MA 366 midterm review problems
Hopefully final version as of March 21st.

- The midterm will be on Wednesday, March 27th, from 8 to 9 pm in ME 1130.
- It will cover all the material from the classes and homework up to that date.
- No notes or books or electronic devices are allowed, but the reference page at the back of this document will be included on the exam.
- Most of the problems on the exam will be closely based on ones from the final version of the list below (but of course the actual exam will be much shorter).
- For each problem, you must justify your answers.
- Please let me know if you have a question or find a mistake, and note that these are not arranged in order of difficulty!
- If you want more insight into phase portraits, the website `http://mathlets.org/mathlets/linear-phase-portraits-matrix-entry/` gives beautiful pictures for matrices of the form
$$\begin{bmatrix} 0 & 1 \\ -q & p \end{bmatrix}$$
and for more pictures see also
`http://mathlets.org/mathlets/linear-phase-portraits-cursor-entry/`

1. Solve the initial value problem
$$y'' - 6y' + 8y = 0, \qquad y(0) = 1,\ y'(0) = a,$$
where a is a real number. What is $\lim_{t\to\infty} y(t)$? How does the answer depend on a?

2. Find the general solution to
$$2y'' + 6y' + 10y = 0.$$

3. Solve the initial value problem
$$y'' + 9y = 5t + 1, \qquad y(0) = 0,\ y'(0) = 0.$$

4. A critically damped spring-mass system is governed by
$$mu'' + \gamma u' + ku = 0, \qquad u(0) = a,\ u'(0) = b,$$
where $m,\ \gamma,\ k$, and a are given positive constants. For which values of b do we have $u(t) > 0$ for all $t > 0$?

5. Find the general solution to the differential equation
$$y^{(4)} - 8y' = 0.$$

6. Let a be a real number. For which values of a does the equation

$$y^{(8)} + y^{(6)} + y^{(4)} + y'' + ay = 0$$

have a real-valued solution which is never zero?

7. Find the general solution to the differential equation.

$$y^{(4)} - y = e^t.$$

8. Find the general solution to the system

$$\begin{aligned} x_1' &= 7x_1 - 4x_2, \\ x_2' &= 8x_1 - 5x_2, \end{aligned}$$

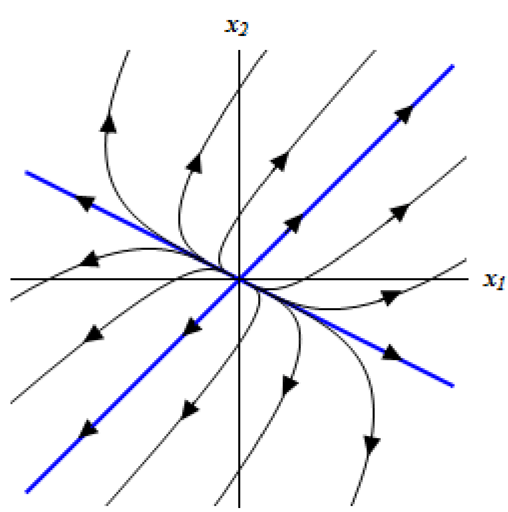
and sketch a phase portrait. Label all equilibrium and straight line solutions. If you know that $x_1(0) = 1$ and $x_2(0) = 3$, find $x_1(1)$ and $x_2(1)$.

9. Find the general solution to the system

$$x'(t) = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix} x(t),$$

and sketch a phase portrait. Find the solution satisfying $x(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. What are the nearest and farthest points to the origin that this solution passes through?

10. Suppose the system $x'(t) = Ax(t)$ has the following phase portrait



where the blue lines are given by $x_1 = x_2$ and $x_1 = -2x_2$, and the matrix A has eigenvalues 1 and 2. Find A.

11. Let α be a real number. For each of the systems below, and for each value of α decide if the origin is a stable node, unstable node, stable improper node, unstable improper node, saddle point, center, stable spiral, unstable spiral, or none of the above.

   (a)
   $$x'(t) = \begin{bmatrix} 1 & \alpha \\ 1 & 1 \end{bmatrix} x(t),$$

   (b)
   $$x'(t) = \begin{bmatrix} \alpha & 1-\alpha \\ 0 & \alpha^{-1} \end{bmatrix} x(t).$$

12. Find the general solution to the system

$$\begin{aligned} x_1' &= x_1 - 4x_2 + 2, \\ x_2' &= x_1 - 3x_2 + 1, \end{aligned}$$

and sketch a phase portrait. Label all equilibrium and straight line solutions.

13. A spring-mass system is governed by

$$u'' + \gamma u' + u = 0, \qquad u(0) = 0, \quad u'(0) = 1,$$

where $\gamma \geq 0$ is a given constant. For which values of γ do we have $u(5\pi) = 0$?

Hint: Consider first the undamped case, then underdamped, then critically damped and overdamped. It is not necessary to completely find u in all cases.

14. A spring-mass system is governed by

$$u'' + u = \cos(\omega t), \qquad u(0) = u'(0) = 0,$$

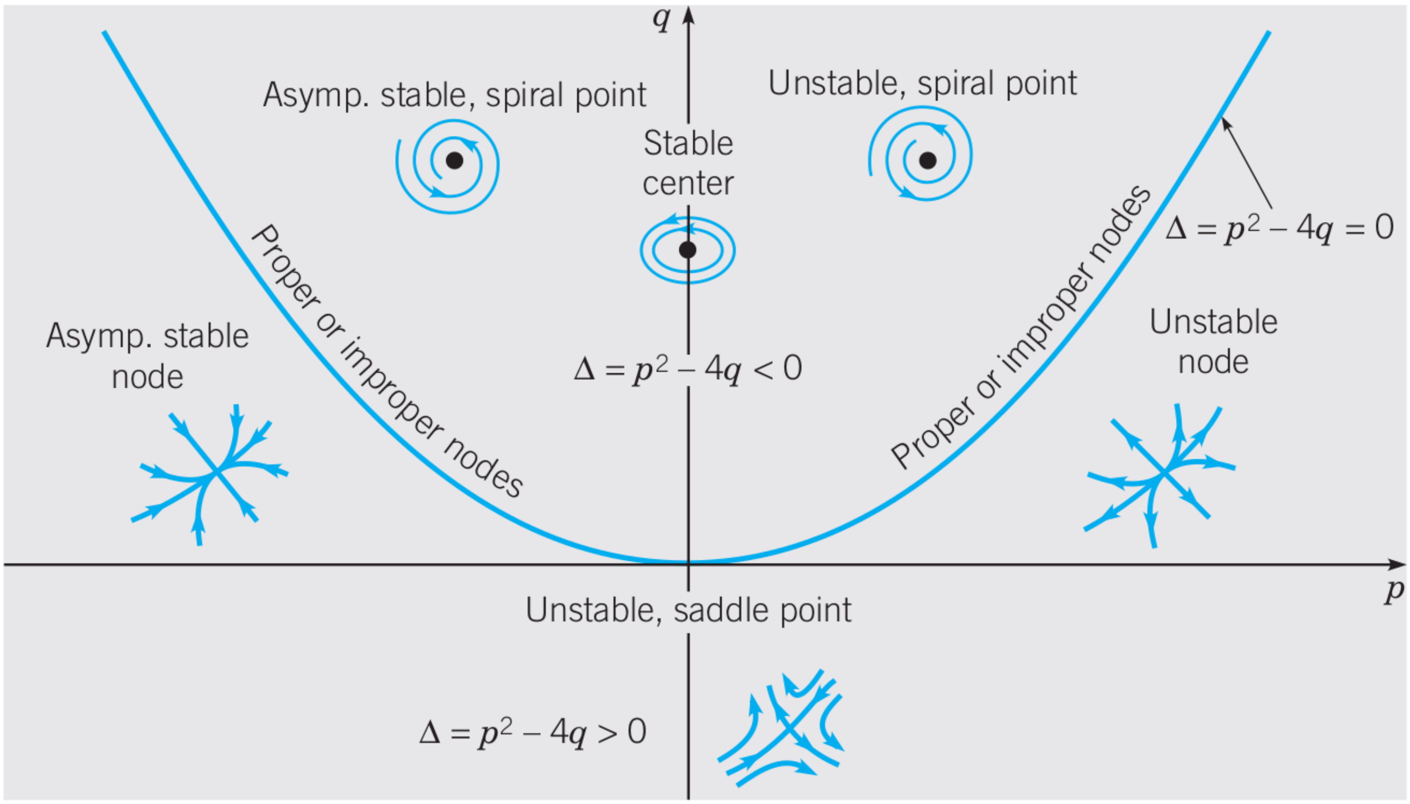
where $\omega \geq 2$ is a given constant.

   (a) Find $u(t)$.

   (b) Show that $|u(t)| \leq 1$ for all t.

   (c) Can you find a constant $A < 1$ such that $|u(t)| \leq A$ for all t? How small an A can you find?

Reference page



Here p = trace $= a + d = \lambda_1 + \lambda_2$ and q = determinant $= ad - bc = \lambda_1\lambda_2$.

If A is defective, then $x' = Ax$ is solved by

$$x(t) = e^{\lambda t}(C_1 v + C_2(tv + w)),$$

where $(A - \lambda)v = 0$ and $(A - \lambda)w = v$.