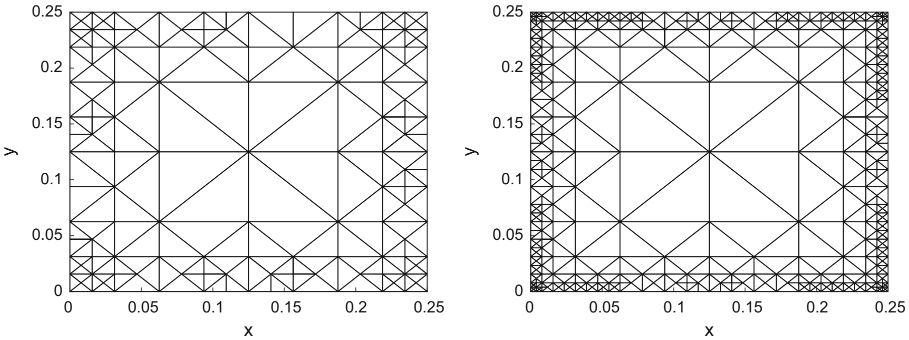

Fig. 7 A mesh with 212 triangles, iteration 8 (left) and a mesh with 675 triangles, iteration 12 (right), generated by η_7, in case of $\gamma = 0.1, \theta = 0.5$

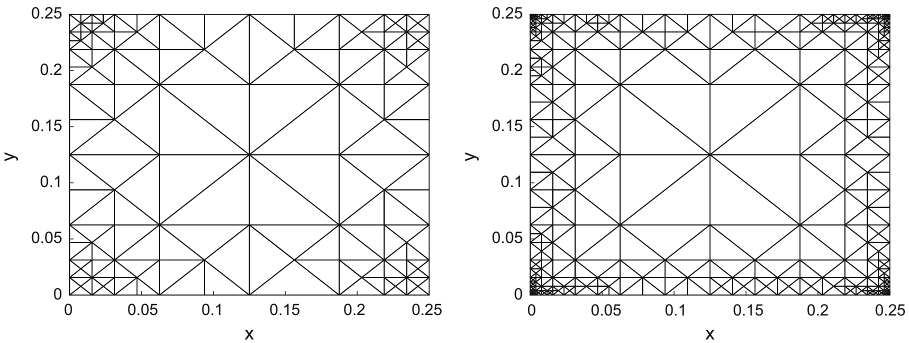

Fig. 8 A mesh with 170 triangles, iteration 8 (left) and a mesh with 594 triangles, iteration 12 (right), generated by η_7, in case of $\gamma = 10^{-4}, \theta = 0.5$

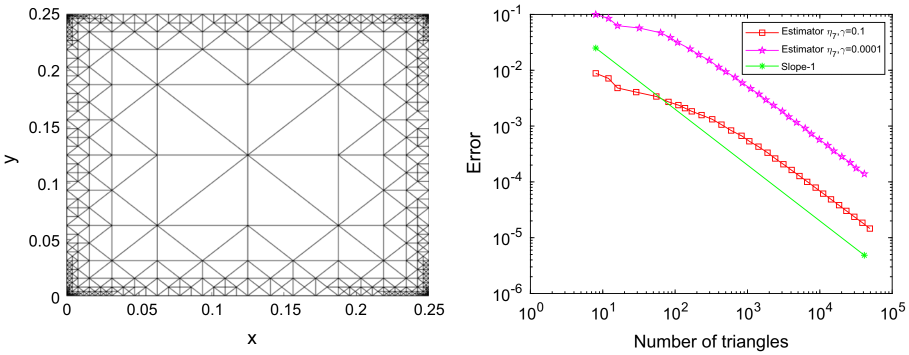

Fig. 9 A mesh with 1061 triangles, iteration 14 (left), generated by η_7, in case of $\gamma = 10^{-4}, \theta = 0.5$; and the estimated η_6 (right) in case of $\gamma = 0.1, 10^{-4}, \theta = 0.5$

Figures 7, 8, and the first picture in 9 show the meshes generated by the indicator η_7 in case of $\gamma = 0.1$ and $\gamma = 10^{-4}$, respectively. It can be seen that the indicator η_7 is a good guidance to refine the mesh around the four corners of the boundary.

The second picture in 9 reports the estimated value of the indicator η_7, including a convergence line Slope−1. This suggests that the indicator η_7 is of convergence order 1 in numbers of elements for boundary concentrated meshes, and that the adaptive method based on η_7 performs as good as the known a priori technique, since the boundary indicator η_7 reflects the gap between the approximation to the control and the normal derivative of the approximation to the adjoint state in terms of the control rule, it can be understood as an indicator the numerical errors for the control variable.

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