A SPACE SYMMETRY IS DETERMINED BY THE IMAGES OF FOUR NON-COPLANAR POINTS

A *symmetry* of a subset M of euclidean space is a bijective map $S\colon M \to M$ such that for all points $A, B \in M$, the distance from $S(A)$ to $S(B)$ is the same as the distance from A to B. These symmetries form a subgroup of $\mathrm{Perm}(M)$.

The title means that *if A, B, C, D are four points in 3-space, not all lying in a single plane, and if S and T are two 3-space symmetries such that*

$$S(A) = T(A), \quad S(B) = T(B), \quad S(C) = T(C), \quad \text{and } S(D) = T(D),$$

then $S = T$.

Saying that $S = T$ is the same as saying that the symmetry $U = S^{-1}T$ is the identity. So our problem is to show that $U(X) = X$ for every point X.

Suppose there is some point X such that $X' = U(X)$ is different from X. That leads to a contradiction (thereby proving what we want), as follows.

Note that

$$U(A) = S^{-1}T(A) = S^{-1}S(A) = A,$$

and similarly $U(B) = B$, $U(C) = C$, and $U(D) = D$. Since U is distance preserving, the distance from A to X is the same as the distance from $U(A)$ to $U(X)$, i.e., from A to X'; therefore A, being equidistant from X and X', must lie in the perpendicular bisector of the line segment $\overline{XX'}$, i.e., in the plane passing through the midpoint of $\overline{XX'}$ and perpendicular to $\overline{XX'}$. Similarly B, C, and D lie in that plane, contradicting the assumption that A, B, C, and D are not coplanar. □

Practice exercises (in increasing order of difficulty).

• Test your understanding of the preceding argument by making up a similar one to prove the simpler fact that a plane symmetry is determined by the images of three non-collinear points.

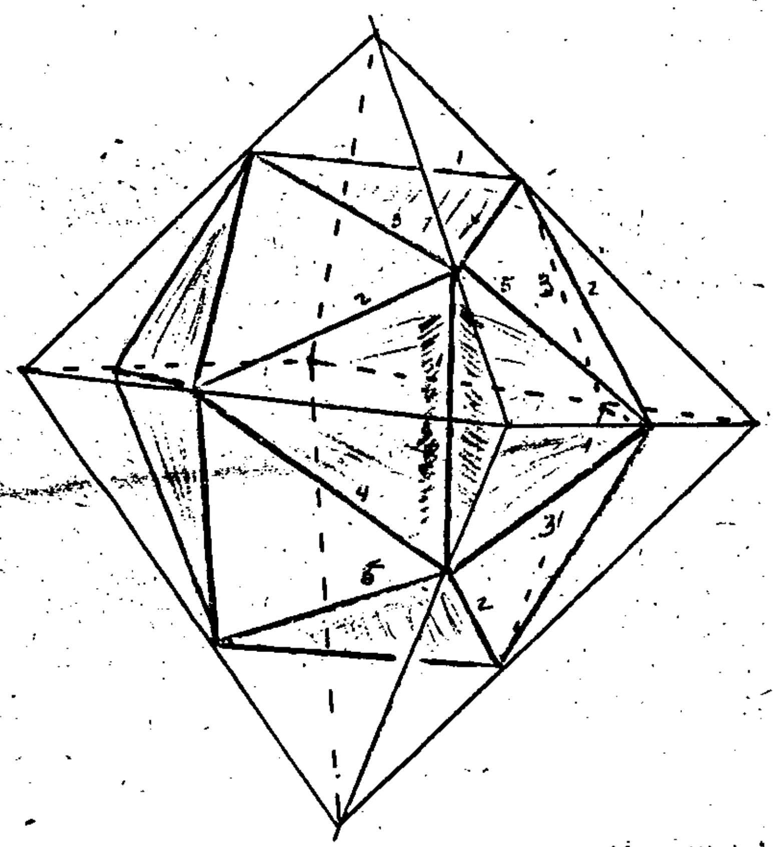
• Show that any space symmetry leaving the origin fixed is a linear transformation given by an orthogonal matrix.

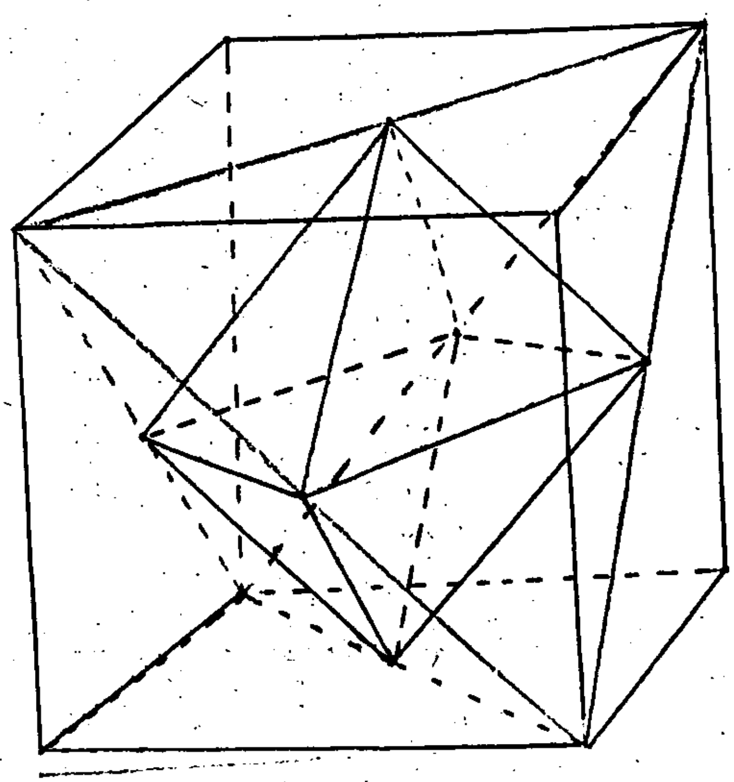
• Prove that if A, B, C are three non-collinear points in space and S and T are 3-space symmetries such that $S(A) = T(A)$, $S(B) = T(B)$, and $S(C) = T(C)$, then *either* $S = T$ *or* $S = RT$ where R is reflection in the (unique) plane through $T(A)$, $T(B)$, and $T(C)$.

(Thus a space symmetry is *almost* determined by the images of three non-collinear points.)

• An octahedron is two pyramids with their bases pasted together. Show that the group of symmetries of an octahedron is the same as the group of symmetries of a cube. (One way to do this is to think about the second picture below—note how any symmetry of the cube induces one of the inscribed octahedron. While you're at it, which symmetries of the cube produce symmetries of the inscribed tetrahedron?)

Octahedron (one of 5) about an icosahedron





Octahedron < Tetraheadron < Cube.