MA 527 final review problems

Hopefully final version as of December 2nd

- The final will be on Monday, December 11th, from 7 to 9 pm, in MATH 175.
- It will cover all the material we have done from the whole semester, minus sections 4.6 and 12.12, with material from all parts of the semester represented approximately equally.
- Most of the problems on the exam will be closely based on ones from the list below, and from the Midterm 1 and Midterm 2 exams and review problems (but the actual exam will be much shorter).
- For each problem, you must explain your reasoning.
- The reference pages at the end will also be provided on the actual exam.
- Note that these are not arranged in order of difficulty!

1. In class we derived the formulas

$$u(x,y) = \sum_{n=1}^{\infty} C_n \left(e^{ny} - e^{-ny}\right) \sin nx$$

for the solution to

$$\partial_x^2 u(x,y) + \partial_y^2 u(x,y) = 0,$$

with boundary conditions

$$u(x,0) = u(0,y) = u(\pi,y) = 0, \qquad u(x,\pi) = f(x),$$

where

$$C_n = \frac{2}{\pi\left(e^{n\pi} - e^{-n\pi}\right)} \int_0^\pi f(x) \sin nx dx.$$

Derive a corresponding formula for the solution to

$$\partial_x^2 u(x,y) + \partial_y^2 u(x,y) = 0,$$

with boundary conditions

$$\partial_y u(x,0) = \partial_x u(0,y) = u(x,\pi) = 0, \qquad u(\pi,y) = f(x),$$

2. Use the method of separation of variables for the partial differential equation

$$\partial_x^2 u(x,y,z) + \partial_y^2 u(x,y,z) + \partial_z^2 u(x,y,z) = 0$$

to derive three ordinary differential equations, involving two constants of separation. You do not need to solve the resulting ordinary differential equations.

3. (a) Let $a > 0$ be given. Find the Fourier transform of

$$f(x) = \begin{cases} 1 - a|x|, & |x| \leq 1/a, \\ 0, & \text{otherwise.} \end{cases}$$

(b) Let $b > 0$ be given. Use your answer to part (a) to find the Fourier transform of

$$g(x) = \frac{1 - \cos(bx)}{x^2}.$$

4. In class we derived the formula

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(w) e^{-w^2 t} e^{iwx} dw,$$

for the solution to

$$\partial_t u(x,t) = \partial_x^2 u(x,t), \qquad u(x,0) = f(x).$$

Derive the corresponding formula for the solution to

$$\partial_t u(x,t) = \partial_x^6 u(x,t), \qquad u(x,0) = f(x).$$

Reference pages

- The **span** of a set of vectors is the set of all possible linear combinations of those vectors.
- A set of vectors is **linearly dependent** if one of the vectors is a linear combination of the others; otherwise it is **linearly independent**.
- A **basis** of a vector space is a linearly independent set of vectors which span the space.
- The **dimension** of the space is the number of elements in a basis.
- The **row space** of a matrix is the span of the rows; **column space** is defined similarly.
- The **rank** of the matrix is the dimension of the row or column space.
- The **nullspace** of a matrix A is the space of solutions to $Ax = 0$.
- The column space is also the space of b such that $Ax = b$ has a solution.
- The **nullity** of a matrix is the dimension of the nullspace.
- The rank and nullity of a matrix add up to the number of columns.
- If $Ax = b$ has a solution, then the rank of A is the number of bound variables and the nullity is the number of free variables.
- If $Ax = b$ has a solution x_0, then the general solution is $x_0 + x_h$, where x_h is the general solution to $Ax = 0$.
- Multiplying a row of a matrix by a constant multiplies the det. by the same constant.
- Switching two rows of a matrix multiplies the determinant by -1.
- $(AB)^T = B^T A^T$ and $(AB)^{-1} = B^{-1}A^{-1}$.
- A square matrix is **invertible** if and only if the determinant is nonzero.
- $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$ and $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = (ad-bc)^{-1} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.
- $\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = aei + bfg + cdh - ceg - bdi - afh.$
- If $A = PDP^{-1}$ with D diagonal then D consists of the eigenvalues of A and the columns of P are the corresponding eigenvectors. If A is symmetric then P can be chosen orthogonal, which means $P^{-1} = P^T$ and all the columns of P are unit length and mutually orthogonal.
- The general solution to $y' = Ay$, where A is a diagonalizable 2x2 matrix and y a 2x1 vector is $y = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2$, where v_1 and v_2 are independent eigenvectors of A and λ_1 and λ_2 are the corresponding eigenvalues.
- The origin is an equilibrium point of $y' = Ay$. If the eigenvalues all have nonpositive real part, then it is stable, and if the real part is negative then it is attractive. If the

eigenvalues are real and opposite signs it is a saddle, if they are real and same sign it is a node (improper if different, proper if the same and there are two eigenvectors). If the eigenvalues are complex it is a spiral, unless the real part is zero in which case it is a center. These properties all carry over to nonlinear systems, except that a center can become a spiral and a proper node can also become a spiral of the same stability/attractiveness.

- The Laplace transform is defined by $\mathcal{L}[f(t)](s) = F(s) = \int_0^\infty e^{-ts} f(t)dt$.

$f(t)$	$F(s)$		$f(t)$	$F(s)$
t^n	$n!s^{-n-1}$		$e^{at}f(t)$	$F(s-a)$
y'	$sY(s) - y(0)$		$u(t-a)f(t-a)$	$e^{-as}F(s)$
y''	$s^2Y(s) - sy(0) - y'(0)$		$\delta(t-a)$	e^{-as}
$\cos bt$	$s/(s^2+b^2)$		$f * g(t) = \int_0^t f(\tau)g(t-\tau)d\tau$	$F(s)G(s)$
$\sin bt$	$b/(s^2+b^2)$		$tf(t)$	$-F'(s)$

- If f has period $2L$, then the Fourier series for f is $a_0 + \sum_{n=1}^\infty a_n \cos(n\pi x/L) + b_n \sin(n\pi x/L)$, with $a_0 = (2L)^{-1} \int_{-L}^L f(x)dx$, $a_n = L^{-1} \int_{-L}^L f(x)\cos(n\pi x/L)dx$, $b_n = L^{-1} \int_{-L}^L f(x)\sin(n\pi x/L)dx$.
- If $g(x) = a_0 + \sum_{n=1}^N a_n \cos(n\pi x/L) + b_n \sin(n\pi x/L)$, then $E = \int_{-L}^L (f(x) - g(x))^2 dx = \int_{-L}^L f(x)^2 dx - 2La_0^2 - L\sum_1^N a_n^2 + b_n^2$.
- The Sturm–Liouville problem

$$(py')' + qy + \lambda ry = 0$$

with boundary conditions $k_1 y(a) + k_2 y'(a) = 0$ and $l_1 y(b) + l_2 y'(b) = 0$ has eigenfunctions orthogonal with respect to the inner product $(f, g) = \int_a^b f(x)g(x)r(x)dx$.

- The differential equation $y'' + ay' + by = 0$ has the general solution
  1. $y(x) = Ae^{r_1 x} + Be^{r_2 x}$ if r_1 and r_2 are the distinct roots of $r^2 + ar + b = 0$,
  2. $y(x) = Ae^{r_0 x} + Bxe^{r_0 x}$ if r_0 is the unique repeated root of $r^2 + ar + b = 0$,
  3. $y(x) = Ae^{cx}\cos(\omega x) + Be^{cx}\sin(\omega x)$ if $c \pm i\omega$ are the distinct roots of $r^2 + ar + b = 0$.
- $\sin 0 = 0$, $\sin \pi/2 = 1$, $\sin \pi = 0$, $\sin 3\pi/2 = -1$, $\sin 2\pi = 0$, etc.
- $\cos 0 = 1$, $\cos \pi/2 = 0$, $\cos \pi = -1$, $\cos 3\pi/2 = 0$, $\cos 2\pi = 1$, etc.
- The Fourier transform and its inverse are $\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty f(x)e^{-iwx}dx$, and $f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty \hat{f}(w)e^{iwx}dw$.
- $\widehat{\frac{d}{dx}f}(w) = iw\hat{f}(w)$ and $\frac{d}{dw}\hat{f} = \widehat{-ixf}$.