

MA 16200

Study Guide - Exam # 2

(1) Trigonometric Substitutions:

<i>Expression*</i>	<i>Trig Substitution</i>	<i>Identity needed</i>
$(a^2 - x^2)^{\frac{1}{2}}$	$x = a \sin \theta$	$1 - \sin^2 \theta = \cos^2 \theta$
$(a^2 + x^2)^{\frac{1}{2}}$	$x = a \tan \theta$	$1 + \tan^2 \theta = \sec^2 \theta$
$(x^2 - a^2)^{\frac{1}{2}}$	$x = a \sec \theta$	$\sec^2 \theta - 1 = \tan^2 \theta$

* Or powers of these expressions.

(2) Integration via Partial Fractions: Use for (proper) rational functions $\frac{R(x)}{Q(x)}$;

If $\text{degree } R(x) \geq \text{degree } Q(x)$, i.e. rational function is improper, then do long division before using partial fractions.

(3) Integration via Clever Substitutions/Using Integral Tables: Use a substitution to transform integral into a form to be able to use another integral techniques (**Simple Substitution**, **Integration by Parts**, **Trig Integrals**, **Trig Substitution**, **Partial Fractions**, etc) or use integral tables.

(4) Approximating definite integrals $\int_a^b f(x) dx$.

Let $\Delta x = \frac{b-a}{n}$, $x_k = a + k \Delta x$ and $\bar{x}_k = \frac{1}{2}(x_{k-1} + x_k)$ (Note that $x_0 = a$ and $x_n = b$)

(a) Midpoint Rule: $\int_a^b f(x) dx \approx M_n = (\Delta x) [f(\bar{x}_1) + f(\bar{x}_2) + \cdots + f(\bar{x}_n)]$

(b) Trapezoidal Rule: $\int_a^b f(x) dx \approx T_n = \left(\frac{\Delta x}{2}\right) [f(x_0) + 2f(x_2) + 2f(x_3) + \cdots + 2f(x_{n-1}) + f(x_n)]$

(c) Simpson's Rule: Only works for n even.

$$\int_a^b f(x) dx \approx S_n = \left(\frac{\Delta x}{3}\right) [f(x_0) + 4f(x_2) + 2f(x_3) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)]$$

(5) Improper integrals: **Type I** (unbounded intervals) $\int_a^\infty f(x) dx$, $\int_{-\infty}^b f(x) dx$ or $\int_{-\infty}^\infty f(x) dx$;

Improper integrals of **Type II** (discontinuous integrand at one or both endpoints) $\int_a^b f(x) dx$.

(a) Comparison Theorem : Let $f(x)$ and $g(x)$ be continuous for $x \geq a$.

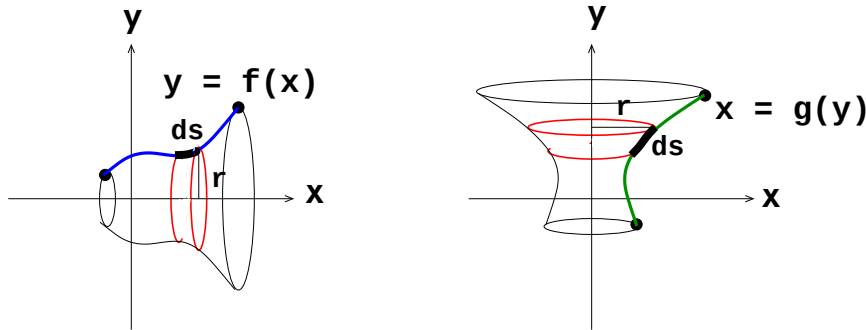
(i) If $0 \leq f(x) \leq g(x)$ for $x \geq a$ and $\int_a^\infty g(x) dx$ converges $\implies \int_a^\infty f(x) dx$ also converges.

(ii) If $0 \leq g(x) \leq f(x)$ for $x \geq a$ and $\int_a^\infty g(x) dx$ diverges $\implies \int_a^\infty f(x) dx$ also diverges.

(b) $\int_1^\infty \frac{1}{x^p} dx$ Theorem : The improper integral $\int_1^\infty \frac{1}{x^p} dx$ will $\begin{cases} \text{Converge if } p > 1 \\ \text{Diverge if } p \leq 1 \end{cases}$

(6) Arc length $L = \int_a^b \sqrt{1 + (f'(x))^2} dx$ or $L = \int_c^d \sqrt{1 + (g'(y))^2} dy$.

(7) Surface area of revolution: $S = \int 2\pi \{\text{radius}\} ds$ or $S = \int 2\pi r ds$,
 where $ds = \sqrt{1 + (f'(x))^2} dx$ or $ds = \sqrt{1 + (g'(y))^2} dy$.



(8) Center of mass of a system of discrete masses $m_1, m_2, \dots, m_n$ located at $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$ is $(\bar{x}, \bar{y})$, where

$$\bar{x} = \frac{M_y}{M} = \frac{\sum_{k=1}^n m_k x_k}{\sum_{k=1}^n m_k}, \quad \bar{y} = \frac{M_x}{M} = \frac{\sum_{k=1}^n m_k y_k}{\sum_{k=1}^n m_k}$$

M_x = moment of system about the x -axis; M_y = moment of system about the y -axis;
 M = total mass of the system.

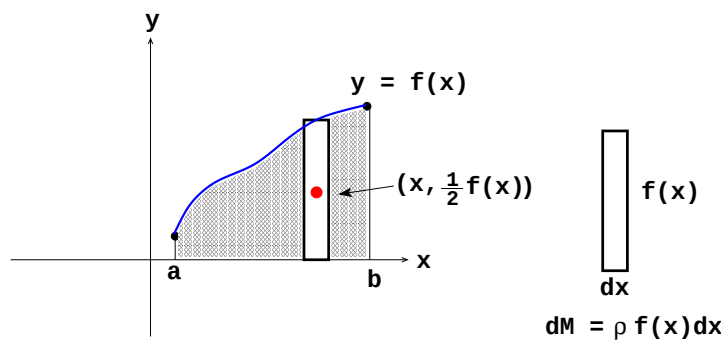
(9) Moments, center of mass (center of mass = *centroid* if density ρ = constant).

(a) Lamina defined by $y = f(x)$, $a \leq x \leq b$ and ρ = constant:

$$\bar{x} = \frac{M_y}{M} = \frac{\int_a^b x \rho f(x) dx}{\int_a^b \rho f(x) dx} = \frac{\int_a^b x f(x) dx}{\int_a^b f(x) dx} = \frac{\int_a^b x f(x) dx}{A}$$

$$\bar{y} = \frac{M_x}{M} = \frac{\int_a^b \frac{1}{2} \rho \{f(x)\}^2 dx}{\int_a^b \rho f(x) dx} = \frac{\int_a^b \frac{1}{2} \{f(x)\}^2 dx}{\int_a^b f(x) dx} = \frac{\int_a^b \frac{1}{2} \{f(x)\}^2 dx}{A}$$

where A = area under the curve $f(x)$ when $a \leq x \leq b$.

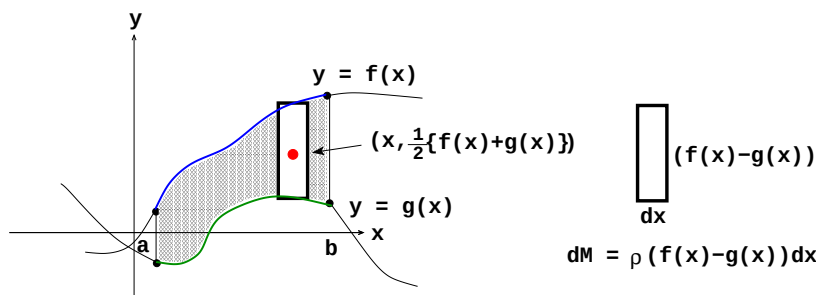


(b) Lamina between two curves by $y = f(x)$, $y = g(x)$, $a \leq x \leq b$ and $\rho = \text{constant}$:

$$\bar{x} = \frac{M_y}{M} = \frac{\int_a^b x \rho (f(x) - g(x)) dx}{\int_a^b \rho (f(x) - g(x)) dx} = \frac{\int_a^b x (f(x) - g(x)) dx}{\int_a^b (f(x) - g(x)) dx} = \frac{\int_a^b x (f(x) - g(x)) dx}{A}$$

$$\bar{y} = \frac{M_x}{M} = \frac{\int_a^b \frac{1}{2} \rho (\{f(x)\}^2 - \{g(x)\}^2) dx}{\int_a^b \rho (f(x) - g(x)) dx} = \frac{\int_a^b \frac{1}{2} (\{f(x)\}^2 - \{g(x)\}^2) dx}{\int_a^b (f(x) - g(x)) dx} = \frac{\int_a^b \frac{1}{2} (\{f(x)\}^2 - \{g(x)\}^2) dx}{A}$$

where $A = \text{area between the curves } y = f(x) \text{ and } y = g(x) \text{ when } a \leq x \leq b$



(10) Sequences; limits of sequences; Limit Laws for Sequences; monotone sequences (increasing and decreasing); bounded sequences; **Monotone Sequence Theorem**.

(11) Additional useful limit theorems:

(a) Theorem: If $\lim_{x \rightarrow \infty} f(x) = L$ and $f(n) = a_n$, then $\lim_{n \rightarrow \infty} a_n = L$.

(b) Squeeze Theorem for Sequences: If $a_n \leq b_n \leq c_n$ for all $n \geq N_0$ with $a_n \rightarrow L$ and $c_n \rightarrow L$, then $b_n \rightarrow L$.

(c) Theorem: If $a_n \rightarrow L$ and f is continuous at L , then $f(a_n) \rightarrow f(L)$.

(12) Infinite series $\sum_{n=1}^{\infty} a_n$; n^{th} partial sum $s_n = \sum_{k=1}^n a_k$; the infinite series $\sum_{n=1}^{\infty} a_n$ **converges** to s if $s_n \rightarrow s$; the infinite series **diverges** if $\{s_n\}$ does not have a limit.

(13) Divergence Test for Series: If $\lim_{n \rightarrow \infty} a_n \neq 0$ or limit fails to exist $\Rightarrow \sum_{n=1}^{\infty} a_n$ DIVERGES.

(14) Special Infinite Series:

(a) Harmonic Series: $\sum_{n=1}^{\infty} \frac{1}{n}$. This series DIVERGES.

(b) Geometric Series: $\sum_{n=1}^{\infty} ar^{n-1}$ converges if $|r| < 1$; diverges if $|r| \geq 1$:

(i) $\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \dots = a(1 + r + r^2 + r^3 + \dots) = \frac{a}{1-r}$, if $|r| < 1$.

(ii) $\sum_{n=1}^{\infty} ar^{n-1}$ will DIVERGE if $|r| \geq 1$.