Quiz 3 — Solutions

MA 262
Artur's Class

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Problem 1

Give all values of n, c such that

$$f(x, y) = x^n + cy^2$$

is homogeneous.

Solution

A function f of two variables x, y is homogeneous when $f(tx, ty) = t^k f(x, y)$, where k is the degree of homogeneity. (It is homogeneous of degree zero when $f(tx, ty) = t^0 f(x, y) = f(x, y)$.)

Try to enforce the identity:

$$f(tx, ty) = t^k f(x, y)$$
$$t^n x^n + ct^2 y^2 = t^k x^n + ct^k y^2$$

This means that $t^n = t^k$ and $ct^2 = ct^k$. Since these must be valid for all t, the first equation implies $n = k$. The implications of the second equation depend on the value of c; more precisely, it depends on whether c is zero or not.

There are two (families of) possibilities:

- If c is nonzero, then $n = k = 2$. $n = 2$ is also a possibility when $c = 0$. So one family is $c \in \mathbb{R}$, $n = 2$.
- When $c = 0$ we have $n = k$ but, k has no restrictions. Thus another family is $c = 0$, $n \in \mathbb{R}$.

Problem 2

Give all values α such that

$$dy + 3xy(4 + y^\alpha)\ dx = 0$$

is

(i) Bernoulli,

(ii) Linear.

DO NOT SOLVE.

Solution

It is a first order equation. Rearrange slightly:

$$\frac{dy}{dx} + 12xy = -3y^{1+\alpha}$$

This equation is Bernoulli for any value of the exponent on the right hand side, i.e., for all real α This equation is linear when $y^{1+\alpha} = y$ or $y^{1+\alpha}$ is a constant. The first case happens when $\alpha = -1$ and the second when $\alpha = 0$.

Problem 3

Is

$$2xy\ dx + (x^2 + 1)\ dy = 0$$

exact? Why?

Solution

Write $M = 2xy$ and $N = x^2 + 1$. It is easy to see that $\partial M/\partial y = \partial N/\partial x$. But this is precisely the criteria for exactness.[1]

Problem 4

Find potential function $\phi = \phi(x, y)$ for the equation in #3.

[1]More precisely, this is the criteria for exactness on a simply connected domain.

Solution

The potential function should satisfy $\partial\phi/\partial x = M$. So for a fixed y we should have

$$\phi(x, y) = \int M dx = \int 2xy \; dx = x^2 y + h(y),$$

for some constant $h(y)$. As y varies, so does $h(y)$ so we may view h as a function. If a potential function exists, then h must be differentiable. Since we want $\partial\phi/\partial y = N$ we must have that $h'(y) = 1$. But $h(y) = y$ is a solution to this. So

$$\phi(x, y) = x^2 y + y$$

is a potential function for the exact equation.