Fall 2012

MA 16200

Study Guide - Exam # 3

(1) Sequences; limits of sequences; Limit Laws for Sequences; Squeeze Theorem; monotone sequences; bounded sequences; Monotone Sequence Theorem.

(2) Infinite series $\sum_{n=1}^{\infty} a_n$; sequence of partial sums $s_n = \sum_{k=1}^{n} a_k$; the series $\sum_{n=1}^{\infty} a_n$ converges to s if its sequence of partials sums $s_n \to s$.

(3) **GEOMETRIC SERIES**:

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \cdots = a(1 + r + r^2 + r^3 + \cdots) = \frac{a}{1-r}, \text{ if } |r| < 1.$$

The Geometric Series diverges if $|r| \geq 1$.

(4) **p - SERIES**: $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges when $p > 1$; diverges when $p \leq 1$.

(5) **HARMONIC SERIES**: $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

(6) LIST OF CONVERGENCE TESTS FOR $\sum_{n=1}^{\infty} a_n$:

- [0] **Divergence Test**
- [1] **Integral Test**
- [2] **Comparison Test**
- [3] **Limit Comparison Test**
- [4] **Alternating Series Test**
- [5] **Ratio Test**
- [6] **Root Test**

(A useful inequality: $\ln x < x^{\alpha}$, for any fixed constant $\alpha \geq \frac{1}{2}$.)

(7) STRATEGY FOR CONVERGENCE/DIVERGENCE OF INFINITE SERIES :

Usually first look at the form of the series $\sum a_n$:

(i) If $\lim_{n\to\infty} a_n \neq 0$ or DNE, the use **Divergence Test**.

(ii) If series is a p-Series $\sum \frac{1}{n^p}$, use *p-Series* conclusions; if series is a Geometric series $\sum a\, r^{n-1}$, use *Geometric Series* conclusions.

(iii) If $\sum a_n$ looks like a p-Series or Geometric series, use **Comparison Test** or **Limit Comparison Test**.

(iv) If a_n involves factorials, use **Ratio Test**.

(v) If a_n involves n^{th} powers, use **Root Test**.

(vi) If $\sum a_n$ is an alternating series, use **Alternating Series Test** (or use **Ratio/Root Test** to show absolute convergence).

(vii) The last resort is usually to use the **Integral Test**.

Convergence Tests

0 **DIVERGENCE TEST**: If $\lim_{n\to\infty} a_n \neq 0$ or limit does not exist $\Longrightarrow \sum_{n=1}^{\infty} a_n$ diverges

1 **INTEGRAL TEST**: Suppose $f(x)$ is continuous, decreasing, and positive on $[1, \infty)$ and $f(n) = a_n$.

(i) If $\int_1^{\infty} f(x)\,dx$ converges $\Longrightarrow \sum_{n=1}^{\infty} a_n$ converges.

(ii) If $\int_1^{\infty} f(x)\,dx$ diverges $\Longrightarrow \sum_{n=1}^{\infty} a_n$ diverges.

(i.e., the infinite series $\sum_{n=1}^{\infty} a_n$ converges if and only if the improper integral $\int_1^{\infty} f(x)\,dx$ converges.)

2 **COMPARISON TEST**: Suppose $\sum a_n$ and $\sum b_n$ are series with $a_n, b_n > 0$.

(i) If $a_n \leq b_n$ for all n and $\sum b_n$ converges $\Longrightarrow \sum a_n$ converges.

(ii) If $a_n \geq b_n$ for all n and $\sum b_n$ diverges $\Longrightarrow \sum a_n$ diverges.

3 **LIMIT COMPARISON TEST**: Suppose $\sum a_n$ and $\sum b_n$ are series with $a_n,\ b_n > 0$.
If $\lim_{n\to\infty} \frac{a_n}{b_n} = c$, where $0 < c < \infty$, then either both series converge or both diverge.

4 **ALTERNATING SERIES TEST**: Given an alternating series $\sum_{n=1}^{\infty} (-1)^{n-1} b_n$.

$$\text{If } \begin{cases} b_n > 0 \\ \{b_n\} \text{ is a decreasing sequence} \\ \lim_{n\to\infty} b_n = 0 \end{cases}, \text{ then the alternating series } \sum_{n=1}^{\infty} (-1)^{n-1} b_n \text{ converges.}$$

5 **RATIO TEST**: Let $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L$.

(i) If $L < 1 \Longrightarrow \sum_{n=1}^{\infty} a_n$ converges absolutely, hence converges.

(ii) If $L > 1 \Longrightarrow \sum_{n=1}^{\infty} a_n$ diverges.

(If $L = 1$, then test is *inconclusive*, try another convergence test.)

6 **ROOT TEST**: Let $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L$.

(i) If $L < 1 \Longrightarrow \sum_{n=1}^{\infty} a_n$ converges absolutely, hence converges.

(ii) If $L > 1 \Longrightarrow \sum_{n=1}^{\infty} a_n$ diverges.

(If $L = 1$, then test is *inconclusive*, try another convergence test.)

(8) Alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}b_n$; *Estimating Alternating Series* : If $b_n > 0$, $\{b_n\}$ is decreasing, and $b_n \to 0$ and if $s = \sum_{n=1}^{\infty}(-1)^{n-1}b_n$, then $|s - s_n| \leq b_{n+1}$

(9) Absolute convergence of $\sum a_n$ (i.e. $\sum |a_n|$ converges). Conditional convergence (i.e.,$\sum a_n$ converges, but $\sum |a_n|$ diverges);
NOTE: Absolute Convergence of $\sum a_n \Longrightarrow$ Convergence of $\sum a_n$
(i.e., If $\sum |a_n|$ converges, then $\sum a_n$ converges).

(10) Power series about a: $\sum_{n=0}^{\infty} c_n(x-a)^n$; **Radius of Convergence (ROC)** and **Interval of Convergence (IOC)**; usually use Ratio Test (or Root Test) to determine **ROC** and **IOC**.

IMPORTANT: To find complete IOC, don't forget to check the endpoints of the interval !

(11) Operations on power series: multiplication, differentiation, integration of power series.

(12) **Taylor Series about *a***: $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$.

(13) **Maclaurin Series**: $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n$ (i.e., Taylor Series about 0).

(14) ***n*^th^- degree Taylor polynomial**: $T_n = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k \approx f(x)$, near $x = a$.

(15) **Binomial Series**: $(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n$, where $\binom{k}{n} = \frac{k!}{n!(k-n)!}$.

(16) **Useful Maclaurin Series**

(a) $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$, valid for $-1 < x < 1$

(b) $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$, valid for $-\infty < x < \infty$

(c) $\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$, valid for $-\infty < x < \infty$

(d) $\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$, valid for $-\infty < x < \infty$

(e) $\tan^{-1} x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$, valid for $-1 < x < 1$

(f) $(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots$, for $-1 < x < 1$

(17) **Parametric Curves**. $C : \begin{cases} x = x(t) \\ y = y(t) \end{cases}$; sketching parametric curves; tangents to curves;

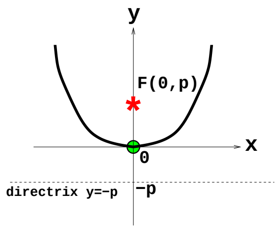
$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \quad \text{and} \quad \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}, \qquad \text{provided} \quad \frac{dx}{dt} \neq 0$$

Arc length $L = \int_\alpha^\beta ds$, where $ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$; Surface Area $S = \int_\alpha^\beta 2\pi r\, ds$.

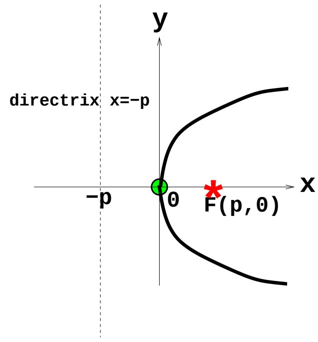
(18) **Polar Coordinates**. Graphs of polar curves $r = f(\theta)$; symmetry; tangents to polar curves; polar curves $r = f(\theta)$ written parametrically: $C : \begin{cases} x = f(\theta)\cos\theta \\ y = f(\theta)\sin\theta \end{cases}$.

(19) **Conic Sections**.

(a) Parabolas: *vertex* $= (0,0)$

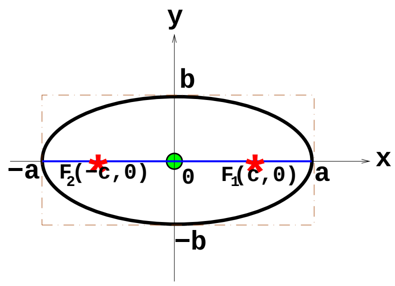


$$\boldsymbol{x^2 = 4py}$$

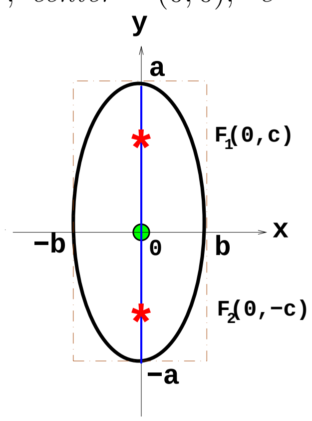


$$\boldsymbol{y^2 = 4px}$$

(b) Ellipses: $a \geq b > 0$; *major axis* $= 2a$; *center* $= (0,0)$; $c = \sqrt{a^2 - b^2}$

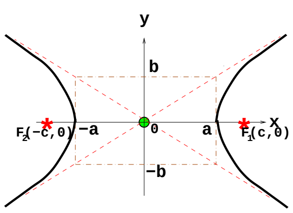


$$\boldsymbol{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1}$$



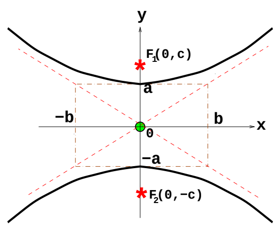
$$\boldsymbol{\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1}$$

(c) Hyperbolas: $a > 0$ and $b > 0$; *center* $= (0,0)$; $c = \sqrt{a^2 + b^2}$



asymptotes : $y = \pm \frac{b}{a} x$

$$\boldsymbol{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}$$



asymptotes : $y = \pm \frac{a}{b} x$

$$\boldsymbol{\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1}$$

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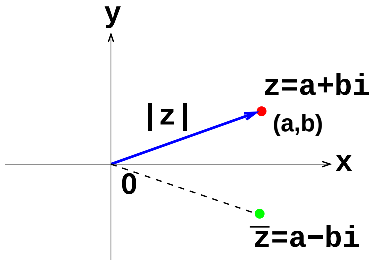
(20) Review of Complex Numbers $\mathbb{C}$

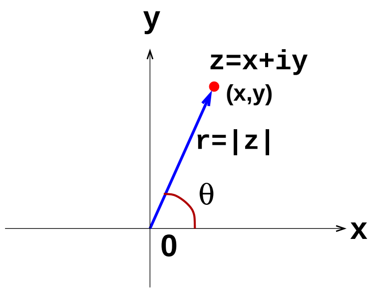
(a) Complex number $z = a + bi$, where $a, b \in \mathbb{R}$ and $i = \sqrt{-1}$ (hence $i^2 = -1$). The real number $a = \Re e\{z\}$ (the *real part* of z), and the real number $b = \Im m\{z\}$ (the *imaginary part* of z); $|z| = \sqrt{a^2 + b^2}$ is the *modulus* of z; the *conjugate* of $z = a + bi$ is the complex number $\overline{z} = a - bi$:



(b) Addition and multiplication of complex numbers.

(c) Important Identity: $\boxed{|z|^2 = z\,\overline{z}}$; thus $\dfrac{1}{z} = \dfrac{\overline{z}}{|z|^2}$.

(d) Polar Form of $z = x + iy$:



Hence $z = \underbrace{x + iy}_{\text{Rectangular Form}} = \underbrace{(r\cos\theta) + i\,(r\sin\theta)}_{\text{Polar Form}}$

$r = |z| = \sqrt{x^2 + y^2}$ modulus; θ = an argument of z, where $\tan\theta = \dfrac{y}{x}$.

Hence if $\begin{cases} z_1 = r_1\,(\cos\theta_1 + i\sin\theta_1) \\ z_2 = r_2\,(\cos\theta_2 + i\sin\theta_2) \end{cases}$, then $z_1 z_2 = r_1 r_2\,(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2))$

(e) Recall *Euler's Formula* : $\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$

(f) Exponential Form of $z = x + iy$:

Hence $z = \underbrace{x + iy}_{\text{Rectangular Form}} = \underbrace{(r\cos\theta) + i\,(r\sin\theta)}_{\text{Polar Form}} = \underbrace{re^{i\theta}}_{\text{Exponential Form}}$

(g) **deMoivre's Law**: If n is an integer (positive or negative), then $\left(re^{i\theta}\right)^n = r^n e^{in\theta}$ i.e., $\{r(\cos\theta + i\sin\theta)\}^n = r^n\,(\cos n\theta + i\sin n\theta)$