Quiz 12 Solutions

MA 262
Artur's Class

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Problem 1

Solve the following linear system, where ω is a constant.

$$\begin{aligned} x' &= y \qquad (1) \\ y' &= -\omega^2 x. \qquad (2) \end{aligned}$$

Solution

Let us use t for the independent variable, i.e., x and y are functions of t. Then the first equation, literally, means $y(t) = dx/dt$. Substituting this into the second equation gives

$$x''(t) + \omega^2 x(t) = 0.$$

But this is an equation we know how to solve; it's general solution is

$$x(t) = c_1 \cos \omega t + c_2 \sin \omega t.$$

Since $y(t) = x'(t)$, we get

$$y(t) = -c_1 \omega \sin \omega t + c_2 \omega \cos \omega t.$$

Thus the vector solution is

$$\mathbf{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} c_1 \cos \omega t + c_2 \sin \omega t \\ -c_1 \omega \sin \omega t + c_2 \omega \cos \omega t \end{pmatrix}.$$

Or, written differently

$$\mathbf{x}(t) = c_1 \begin{pmatrix} \cos \omega t \\ -\omega \sin \omega t \end{pmatrix} + c_2 \begin{pmatrix} \sin \omega t \\ \omega \cos \omega t \end{pmatrix}$$

Remark

From this form you should be able to see what the motion 'looks like'. Try plotting it on your paper. (It is ellipsoidal.) This is the phase space trajectory for a pendulum, or if you like quantum mechanics: a harmonic oscillator.

Problem 2

Suppose $f(t)$ and $g(t)$ are arbitrary real valued functions on $(-\infty, \infty)$. What must be true about $f(t)$ or $g(t)$ so that the vector valued functions

$$\mathbf{x}_1(t) = \begin{pmatrix} f(t) \\ g(t) \end{pmatrix}, \quad \mathbf{x}_2(t) = \begin{pmatrix} f(t)+1 \\ g(t) \end{pmatrix},$$

are linearly independent?

Solution

Use Wronskian. We know that the vector valued functions $\mathbf{x}_1$ and $\mathbf{x}_2$ are independent if their Wronskian is *not* zero in at least a single value of $t \in (-\infty, +\infty)$. Compute:

$$\text{Wronskian} = f(t)g(t) - (f(t)+1)g(t) = -g(t).$$

Thus:

> When the function g is nonzero at at least one value of t, the two vector valued functions are independent.

(And there is no condition on f, i.e., f can be anything!)

Alternate Solution

Just write $\mathbf{x}_1(t) = c\,\mathbf{x}_2(t)$, for some nonzero $c \in \mathbb{R}$. And notice, after expanding, that this is only possible if $g(t) = 0$ everywhere.

Remark

Be very careful when using quantifiers. The correct solution to this problem (i.e., a condition which is equivalent to the vector valued functions being linearly independent) is either 'The function g is not identically zero' or '$g(t)$ is nonzero for *some* value of t'. Note that this is very different from saying 'the function $g(t) \neq 0$ for *all* t.'