Sequences

Let $\vec{w}$ be a 2x1 vector, and A a 2x2 diagonalizable matrix. Let $(\vec{v}_1, \vec{v}_2)$ be a basis of eigenvectors of A, and (λ_1, λ_2) the corresponding eigenvectors. If

$$\vec{w} = c_1\vec{v}_1 + c_2\vec{v}_2,$$

then

$$A^n\vec{w} = c_1\lambda_1^n\vec{v}_1 + c_2\lambda_2^n\vec{v}_2.$$

In other words there are constants a, b, c, d such that

$$A^n\vec{w} = \begin{bmatrix} a\lambda_1^n + b\lambda_2^n \\ c\lambda_1^n + d\lambda_2^n. \end{bmatrix} \tag{1}$$

We can use this to solve some recurrence problems for sequences by calculating just the eigenvalues and not the eigenvectors.

Example. Find $A^{1000}\vec{w}$, where

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \qquad \vec{w} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}. \tag{2}$$

The eigenvalues are obtained by solving $(1-\lambda)(2-\lambda) - 6 = 0$, which gives $\lambda_{1,2} = -1, 4$. To find a, b, c, d, plug in equation (2) into equation (1) with $n = 0$ to get

$$a + b = 3, \quad c + d = 1,$$

and plug in equation (2) into equation (1) with $n = 1$ to get

$$-a + 4b = 5, \quad -c + 4d = 11.$$

That gives

$$a = 7/5, \quad b = 8/5, \quad c = -7/5, \quad d = 12/5.$$

Then

$$A^{1000}\vec{w} = \begin{bmatrix} (7/5)(-1)^{1000} + (8/5)4^{1000} \\ (-7/5)(-1)^{1000} + (12/5)4^{1000} \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 7 + 8 \cdot 4^{1000} \\ -7 + 12 \cdot 4^{1000} \end{bmatrix}.$$

Exercise.

Find $A^{123}\vec{w}$, where

$$A = \begin{bmatrix} -4 & 2 \\ 3 & -3 \end{bmatrix} \qquad \vec{w} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Simplify your answer similarly to the example.

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