Simplicity of A_n $(n \geq 5)$.

Lemma 1. *If $n \geq 3$ then any proper normal subgroup of the alternating group A_n has index divisible by 3.*

Proof. Let $p = |A_n/H|$. Then every p-th power in A_n lies in H. So if p is not divisible by 3 then every 3-cycle lies in H, whence $H = A_n$. (Any 3-cycle c satisfies $c = c^{3n+1} = (c^{-1})^{3n+2}$, for any integer n.)

Lemma 2. *A_5 is simple.*

Proof. By Lemma 1, any proper $H \triangleleft A_5$ has order dividing 20. So H cannot contain any order-3 element, i.e., 3-cycle; and also H cannot contain any 5-cycle, since any such has 6 conjugates, and 6 doesn't divide 20. The only remaining nontrivial even permutations are the 15 products $(ab)(cd)$ of two disjoint 2-cycles, any two of which are conjugate in S_5, hence in A_5 (since 15 is odd); and since 15 doesn't divide 20, these can't be in H either. Thus $|H| = 1$.

Theorem. *A_n is simple for $n \geq 5$.*

Proof. Proceed by induction, the case $n = 5$ being given above. So suppose $n > 5$, A_{n-1} is simple, and $H \triangleleft A_n$. Then $H \cap A_{n-1}$, being normal in A_{n-1}, is either A_{n-1} or trivial.

In the former case, any 3-cycle, being conjugate to one in A_{n-1}, lies in H, making $H = A_n$. The same holds if H contains any conjugate of A_{n-1}.

The remaining possibility is that H intersects any conjugate of A_{n-1} trivially, i.e., no nonidentity permutation $h \in H$ has a fixed point. Writing h and its powers (none of which have fixed points) as products of cycles, one sees then that h is a product of p q-cycles for some p and q such that $pq = n$. Any element in the centralizer C_h of h produces a permutation of these cycles, and thus there is a surjective homomorphism $C_h \twoheadrightarrow S_p$. The kernel consists of all elements that are products of powers of these cycles, and so has cardinality q^p. Thus $|C_h| = (q^p)(p!)$, whence the number of S_n-conjugates of h is $n!/(q^p)(p!)$. The number of A_n-conjugates is at least half of that, and—since all such conjugates lie in H—must be less than

$$|H| = |H|/|H \cap A_{n-1}| < |A_n|/|A_{n-1}| = n.$$

Now $n!/(q^p)(p!)$ is the product of all numbers $< n$ which are not multiples of q, which product is at least $(n-1)(n-2)$ if $q > 2$, or $(n-1)(n-3)$ if $q = 2$. In either case, since $(n-1)(n-2)/2 > (n-1)(n-3)/2 > n$ when $n > 5$, we must have $q = 1$. Thus $|H| = 1$. $\square$

For another proof, see Clark, §**83**.