MA 351 first midterm review problems
Version as of February 15th.

The first midterm will be in class on Monday, February 17th. No notes, books, or electronic devices will be allowed. Most of the exam will be closely based on problems, or on parts of problems, from the list below. Justify your answers. Please let me know if you have a question or find a mistake.

1. Solve each of the following systems. If there is a unique solution, find it. If there is no solution, explain why. If there are infinitely many solutions, parametrize them.

   (a)
   $$2x - 4y = 3$$
   $$3x + 2y = 1$$

   (b)
   $$x + 2y - 4z = 6$$
   $$2x + 3y - 6z = 9$$
   $$x + 3y - 6z = 9$$

   (c)
   $$2x + 3y + 4z = 5$$
   $$4x + 6y + 8z = 9$$

2. Consider the filter
   $$y_1 = 3x_1 + 2x_2 - 2x_3 + 3x_4$$
   $$y_2 = 4x_1 + 3x_2 - 2x_3 + 2x_4$$
   $$y_3 = 2x_1 + x_2 + \alpha x_3 + 4x_4,$$
   where α is a real parameter.

   (a) For which values of α is every column $\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ in the range of the filter?

   (b) Let α be a value such that the range of the filter is a plane. Give an equation for the plane.

3. Find all 2x2 matrices M with nonnegative integer entries such that $M^2 = M$.

4. Let $A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

   (a) Solve $MA = B$ for M.

   (b) Solve $AMA = B$ for M.

5. Plot the solutions to the following equations in the (x, y) plane:

   (a) $\begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

(b) $\begin{bmatrix} 2 & -4 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & -4 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

6. Let $M = \begin{bmatrix} -1 & 3 \\ 2 & 4 \end{bmatrix}$ and let $V = \begin{bmatrix} 1 & b \\ 2 & 1 \end{bmatrix}$, where b is a real parameter. For which values of b is $V^{-1}MV$ diagonal?

7. Find a, b, c, d, such that the linear filter

$$y_1 = ax_1 + bx_2,$$

$$y_2 = cx_1 + dx_2,$$

scales by 2 in the $(1, 2)$ direction and by -2 in the $(-1, 2)$ direction.

8. Let $M = \begin{bmatrix} -4 & 1 \\ 9 & -4 \end{bmatrix}$. Find a matrix V such that $V^{-1}MV$ is diagonal.

9. Find the eigenvalues of $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, and find an eigenvector for each eigenvalue.

Here are short answers; a proper solution includes some clear steps leading to these answers.

1. (a) $x = 5/8$ and $y = -7/16$, which could also be written as $(5/8, -7/16)$ or $\begin{bmatrix} 5/8 \\ -7/16 \end{bmatrix}$.

   (b) There are infinitely many solutions, which could be written $x = 0$ and $z = (y-3)/2$ where y ranges over all real numbers, or $(0, y, (y-3)/2)$ where y ranges over all real numbers, or $(0, y, (y-3)/2) = (0, 0, -3/2) + y(0, 1, 1/2)$ where y ranges over all real numbers or $\begin{bmatrix} 0 \\ y \\ (y-3)/2 \end{bmatrix}$ where y ranges over all real numbers, or $\begin{bmatrix} 0 \\ 0 \\ -3/2 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ 1/2 \end{bmatrix}$ where y ranges over all real numbers, or other similar ways.

   (c) There are no solutions, because taking twice the first equation minus the second yields a bad equation.

2. (a) For all $\alpha \neq -2$.

   (b) If $\alpha = -2$ then the range is $2y_1 - y_2 - y_3 = 0$.

3. The possiblities are $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & n \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ n & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ n & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & n \\ 0 & 1 \end{bmatrix}$ where n is any nonnegative integer.

4. (a) $M = BA^{-1} = \begin{bmatrix} 2 & -1 \\ -5/2 & 3/2 \end{bmatrix}$

   (b) $M = A^{-1}BA^{-1} = \begin{bmatrix} -35/4 & 19/4 \\ 13/2 & -7/2 \end{bmatrix}$.

5. (a) Plot the point $(1, -1)$.

   (b) Plot the line $y = (x-3)/2$.

6. $b = -3$.

7. The matrix of the filter is $\begin{bmatrix} 1 & -1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 \\ 4 & 0 \end{bmatrix}$, so $a = d = 0$, $b = 1$, $c = 4$.

8. One possibility is $V = \begin{bmatrix} 1 & 1 \\ 3 & -3 \end{bmatrix}$.

9. The eigenvalues are 0 and 2, and some corresponding eigenvectors are $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.