ONE-PARAMETER CONTINUOUS FIELDS OF KIRCHBERG ALGEBRAS. II

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ABSTRACT. Parallel to the first two authors' earlier classification of separable unital one-parameter continuous fields of Kirchberg algebras with torsion free K-groups supported in one dimension, one-parameter separable unital continuous fields of AF-algebras are classified by their ordered K_0-sheaves. We prove Effros-Handelman-Shen type theorems for separable unital one-parameter continuous fields of AF-algebras and Kirchberg algebras.

CONTENTS

1. INTRODUCTION

One-parameter separable unital continuous fields of Kirchberg algebras with torsion free K_i-groups and trivial K_{i+1}-groups ($i \in \{0, 1\}$ fixed) were classified in [2] by their K_i-sheaves. Using the semi-projectivity of the Kirchberg algebras with finitely generated torsion free K_0-groups and trivial K_1-groups, these continuous fields were shown by the authors to

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be inductive limits of fields with finitely many singular points (the so-called elementary fields). Using the classification results of Kirchberg and Phillips for Kirchberg algebras, a uniqueness theorem and an existence theorem for elementary fields were proved, and hence a classification theorem for inductive limit continuous fields was obtained.

In this note, we shall show that we can use a similar procedure to get a classification of separable unital one-parameter continuous fields of AF-algebras. More precisely, noting that finite dimensional C*-algebras are semi-projective, we apply the methods of [2] to represent any separable unital continuous field of AF-algebras as an inductive limit of continuous fields of finite dimensional C*-algebras with finitely many singularities. Very much parallel to [2], we obtain a classification of one-parameter separable unital continuous fields of AF-algebras by their ordered K_0-sheaves pointed by the class of the unit.

The K_0-presheaves of continuous fields of C*-algebras over $[0,1]$ are always continuous in the sense that, if $\mathcal{S}$ is a such presheaf, then for any closed subinterval $[a,b]$ and any decreasing sequence of closed intervals $([a_i,b_i])_{i=1}^{i=\infty}$ with $\bigcap_{i=1}^{i=\infty}[a_i,b_i]=[a,b]$, the canonical map from the inductive limit $\varinjlim \mathcal{S}[a_i,b_i]$ to $\mathcal{S}[a,b]$ is an isomorphism. Moreover, the stalk $\mathcal{S}_x$ is isomorphic to the K_0-group of the fibre algebra at x. If A is a unital continuous field of stably finite C*-algebras over $[0,1]$, then the projection map from A onto any fibre is strictly positive at the level of the ordered K_0-groups.

The continuity condition is a key ingredient for Effros-Handelman-Shen type theorems for both one-parameter continuous fields of AF-algebras and one-parameter continuous fields of Kirchberg algebras with trivial K_1-groups (see Theorem 5.7 and Theorem 5.8). In the AF case, the strict positivity condition is equally important. More precisely, if a sheaf $\mathcal{S}$ of pointed ordered groups on $[0,1]$ satisfies the continuity condition and if it is strictly positive and any stalk is a dimension group, then there is a separable unital continuous field of AF-algebras the K_0-sheaf of which is isomorphic to $\mathcal{S}$. The statement of the theorem for continuous fields of Kirchberg algebras is similar, except that positivity plays no role. In the proof of these theorems, we use the continuity condition (and the strict positivity condition in the AF case) to decompose the given sheaf into an inductive limit of sheaves with finitely many singular points. Any morphism of such (elementary) sheaves lifts to a morphism of elementary continuous fields. Thus, the inductive limit of these elementary fields of C*-algebras has the given sheaf as its K_0-sheaf.

In the last part of the paper, we study the K_0-sheaves of separable one-parameter continuous fields whose fibres are unital hereditary sub-C*-algebras of $\mathcal{O}_\infty$. These sheaves can be viewed as sheaves of integer valued functions satisfying certain properties (see Corollary 6.4). The set of their zero points is studied in detail.

2. Continuous fields of C*-algebras and the invariant

Definition 2.1 ([3])**.** Let $\mathcal{C}$ be a class of C*-algebras and let T be a locally compact topological space. A *continuous field* A of C*-algebras over T with fibres in $\mathcal{C}$ is a family $(A(t))_{t\in T}$ of C*-algebras in $\mathcal{C}$, together with a set $\Gamma \subset \prod_{t\in T} A(t)$ of vector fields such that:

1. Γ is a $*$-subalgebra of $\prod_{t\in T} A(t)$;
2. For any $t \in T$, the set $\{x(t) : x \in \Gamma\}$ is dense in $A(t)$;
3. For any $x \in \Gamma$, the function $t \mapsto ||x(t)||$ is continuous;
4. Let $x \in \prod_{t\in T} A(t)$ be a vector field; if, for any $t \in T$ and every $\varepsilon > 0$, there exists $x' \in \Gamma$ such that $||x(s)-x'(s)|| < \varepsilon$ for all s in some neighbourhood of t, then $x \in \Gamma$.

The subset $\Gamma_0(A)$ of $x \in \Gamma$ such that $t \mapsto ||x(t)||$ vanishes at infinity on T, with norm

$$||x|| = \sup_{t\in T} ||x(t)||$$

is a C*-algebra, called *the C*-algebra of the continuous field A*. $\Gamma_0(A)$ is a continuous $C(X)$-algebra in the sense of Kasparov as well as a continuous C*-bundle in the sense of [1].

In this paper we will be concerned mainly with continuous fields over the unit interval, i.e. one-parameter continuous fields. Abusing the terminology, we will use the same notation for a continuous field and for its C*-algebra.

2.1. **Sheaves of groups.** Let $\mathcal{U}$ be a category of closed subintervals of $[0,1]$ where the morphisms are the inclusions maps. We assume that $\{V^\circ : V \in \mathcal{U}\}$ is a basis for the topology of $[0,1]$ and if $V_1, V_2 \in \mathcal{U}$ and $V_1 \cap V_2 \neq \varnothing$ then both $V_1 \cup V_2$ and $V_1 \cap V_2$ are in $\mathcal{U}$. For example we take $\mathcal{U}$ to be the category of all dyadic intervals. Some of the intervals of $\mathcal{U}$ can have zero length.

Let $\mathcal{V}$ be the full subcategory of $\mathcal{U}$ consisting of intervals of positive length (nondegenerate intervals). A presheaf of groups on $[0,1]$ is a contravariant functor $\mathcal{S}$ from $\mathcal{V}$ to the category of groups. A morphism $V \subset V'$ is taken by $\mathcal{S}$ to the restriction map $\pi_V^{V'} : \mathcal{S}(V') \to \mathcal{S}(V)$. Let $V, V' \in \mathcal{V}$ be such that $V \cap V' \in \mathcal{V}$. The restriction maps induce a natural map

$$\mathcal{S}(V \cup V') \to \{(f,g) \in \mathcal{S}(V) \oplus \mathcal{S}(V') : \pi^V_{V\cap V'}(f) = \pi^{V'}_{V\cap V'}(g)\}. \tag{2.1}$$

A presheaf $\mathcal{S}$ is a sheaf if the above map is bijective for all V, V'.

Definition 2.2. A presheaf $\mathcal{S}$ is continuous if for any decreasing sequence of closed subintervals $(V_i)_{i=1}^\infty$ whose intersection $\bigcap_{i=1}^\infty V_i = V$ is in $\mathcal{V}$, the inductive limit $\varinjlim \mathcal{S}(V_i)$ is canonically isomorphic to $\mathcal{S}(V)$. If in addition $\mathcal{S}$ satisfies the pullback condition (2.1), then we say that $\mathcal{S}$ is a continuous sheaf.

The stalk of $\mathcal{S}$ at a point $x \in [0,1]$, denoted by $\mathcal{S}_x$, is defined as the inductive limit of the groups $\mathcal{S}(V)$ with x in the interior of V. The restriction map $\mathcal{S}(V) \to \mathcal{S}_x$ is denoted by π_x. A continuous presheaf $\mathcal{S}$ extends naturally to a contravariant functor $\mathcal{S}'$ on $\mathcal{U}$ if we set $\mathcal{S}'(\{x\}) = \mathcal{S}_x$. This extension is unique if we require that $\mathcal{S}'$ be continuous in the sense that $\varinjlim \mathcal{S}'(U_i)$ is canonically isomorphic to $\mathcal{S}'(U)$ for any decreasing sequence $(U_i)_{i=1}^{\infty}$ of elements of $\mathcal{U}$ whose intersection $\bigcap_{i=1}^{\infty} U_i = U$ is nonempty. The functor $\mathcal{S}'$ will be also called a presheaf, or an extended presheaf if we want to emphasize that it is also defined on degenerate intervals. In the sequel we shall identify any continuous presheaf $\mathcal{S}$ on $\mathcal{V}$ with its continuous extension $\mathcal{S}'$ to $\mathcal{U}$.

Remark 2.3. Let us note that if $\mathcal{S}$ is a continuous presheaf, then we have evaluation maps $\pi_x : \mathcal{S}[a, b] \to \mathcal{S}_x$ for every $x \in [a, b]$ and not only for the points x in (a, b). Indeed, by continuity, any element $f \in \mathcal{S}[a, b]$ lifts to an element $f' \in \mathcal{S}(V)$ for some neighbourhood $V \in \mathcal{V}$ of $[a, b]$ and the element $\pi_x(f')$ is independent of f'. If $\mathcal{S}$ is a continuous presheaf, it is easy to verify that the pullback condition (2.1) is equivalent to requiring that for each $a < c < b$ with $[a, c], [c, b] \in \mathcal{V}$ the restriction maps induce an isomorphism

$$\mathcal{S}[a, b] \cong \{(f, g) \in \mathcal{S}[a, c] \oplus \mathcal{S}[c, b] : \pi_c^{[a,c]}(f) = \pi_c^{[c,b]}(g)\}. \tag{2.2}$$

We are going to give an equivalent description of continuous sheaves on $[0, 1]$. Let $(G_x)_{x\in[0,1]}$ be a family of abelian groups. Suppose that for each $U \in \mathcal{V}$ a subgroup $\mathcal{F}(U) \subset \prod_{x\in U} G_x$ is given. The elements of $\mathcal{F}(U)$ are functions and hence there is a natural restriction map $\mathcal{F}(U) \to \prod_{x\in V} G_x$, $f \mapsto f|_V$, whenever $V \subset U$. Consider the following conditions:

(i) If $V \subset U$ and $f \in \mathcal{F}(U)$, then $f|_V \in \mathcal{F}(V)$;

(ii) For any $x \in [0, 1]$ and any $a \in G_x$, there is a neighbourhood $U \in \mathcal{V}$ of x and there is $f \in \mathcal{F}(U)$ such that $f(x) = a$;

(iii) For any $U \in \mathcal{V}$ and $f \in \mathcal{F}(U)$, the null set of f, $\mathrm{null}(f) = \{x \in U : f(x) = 0\}$ is open in U;

(iv) $\mathcal{F}[a, b] \cong \{(f, g) \in \mathcal{F}[a, c] \oplus \mathcal{F}[c, b] : f(c) = g(c)\}$ (canonically), for $a < c < b$.

We shall view the family of groups $U \mapsto \mathcal{F}(U)$ together with the corresponding restriction maps as a presheaf on $\mathcal{V}$.

Proposition 2.4. *If the conditions (i) through (iv) are satisfied, then $\mathcal{F}$ is a continuous sheaf whose stalk at x is G_x. Conversely, any continuous sheaf of abelian groups on $[0, 1]$ is obtained in this manner, up to an isomorphism.*

Proof. For the sake of simplicity we assume that $\mathcal{V}$ consists of all nondegenerate closed subintervals of $[0, 1]$. ($\Rightarrow$) $\mathcal{F}$ is clearly a presheaf. To compute its stalks, we observe that the canonical map $\mathcal{F}_x \to G_x$ is surjective by the condition (ii) and injective by the condition (iii). Let $0 < a < b \leq 1$. We are going to show that the canonical map $\theta : \varinjlim \mathcal{F}[a - 1/n, b] \to \mathcal{F}[a, b]$ is bijective. The general continuity of $\mathcal{F}$ is verified by similar arguments. Let $f \in \mathcal{F}[a, b]$. By the condition (ii) there is $g \in \mathcal{F}[a - 1/n, a + 1/n]$ for some n such that $g(a) = f(a)$. By the condition (iv) the restriction g to $[a - 1/n, a]$ glues with f to give rise to an element $h \in \mathcal{F}[a - 1/n, b]$ whose restriction to $[a, b]$ is equal to f. Thus θ

is surjective. Let $h \in \mathcal{F}[a-1/n, b]$ be such that its restriction to $[a, b]$ is the zero function. By the condition (iii), null(h) is open in $[a-1/n, b]$ and hence h must vanish on $[a-1/m, b]$ for some $m \geq n$. Therefore θ is injective.

($\Leftarrow$) Given a continuous sheaf $\mathcal{S}$ on $[0,1]$ we set $G_x = \mathcal{S}_x$. For $U \in \mathcal{V}$ and $s \in \mathcal{S}(U)$ we define $\hat{s} \in \prod_{x \in U} G_x$ by $\hat{s}(x) = \pi_x(s)$ and set $\mathcal{F}(U) = \{\hat{s} : s \in \mathcal{S}(U)\}$. We leave it for the reader to verify that the family $\{\mathcal{F}(U)\}$ satisfies the conditions (i) through (iv) and that the correspondence $\mathcal{S}(U) \mapsto \mathcal{F}(U)$ is bijective. □

Let A be a continuous field over X. For any closed subset X' of X, the restriction of A to X' is a continuous field of C*-algebras over X'. Denote by $A(X')$ the C*-algebra of this continuous field. Then there is a canonical *-homomorphism $\pi^X_{X'} : A(X) \to A(X')$. Let $\mathcal{U}$ be a basis for the topology of X consisting of closed subsets. If we set $\mathcal{S}(U) = \mathrm{K}_0(A(U))$, with restriction maps $\mathrm{K}_0(\pi^U_V) : \mathrm{K}_0(A(U)) \to \mathrm{K}_0(A(V))$, $V \subset U$, then $\mathcal{S}$ is a presheaf on $\mathcal{U}$. Sometimes we will write $\mathcal{S}(U) = \mathbb{K}_A(U)$. If A is a one-parameter continuous field of C*-algebras with trivial K_1-group, and $\mathcal{U}$ consists of non-degenerate closed subintervals, then $\mathcal{S}$ is in fact a sheaf on $\mathcal{U}$ by Proposition 4.1 of [2]. In this case, we shall refer to it as the K_0-sheaf of A. This is the invariant to be studied in this paper. See Section 4 of [2] for a background discussion.

3. A classification of continuous fields of AF-algebras

Let A be a separable unital continuous field of AF-algebras over $[0,1]$. Since finite dimensional C*-algebras are semiprojective, we can use the same arguments as in [2] to show that A is an inductive limit of continuous fields of finite dimensional C*-algebras, with finitely many singularities.

3.1. **Basic building blocks.** We study certain elementary unital continuous fields of finite dimensional C*-algebras which serve as basic building blocks in the study of continuous fields of AF-algebras.

Let $0 = a_0 < a_1 < \cdots < a_{2m} < a_{2m+1} = 1$ be a partition of $[0,1]$. Let us set $Y_i = [a_{2i}, a_{2i+1}]$, $Z_i = [a_{2i+1}, a_{2i+2}]$, $Y = [a_0, a_1] \cup [a_2, a_3] \cup ... \cup [a_{2m}, a_{2m+1}]$, and $Z = [a_1, a_2] \cup [a_3, a_4] \cup .... \cup [a_{2m-1}, a_{2m}]$; thus $Y \cap Z = \{a_1, a_2, ..., a_{2m}\}$. For the sake of brevity let us refer to the above cover as $\{Y, Z\}$.

Let $\{E_i\}_{i=0}^m$, $\{F_i\}_{i=0}^{m-1}$ be finite dimensional C*-algebras, and let $\{\gamma_{i,i} : F_i \to E_i\}_{i=0}^{m-1}$ and $\{\gamma_{i,i+1} : F_i \to E_{i+1}\}_{i=0}^{m-1}$ be two sets of unital *-monomorphisms. One can form two simple continuous fields by setting

$$E = \bigoplus_{i=0}^{m} \mathrm{C}(Y_i, E_i) \quad \text{and} \quad F = \bigoplus_{i=0}^{m-1} \mathrm{C}(Z_i, F_i).$$

Let G denote the restriction of E to $Y \cap Z$ and let $\pi : E \to G$ be the corresponding restriction map. There is a unital *-homomorphism $\eta : F \to G$ defined by

$$(f_0, \cdots, f_{m-1}) \mapsto (\gamma_{0,0}(f_0(a_1)), \gamma_{0,1}(f_0(a_2)), \cdots, \gamma_{m-1,m-1}(f_{m-1}(a_{2m-1})), \gamma_{m-1,m}(f_{m-1}(a_{2m}))).$$

We then can define a unital continuous field $P_{\mathcal{D}}$ with finitely many singularities, as the pull-back of the diagram $\mathcal{D}$:

$$E \xrightarrow{\pi} G \xleftarrow{\eta} F .$$

More precisely, $P_{\mathcal{D}}$ is defined by

$$\{(e, f) \in E \oplus F; \pi(e) = \eta(f)\}.$$

Since the maps $\gamma_{i,j}$ are unital and injective, $P_{\mathcal{D}}$ is a unital continuous field of finite dimensional C*-algebras over $[0,1]$ which has F_i as fibres on Z_i, and has E_i as fibres on $Y_i \setminus Z$. Moreover, it is locally simple except possibly at the singular points $\{a_1, \cdots, a_{2m}\}$.

As in [2], a diagram $\mathcal{D}$ as above is called *admissible*. If A is a continuous field of C*-algebras over $[0,1]$, we denote by $\mathcal{D}A$ the diagram

$$A(Y) \xrightarrow{\pi} A(Y \cap Z) \xleftarrow{\pi} A(Z)$$

whose pull-back is isomorphic to A. Note that in order to simplify notation we denote by the same symbol π the various restriction maps such as $\pi^Y_{Y\cap Z}$ or $\pi^Z_{Y\cap Z}$. A is called *elementary* if there is an admissible diagram $\mathcal{D}$ and a unital morphism of diagrams $\iota : \mathcal{D}A \to \mathcal{D}$

$$\begin{array}{ccccc} A(Y) & \xrightarrow{\pi} & A(Y \cap Z) & \xleftarrow{\pi} & A(Z) \\ \downarrow{\scriptstyle \iota_Y} & & \downarrow{\scriptstyle \iota_{Y\cap Z}} & & \downarrow{\scriptstyle \iota_Z} \\ E & \xrightarrow{\pi} & G & \xleftarrow{\eta} & F \end{array}$$

which induces a *-isomorphism $A \to P_{\mathcal{D}}$. We call $\iota : \mathcal{D}A \to \mathcal{D}$ a *fibred presentation* of A.

Let A and B be two unital continuous fields of C*-algebras with A elementary. A *fibred morphism* Φ from A to B consists of a fibred presentation of A, $\iota : \mathcal{D}A \to \mathcal{D}$, together with unital morphisms of continuous fields $\phi_Y, \phi_{Y\cap Z}, \phi_Z$ such that the following diagram commutes

$$\begin{array}{ccccc} A(Y) & \xrightarrow{\pi} & A(Y \cap Z) & \xleftarrow{\pi} & A(Z) \\ \downarrow{\scriptstyle \iota_Y} & & \downarrow{\scriptstyle \iota_{Y\cap Z}} & & \downarrow{\scriptstyle \iota_Z} \\ E & \xrightarrow{\pi} & G & \xleftarrow{\eta} & F \\ \downarrow{\scriptstyle \phi_Y} & & \downarrow{\scriptstyle \phi_{Y\cap Z}} & & \downarrow{\scriptstyle \phi_Z} \\ B(Y) & \xrightarrow{\pi} & B(Y \cap Z) & \xleftarrow{\pi} & B(Z) \end{array} .$$

A fibred homomorphism induces a morphism of continuous fields $\hat{\Phi} : A \to B$. Denote by $\mathrm{Hom}_{\mathcal{D}}(A, B)$ the set of all fibred homomorphism from A to B corresponding to a given fibred presentation of A, $\iota : \mathcal{D}_A \to \mathcal{D}$.

3.2. **Inductive limit decomposition.** Finite dimensional C*-algebras are semiprojective, [4]. Using this fact, one can get inductive limit decompositions for continuous fields of AF-algebras, by arguments similar to those of Theorem 6.1 and Theorem 6.2 of [2].

Let A be a C*-algebra. Let $a \in A$ and $\mathcal{F}, \mathcal{G} \subset A$. For $\varepsilon > 0$, we write $a \in_\varepsilon \mathcal{F}$ if there is $b \in \mathcal{F}$ such that $||a - b|| < \varepsilon$, and write $\mathcal{F} \subset_\varepsilon \mathcal{G}$ if $a \in_\varepsilon \mathcal{G}$ for any $a \in \mathcal{F}$.

Theorem 3.1. *Let A be a unital continuous field of AF-algebras over $[0,1]$. For any finite subset $\mathcal{F} \subset A$ and any $\varepsilon > 0$, there is an elementary unital continuous field A_1 of finite dimensional C*-algebras with a fibred presentation $\iota : \mathcal{D}A_1 \to \mathcal{D}$, and a unital fibred morphism $\Phi \in Hom_{\mathcal{D}}(A_1, A)$ such that $\mathcal{F} \subset_\varepsilon \hat{\Phi}(A_1)$.*

Proof. We shall find points $0 = a_0 < a_1 < \cdots < a_{2m+1} = 1$ and finite dimensional C*-algebras E_i, F_j $(0 \le i \le m, 0 \le j \le m-1)$ such that if we set $Y_i = [a_{2i}, a_{2i+1}], Z_j = [a_{2j+1}, a_{2j+2}]$ and $E = \bigoplus_i \mathrm{C}(Y_i, E_i)$, $F = \bigoplus_j \mathrm{C}(Z_j, F_j)$, $Y = \bigcup Y_i$, $Z = \bigcup Z_j$, then there are unital fibrewise injective *-homomorphisms $\phi : E \to A(Y)$, $\psi : F \to A(Z)$ of continuous fields such that

$$\pi^Z_{Y\cap Z}(\psi(F)) \subset \pi^Y_{Y\cap Z}(\phi(E))$$

and

$$\pi_Y(\mathcal{F}) \subset_\varepsilon \phi(E), \quad \pi_Z(\mathcal{F}) \subset_\varepsilon \psi(F).$$

If A_1 is the pull-back of the map ϕ and ψ, and $\mathcal{D}$ is defined by

$$E \xrightarrow{\pi} G \xleftarrow{\eta} F$$

where $G = E(Y \cap Z)$ and η is obtained as the composition

$$F(Z) \xrightarrow{\pi^Z_{Y\cap Z}} F(Y \cap Z) \xrightarrow{\gamma} E(Y \cap Z) = G$$

where $\gamma(f) = (\phi^{-1}\psi)|_{Y\cap Z}(f)$, then as in [2], there is a fibred homomorphism $\Phi \in \mathrm{Hom}_{\mathcal{D}}(A_1, A)$ induced by the pair ϕ, ψ, which satisfies the theorem.

We are going to construct E, F, ϕ and ψ. For any $x \in [0,1]$, since $A(x)$ is a unital AF-algebra, there is a finite dimensional unital sub-C*-algebra $F_x \subset A(x)$ such that $\pi_x(\mathcal{F}) \subset_{\varepsilon/4} F_x$. Let $\mathcal{H}$ be a finite subset of F_x such that for any $a \in \mathcal{F}$ there is $h_a \in \mathcal{H}$ such that $||h_a - \pi_x(a)|| < \varepsilon/4$. Since F_x is semiprojective and since A is a continuous field, there is a closed neighbourhood U_x of x and a fibrewise injective unital *-homomorphism $\eta_x : F_x \to A(U_x)$ such that $||\pi_x\eta_x(h) - h|| < \varepsilon/4$ for any $h \in \mathcal{H}$. Therefore, we have that $||\pi_x(\eta_x(h_a)) - \pi_x(a)|| < \varepsilon/2$ for all $a \in \mathcal{F}$. Since A is a continuous field of C*-algebras, after passing to a smaller neighbourhood, we have that $||\eta_x(h_a) - \pi_{U_x}(a)|| < \varepsilon/2$ for any $a \in \mathcal{F}$. In particular, $\pi_{U_x}(\mathcal{F}) \subset_{\varepsilon/2} \eta_x(F_x)$.

By compactness of $[0,1]$, there are points $0 = y_0 < y_1 < \cdots < y_m = 1$, finite dimensional C*-algebras F_j $(0 \le j \le m-1)$, fibrewise injective unital *-homomorphisms $\eta_j : F_j \to$

$A[y_j, y_{j+1}]$, and finite sets $\mathcal{F}_j$ of F_j such that

$$\pi_{[y_j,y_{j+1}]}(\mathcal{F}) \subset_{\varepsilon/2} \eta_j(\mathcal{F}_j) \quad \text{for all } 0 \leq j \leq m-1.$$

Since each F_j is a finite dimensional C*-algebra, there exist finite subsets $\mathcal{G}_j \subset F_j$ and $\delta_j > 0$ such that for any fibrewise injective unital *-homomorphism ϕ from F_j to a unital C*-algebra B with $\phi(\mathcal{G}_j) \subset_{\delta_j} B'$ where B' is a unital sub-C*-algebra of B, there is a fibrewise injective unital *-homomorphism ψ from F_j to B' with $||\phi(a) - \psi(a)|| \leq \varepsilon/2$ for any $a \in \mathcal{F}_j$.

Repeating the argument from above for each fibre $A(y_i)$, there are mutually disjoint closed intervals $Y_i = [a_{2i}, a_{2i+1}]$ $(0 \leq i \leq m)$, such that Y_i is a neighbourhood of y_i, and such that there are finite dimensional C*-algebras E_i and fibrewise injective unital *-homomorphisms $\phi_i : E_i \to A(Y_i)$ such that $\pi_{Y_i}(\mathcal{F}) \subset_\varepsilon \phi_i(E_i)$ and $\pi_{[y_j,y_{j+1}]\cap Y_i}(\eta_j(\mathcal{G}_j)) \subset_{\delta_j} \phi_i(E_i)$ for any i and $j \in \{i-1, i\}$. Consider the continuous field

$$B = \{a \in A : \pi_x(a) \in \pi_x(\phi_i(E_i)), \text{ for } x \in Y_i, 0 \leq i \leq m\}.$$

By construction, $\eta_j(\mathcal{G}_j) \subset_{\delta_j} B[y_j, y_{j+1}]$ for all j. Therefore, there are fibrewise injective unital *-homomorphisms $\psi_j : F_j \to B[y_j, y_{j+1}]$ such that

$$||\psi_j(a) - \eta_j(a)|| < \varepsilon/2 \quad \text{for any } a \in \mathcal{F}_j.$$

Set $Z_j = [a_{2j+1}, a_{2j+2}] \subset [y_j, y_{j+1}]$. The sets (Z_j) are mutually disjoint and $\pi_{Z_j\cap Y_i}\psi_j(F_j) \subset \pi_{Z_j\cap Y_i}\phi_i(E_i)$ whenever $Z_j \cap Y_i \neq \varnothing$. Extend the maps $\phi_i : E_i \to A(Y_i)$ and $\psi_j : F_j \to A(Z_j)$ to (necessarily injective) morphisms of continuous fields of C*-algebras, and define ϕ, ψ as above. Then ϕ, ψ, Y and Z satisfy the requirements at the beginning of the proof, as desired. □

The theorem above gives us a local approximation of continuous fields of AF-algebras by elementary fields of finite dimensional C*-algebras. Using the same arguments as in the proof of Theorem 6.2 of [2], one can prove the following semiprojectivity property for elementary continuous fields of finite dimensional C*-algebras: Let $\mathcal{D}$ be an admissible diagram with components E_i and F_j finite dimensional C*-algebras and based on a closed cover Y, Z and X. Let

$$\begin{array}{ccccc} E & \xrightarrow{\pi} & G & \xleftarrow{\eta} & F \\ \downarrow{\scriptstyle \phi_Y} & & \downarrow{\scriptstyle \phi_{Y\cap Z}} & & \downarrow{\scriptstyle \phi_Z} \\ A(Y) & \xrightarrow{\pi} & A(Y\cap Z) & \xleftarrow{\pi} & A(Z) \end{array}$$

be a commutative diagram with vertical maps unital morphisms of continuous fields of C*-algebras. Then, for any finite sets $\mathcal{F}_E \subset E$, $\mathcal{F}_F \subset F$ and any $\varepsilon > 0$ there are finite sets $\mathcal{G}_E \subset E$, $\mathcal{G}_F \subset F$ and $\delta > 0$ such that for any sub-continuous-field C*-algebra $B \subset A$ with

$\phi(\mathcal{G}_E) \subset_\delta B(Y)$ and $\psi(\mathcal{G}_F) \subset_\delta B(Z)$, there is a commutative diagram

$$\begin{array}{ccccc} E & \xrightarrow{\pi} & G & \xleftarrow{\eta} & F \\ \downarrow{\scriptstyle \phi'_Y} & & \downarrow{\scriptstyle \phi'_{Y\cap Z}} & & \downarrow{\scriptstyle \phi'_Z} \\ B(Y) & \xrightarrow{\pi} & B(Y \cap Z) & \xleftarrow{\pi} & B(Z) \end{array}$$

such that $||\phi(e) - \phi'(e)|| < \varepsilon$ for all $e \in \mathcal{F}_E$ and $||\psi(f) - \psi'(f)|| < \varepsilon$ for all $f \in \mathcal{F}_F$.

The proof of the above property is a repetition of the proof of Theorem 6.2 of [2], and we omit it here. With this semiprojective property together with Theorem 3.1, we have the following inductive limit decomposition theorem.

Theorem 3.2. *Let A be a unital separable continuous field of AF-algebras over $[0,1]$. There are an inductive system (A_k) of unital elementary continuous fields of finite dimensional C*-algebras and unital fibred morphisms $\Phi_k \in \mathrm{Hom}_{\mathcal{D}_k}(A_k, A_{k+1})$ and $\Phi_{k,\infty} \in \mathrm{Hom}_{\mathcal{D}_k}(A_k, A)$ such that $\widehat{\Phi}_{k+1,\infty} \circ \widehat{\Phi}_k = \widehat{\Phi}_{k,\infty}$ and the maps $(\widehat{\Phi}_{k,\infty})$ induce an isomorphism $\varinjlim(A_k, \widehat{\Phi}_k) \cong A$.*

3.3. **A classification theorem.**

Lemma 3.3. *Let A be a unital separable continuous field of AF-algebras over $[0,1]$. The C*-algebra $A[a,b]$ has stable rank one for any $[a,b]$. In particular, $A[a,b]$ is stably finite and has cancellation of projections.*

Proof. By Theorem 3.1, A can be approximated locally by elementary continuous fields of AF-algebras. These approximating fields have stable rank one as seen by applying [5] to various extensions of stable rank one C*-algebras. It follows that $A[a,b]$ also has stable rank one. □

The following lemma plays role similar to that of Corollary 3.3 of [2].

Lemma 3.4. *Let A be a finite dimensional C*-algebra. Let B be a unital continuous field of AF-algebras over $Z = [z_1, z_2]$. Suppose that there are unital continuous field morphisms $\phi, \psi : C(Z, A) \to B$, and unitaries $u_i \in U(B(z_i))$ and $v \in U(B)$ satisfying*

$$u_i \psi_{z_i}(a) u_i^* = \phi_{z_i}(a) \quad \text{for all } a \in C(Z,A),\ i = 1,2 \quad \text{and} \quad v\,\psi\,v^* = \phi.$$

Then, for any finite subset $\mathcal{F} \subset C(Z,A)$ any $\varepsilon > 0$, there is $u \in U(B)$ such that $u_{z_i} = u_i$ and

$$||u\psi(a)u^* - \phi(a)|| < \varepsilon \quad \text{for all } a \in \mathcal{F}.$$

Proof. Since ψ and ϕ are continuous field morphisms, we may assume that $\mathcal{F}$ is in the unit ball of A. From the given assumptions we deduce immediately that

$$[v_{z_i}^* u_i, \psi_{z_i}(a)] = 0 \quad \text{for all } a \in A.$$

Consider the end-point z_1 and let $z \in (z_1, z_2)$. The relative commutant R of the finite dimensional algebra $\psi_{z_1}(A)$ in the AF-algebras $B(z_1)$ is AF. Therefore there is a continuous path of unitaries $\omega : [0,1] \to U(R)$ such that $\omega(0) = v_{z_1}^* u_1$ and $\omega(1) = 1$. By the homotopy lifting property of the fibration $\pi_{z_1} : U(B[z_1, z]) \to U(B(z_1))$, there is a continuous path of unitaries $\Omega : [0,1] \to U(B[z_1, z])$ such that $\Omega(1) = 1$ and $\pi_{z_1}\Omega(t) = \omega(t)$ for all $t \in [0,1]$. After decreasing z if necessary we may arrange that

$$\|[\Omega(t), \pi_{[z_1,z]}\psi(a)]\| < \varepsilon, \quad \text{for all}\, t \in [0,1] \,\text{and}\, a \in \mathcal{F}.$$

Let $h : [z_1, z] \to [0,1]$ be an affine increasing homeomorphism. Then the formula

$$u'_x = \begin{cases} \pi_x \Omega(h(x)) & \text{if } x \in [z_1, z] \\ 1 & \text{otherwise} \end{cases}$$

defines a unitary u' in B such that and $u'_{z_1} = v_{z_1}^* u_1$ and

$$||[u', \psi(a)]|| < \varepsilon \quad \text{for any } a \in \mathcal{F}.$$

Repeating the same argument at the end-point z_2, we obtain a unitary u'' with similar properties. Then $w = u'u''$ is a unitary in B such that $w_{z_i} = v_{z_i}^* u_i$, $i = 1, 2$, and

$$||[w, \psi(a)]|| < \varepsilon \quad \text{for any } a \in \mathcal{F}.$$

Consider the unitary $u = vw \in \mathrm{U}(B)$. We have that $u_{z_i} = u_i$, $i = 1, 2$, and

$$||u\psi(a)u^* - \phi(a)|| \leq ||v(w\psi(a)w^* - \psi(a))v^*|| + ||v\psi(a)v^* - \phi(a)|| < \varepsilon$$

for any $a \in \mathcal{F}$, as desired. □

Lemma 3.5. *Let A be a finite dimensional C^*-algebra and let D be a AF-algebra. Let B be a unital C^*-algebra with cancellation of projections and let $\pi : B \to D$ be a surjective *-homomorphism. Let $\sigma : A \to D$ be a unital *-homomorphism. Suppose that there is a positive morphism $\alpha : K_0(A) \to K_0(B)$ such that $\alpha[1_A] = [1_B]$ and $\pi_*\alpha = \sigma_*$. Then there is a unital *-homomorphism $\varphi : A \to B$ such that $\varphi_* = \alpha$ and $\pi\varphi = \sigma$.*

Proof. Since A is finite dimensional and since B has cancellation of projections there is a unital *-homomorphism $\psi : A \to B$ such that $\psi_* = \alpha$. Therefore $(\pi\psi)_* = \sigma_*$. Since D is AF, we must have $\pi\psi = u\,\sigma\,u^*$ for some unitary $u \in U(D)$. Since u is homotopic to 1_D in $U(D)$, u lifts to a unitary $v \in U(B)$. We conclude that $\varphi = v^*\,\psi\,v$ is the desired lifting of σ. □

By a *fibred* K_0-morphism from A to B, corresponding to a given fibred presentation $\iota : \mathcal{D}A \to \mathcal{D}$ of A, we mean a triple of positive maps $\alpha = (\alpha_Y, \alpha_{Y\cap Z}, \alpha_Z)$, where α_Y has components $\alpha_i : \mathrm{K}_0(E(Y_i)) \to \mathrm{K}_0(B(Y_i))$, α_Z has components $\alpha_j : \mathrm{K}_0(F(Z_j)) \to$

$\mathrm{K}_0(B(Z_j))$, and $\alpha_{Y\cap Z}$ has components $\alpha_{i,j} : \mathrm{K}_0(E(Y_i \cap Z_j)) \to \mathrm{K}_0(B(Y_i \cap Z_j))$, such that these maps preserve the class of the units and the following diagram is commutative:

$$\begin{array}{ccccc}
\mathrm{K}_0(A(Y)) & \xrightarrow{\pi_*} & \mathrm{K}_0(A(Y\cap Z)) & \xleftarrow{\pi_*} & \mathrm{K}_0(A(Z)) \\
\downarrow{\scriptstyle \iota_*} & & \downarrow{\scriptstyle \iota_*} & & \downarrow{\scriptstyle \iota_*} \\
\mathrm{K}_0(E) & \xrightarrow{\pi_*} & \mathrm{K}_0(G) & \xleftarrow{\eta_*} & \mathrm{K}_0(F) \\
\downarrow{\scriptstyle \alpha_Y} & & \downarrow{\scriptstyle \alpha_{Y\cap Z}} & & \downarrow{\scriptstyle \alpha_Z} \\
\mathrm{K}_0(B(Y)) & \xrightarrow{\pi_*} & \mathrm{K}_0(B(Y\cap Z)) & \xleftarrow{\pi_*} & \mathrm{K}_0(B(Z))
\end{array}$$

Let us summarize the above diagram by the notation

$$\mathrm{K}_0(\mathcal{D}A) \xrightarrow{\mathrm{K}_0(\iota)} \mathrm{K}_0(\mathcal{D}) \xrightarrow{\alpha} \mathrm{K}_0(\mathcal{D}B).$$

The set of fibred K_0-morphisms from A to B corresponding to a given fibred presentation $\iota : \mathcal{D}A \to \mathcal{D}$ of A is denoted by $\mathrm{Hom}(\mathrm{K}_0(\mathcal{D}), \mathrm{K}_0(\mathcal{D}B))$.

Parallel to Theorem 8.1 of [2], we have the following existence and uniqueness theorem.

Theorem 3.6. *Let A and B be unital continuous fields of C*-algebras over $[0, 1]$ with fibres AF-algebras. Assume that A is elementary with fibred presentation $\iota : \mathcal{D}A \to \mathcal{D}$. For any K_0-fibred positive morphism $\alpha \in \mathrm{Hom}(K_0(\mathcal{D}), K_0(\mathcal{D}B))$ which preserves the class of the unit, there is a fibred morphism $\Psi \in Hom_{\mathcal{D}}(A, B)$ such that $K_0(\Psi) = \alpha$. If $\Phi \in Hom_{\mathcal{D}}(A, B)$ is another fibred morphism satisfying $K_0(\Phi) = \alpha$, then Φ is approximately unitarily equivalent to Ψ.*

Proof. By assumption, the components of α are such that

$$\alpha_{Y\cap Z}(\mathrm{K}_0(E(Y_i \cap Z_i))) \subset \mathrm{K}_0(B(Y_i \cap Z_j)),$$

$$\alpha_Y(\mathrm{K}_0(E(Y_i))) \subset \mathrm{K}_0(B(Y_i)), \quad \text{and} \quad \alpha_Z(\mathrm{K}_0(F(Z_j))) \subset \mathrm{K}_0(B(Z_j)).$$

Let α_i^E and α_j^F denote the corresponding components of α_Y and α_Z. Also let $\alpha_{i,j} : \mathrm{K}_0(E_i) \to \mathrm{K}_0(B(Y_i \cap Z_j))$ denote the component of $\alpha_{Y\cap Z}$ corresponding to $Y_i \cap Z_j \neq \varnothing$. Since the fibres of B are AF-algebras, there are unital *-homomorphisms $\psi_{i,j} : E_i \to B(Y_i \cap Z_j)$ such that $\mathrm{K}_0(\psi_{i,j}) = \alpha_{i,j}$. By Lemma 3.3, $B(Y_i)$ has cancellation of projections. Therefore by Lemma 3.5 there is a unital *-homomorphism $\psi_i^E : E_i \to B(Y_i)$, which then can be extended to a morphism of continuous fields $\psi_i^E : \mathrm{C}(Y_i, E_i) \to B(Y_i)$, with $\mathrm{K}_0(\psi_i^E) = \alpha_i^E$ and ψ_i^E extending simultaneously the maps $\psi_{i,i-1} \circ \pi_{Y_i \cap Z_{i-1}}$ and $\psi_{i,i} \circ \pi_{Y_i \cap Z_i}$. Arguing in a similar way, for each Z_j there is a unital morphism $\psi_j^F : \mathrm{C}(Z_j, F_j) \to B(Z_j)$ which extends simultaneously the maps $\psi_{j,j} \circ \eta_{j,j}$ and $\psi_{j+1,j} \circ \eta_{j,j+1}$, and $\mathrm{K}_0(\psi_j^F) = \alpha_j^F$. Then $\psi_Y = (\psi_i^E)$ and $\psi_Z = (\psi_j^F)$ and $\psi_{Y\cap Z} = (\psi_{i,j})$ is the desired lifting of α.

We now show the uniqueness part of the theorem. Let Ψ and Φ be as in the statement, and use the same notation for each component of Ψ and Φ. Since E_i is a finite dimensional

C*-algebra and the range C*-algebras have cancellation of projections, ϕ_i^E and ϕ_j^F are unitarily equivalent to ψ_i^E and ψ_j^F respectively. Denote by $u_i \in B(Y_i)$ and $v_j \in B(Z_j)$ the intertwining unitaries. Their restrictions to $Y_i \cap Z_j$ may not equal. But this can be handled in the same way as the argument of Theorem 8.1 of [2] with Lemma 3.4 playing the role of Corollary 3.3 of [2]. This gives the uniqueness part of the theorem. □

With the decomposition theorem 3.2, and the existence and uniqueness theorem 3.6, we have the following classification theorem.

Theorem 3.7. *Let A, B be separable unital continuous fields of AF-algebras over $[0,1]$. Any isomorphism of K_0-sheaves $\alpha : \mathbb{K}_A \to \mathbb{K}_B$ such that $\alpha_{[0,1]}([1_A]) = [1_B]$ lifts to an isomorphism $A \cong B$ of continuous fields of C*-algebras. The lifting is unique up to approximate unitary equivalence.*

Proof. The proof is entirely similar to the proof of Theorem 8.2 of [2]; hence we omit it. □

4. On K$_0$-presheaves of one-parameter continuous fields of C*-algebras

4.1. **Continuity properties of presheaves.** Let A be a unital separable continuous field C*-algebra over $[0,1]$. For any closed subinterval $[a,b]$, if we define $\mathcal{S}[a,b] = \mathrm{K}_0(A[a,b])$, then $\mathcal{S}$ (also denoted by $\mathbb{K}_A$) is a presheaf with restriction homomorphisms induced by the restriction homomorphisms of A. For any decreasing sequence of closed subintervals $([a_i, b_i])_{i=1}^{\infty}$ with $\bigcap_{i=1}^{\infty} [a_i, b_i] = [a,b]$, there is a canonical homomorphism

$$\phi : \varinjlim(\mathcal{S}[a_i, b_i], \phi_{i,i+1}) \to \mathcal{S}[a,b]$$

where $\phi_{i,i+1}$ is the restriction homomorphism from the abelian group $\mathcal{S}[a_i, b_i]$ to the abelian group $\mathcal{S}[a_{i+1}, b_{i+1}]$, and $\varinjlim(\mathcal{S}[a_i, b_i], \phi_{i,i+1})$ is the inductive limit of $(\mathcal{S}[a_i, b_i], \phi_{i,i+1})$ in the category of countable abelian groups.

Lemma 4.1. *The homomorphism ϕ above is an isomorphism. Moreover, if we denote by V the limit of the positive cones of $\mathcal{S}[a_i, b_i]$, the map ϕ induces an isomorphism from V to the positive cone of $\mathcal{S}[a,b]$. The statement also holds if $[a,b]$ reduces to a point; that is, if a_i, b_i converge to x, then the canonical map $\phi : \varinjlim \mathcal{S}[a_i, b_i] \to K_0(A(x))$ is an isomorphism. Thus the stalk $\mathcal{S}_x$ is canonically isomorphic to $K_0(A(x))$.*

Proof. This follows from the usual stability properties of projections since the C*-algebra $A[a,b]$ is isomorphic to the inductive limit C*-algebra $\varinjlim A[a_i, b_i]$. □

Let A be a continuous field of C*-algebras over $[0,1]$, and denote by $\mathcal{S} = \mathbb{K}_A$ the K$_0$-presheaf of A. There is a representation Φ of $\mathcal{S}$ induced by $(\pi_x)_{x\in[0,1]}$: For any $[a,b]$, define

$$\Phi : \mathrm{K}_0(A[a,b]) \ni f \mapsto ([\pi_x](f))_{x\in[a,b]} \in \prod_{x\in[a,b]} \mathrm{K}_0(A(x)). \tag{4.1}$$

Lemma 4.2. *Let A be a unital continuous field of C^*-algebras over $[0,1]$ such that $K_1(A(x)) = 0$ for all $x \in [0,1]$. Then the representation Φ is faithful. In other words, for any $f \in K_0(A[a,b])$, if $\Phi(f) = 0$, then $f = 0$ in $K_0(A[a,b])$.*

Proof. Since $\mathrm{K}_1(A(x)) = 0$ for all x, the K_0-presheaf of A is in fact a sheaf by Proposition 4.1 of [2]. If $\Phi(f) = 0$ for an element $f \in \mathrm{K}_0(A[a,b])$, by Lemma 4.1, there is a neighbourhood U_x of x such that $[\pi_{U_x}](f) = 0$. By the compactness of $[0,1]$ and the fact that the K_0-presheaf is a sheaf, we conclude that $f = 0$ in $\mathrm{K}_0(A[a,b])$. □

4.2. **Ordered K_0-sheaves of one-parameter continuous fields of AF-algebras.**

Lemma 4.3. *Let A be a unital continuous field of AF-algebras over $[0,1]$. Let $0 \le a < b < c \le 1$, and let $p \in A[a,b]$ and $q \in A[b,c]$ be projections with $[\pi_b(p)] = [\pi_b(q)]$. Then there is a projection e in $A[a,c]$ such that $[\pi_{[a,b]}(e)] = [p]$ and $[\pi_{[b,c]}(e)] = [q]$.*

Proof. Since $[\pi_b(p)] = [\pi_b(q)]$ and $A(b)$ is a unital AF-algebra, there is a unitary $u \in A(b)$ such that $u\pi_b(q)u^* = \pi_b(p)$. Since $U(A(b))$ is path connected, u lifts to a unitary $v \in A[b,c]$. Since the projections p and vqv^* assume the same value in $A(b)$, they glue together to a projection $e \in A[a,c]$ with the desired properties. □

This lemma together with the fact that any continuous field of AF-algebras has stable rank one (therefore, any positive element in K_0-group comes from a projection) shows that the ordered K_0-presheaf is a sheaf of ordered groups (see also Lemma 3.3). Moreover, we have that

Proposition 4.4. *An element $f \in K_0(A[a,b])$ is positive if and only if $\Phi(f)$ is pointwise positive.*

Proof. It is simple to verify that if f is positive then $\Phi(f)$ is pointwise positive. Thus it suffices to prove the converse.

Write $f = [p_1] - [p_2]$ for projections p_1 and p_2 in $M_m(A[a,b])$. For any $x \in [a,b]$, we assert that there is a neighbourhood U_x of x, and a projection $p \in M_N(A(U_x))$ (for some N) such that $[p] = [\pi_{U_x}](f)$. Since $[\pi_x](f) \in \mathrm{K}_0^+(A(x))$, there is a partial isometry v_x in $M_m(A(x))$ such that

$$v_x v_x^* = \pi_x(p_2) \quad \text{and} \quad v_x^* v_x \pi_x(p_1) = v_x^* v_x.$$

Since A is a continuous field, there exist a neighbourhood U_x of x and a partial isometry v in $M_m(A_{U_x})$, such that

$$||vv^* - \pi_{U_x}(p_2)|| < 1 \quad \text{and} \quad ||v^*v\pi_{U_x}(p_1) - v^*v|| < 1/4,$$

from which it follows that $\pi_{U_x}(p_2)$ is unitarily equivalent to vv^* and

$$||\pi_{U_x}(p_1)\, v^*v\, \pi_{U_x}(p_1) - v^*v|| \le ||\pi_{U_x}(p_1)\, v^*v\, \pi_{U_x}(p_1) - \pi_{U_x}(p_1)v^*v|| + ||\pi_{U_x}(p_1)v^*v - v^*v|| < 1/2.$$

Therefore, there is a projection in the hereditary sub-C*-algebra of $M_m(A(U_x))$ generated by the projection $\pi_{U_x}(p_1)$ which is unitarily equivalent to v^*v. Thus, $\pi_{U_x}(p_2)$ is Murray-von Neumann equivalent to a subprojection of $\pi_{U_x}(p_1)$. This proves the assertion.

By the compactness of $[0,1]$, there is a partition $0 = a_0 < a_1 < \cdots < a_n = 1$ and m such that there is a projection p_i in each $M_m(A[a_i, a_{i+1}])$ such that $[p_i] = [\pi_{[a_i,a_{i+1}]}](f)$. Applying Lemma 4.3 repeatedly, we obtain a projection p such that $[\pi_{[a_i,a_{i+1}]}(p)] = [\pi_{a_i,a_{i+1}}](f)$ which implies $[p] = f$ (since the K_0-presheaf is a sheaf), as desired. □

Recall ([6]) that an ordered group (G, G^+) (we assume that $0 \in G^+$) has the *Riesz decomposition property* if for any positive elements a, b, and c with $a \leq b + c$, there exist positive elements $0 \leq b' \leq b$ and $0 \leq c' \leq c$ such that $a = b' + c'$. We write $a > b$ if $a - b \in G^+ \setminus \{0\}$. An ordered group (G, G^+) is called *unperforated* if for any element a, $na \in G^+$ for some natural number n implies $a \in G^+$. Note that any unperforated ordered group is torsion free ($na = 0 \Rightarrow a \geq 0$; however $na = 0 \Rightarrow n(-a) = 0 \Rightarrow -a \geq 0$). A unperforated ordered group with the Riesz decomposition property is referred as a *dimension group*. We say a morphism $f : G \to G'$ of ordered groups is *strictly positive* if $f(a) > 0$ whenever $a > 0$.

Any totally ordered group has the Riesz decomposition property. Ordered K_0-groups of AF-algebras are dimension groups. However, the ordered K_0-group of a unital continuous field of AF-algebras may fail to have the Riesz decomposition property. For example, the splitting interval algebra

$$S := \{f \in \mathrm{M}_2(\mathrm{C}[0,1]); f(0) \in \mathbb{C} \oplus \mathbb{C}, f(1) \in \mathbb{C} \oplus \mathbb{C}\}$$

is a continuous field over $[0,1]$ with fibre $\mathrm{M}_2(\mathbb{C})$ between 0 and 1, and $\mathbb{C} \oplus \mathbb{C}$ at 0 or 1. The K_0-group of S does not have the Riesz decomposition property ([8]).

Nevertheless, if the K_0-group of each fibre C*-algebra is totally ordered, then the K_0-group of the global C*-algebra is also totally ordered. In particular, it has the Riesz decomposition property.

Proposition 4.5. *Let A be a unital separable continuous fields of AF-algebras over $[0,1]$. If $K_0(A(x))$ is totally ordered, then the ordered group $K_0(A[a,b])$ is totally ordered for any subinterval $[a,b]$. Moreover, the map $[\pi_x] : K_0(A) \to K_0(A(x))$ is injective for any $x \in [0,1]$.*

Proof. We may assume that $[a,b] = [0,1]$. For any projections p and q in $A[0,1]$, define the sets

$$U := \{x \in [0,1]; [\pi_x(p)] > [\pi_x(q)]\}, \quad V := \{x \in [0,1]; [\pi_x(p)] < [\pi_x(q)]\},$$

and

$$W := \{x \in [0,1]; [\pi_x(p)] = [\pi_x(q)]\}.$$

Since $\mathrm{K}_0(A(x))$ is totally ordered for any x, one has that U, V, W are mutually disjoint and

$$U \cup V \cup W = [0,1].$$

Note that if e is a projection in A such that $[e_x] \neq 0$ for some x, then $[e_y] \neq 0$ for all y in some neighbourhood of x. Therefore, arguing as in Proposition 4.4, we see that the subsets U, V, W are open subsets of $[0,1]$. Since $[0,1]$ is connected, only one of them is nonempty, and hence it is equal to $[0,1]$. If U or V is nonempty then $[p] > [q]$ or $[p] < [q]$ by Proposition 4.4. If W is nonempty and hence $W = [0,1]$, one has that $[p] = [q]$ by Lemma 4.2. Therefore, we always have that $[p] \geq [q]$ or $[p] \leq [q]$. Moreover, if $[\pi_x(p)] = [\pi_x(q)]$ for some $x \in [0,1]$, then one has that $W = [0,1]$ and hence $[p] = [q]$. Therefore the map $[\pi_x]$ is injective, as desired. □

In general, the map $[\pi_x]$ is not injective as illustrated by the case of the splitting interval algebra mentioned. On the other hand, if A is a unital continuous field of stably finite C*-algebras, then the map $[\pi_x]$ is always strictly positive.

Recall that a C*-algebra A is stably finite if for any projections e and f in a matrix algebra over A such that $e \leq f$ and e is Murray-von Neumann equivalent to f, one has $e = f$.

Proposition 4.6. *Let A be a unital continuous field of stably finite C*-algebras over $[0,1]$. Then A is stably finite and the map $[\pi_x] : K_0(A) \to K_0(A(x))$ is strictly positive for any $x \in [0,1]$, i.e. if $g > 0$ then $[\pi_x](g) > 0$.*

Proof. Let v be a partial isometry in some matrix algebra over A such that $vv^* \leq v^*v$. Then $\pi_x(v)\pi_x(v^*) \leq \pi_x(v^*)\pi_x(v)$ holds at each fibre. Since $A(x)$ is stably finite, we have that $\pi_x(v)\pi_x(v^*) = \pi_x(v^*)\pi_x(v)$ for all $x \in [0,1]$. Therefore $\pi_x(vv^* - v^*v) = 0$ for all $x \in [0,1]$, and hence $vv^* = v^*v$. Thus A is stably finite.

Let p be a projection in $M_m(A)$. If $[\pi_{x_0}(p)] = 0$ in $\mathrm{K}_0(A(x_0))$ for some $x_0 \in [0,1]$, since $A(x_0)$ is stably finite, one has that $\pi_{x_0}(p) = 0$. Since p is a projection, we also have that $||\pi_x(p)|| = 0$ or 1 for any $x \in [0,1]$. By the continuity of the field, $||\pi_x(p)||$ is continuous with respect to x. Since $[0,1]$ is connected, this implies that $\pi_x(p) = 0$ for all $x \in [0,1]$ and hence $p = 0$, as desired. □

5. The range of the invariant

5.1. **Elementary covers.** An *elementary cover* $\mathcal{C}$ of an interval $[0,1]$ is a full subcategory of $\mathcal{U}$ satisfying the following conditions. $\mathcal{C}$ has finitely many objects grouped into three coloured families $\{Y_i\}_{i\in I}$, $\{Z_j\}_{j\in J}$,$\{y_{ij}\}$ where the intervals in each family are mutually disjoint, Y_i and Z_j have positive length and $y_{ij} = Y_i \cap Z_j$. It is convenient to regard Y_i and y_{ij} as being coloured with the colour Y and Z_j coloured with the colour Z. The typical example of an elementary cover $\mathcal{C}$ arises from a set of points $0 = x_0 < x_1 < ... < x_n = 1$ by setting $Y_0 = [x_{0,}x_1]$, $Z_1 = [x_1, x_2]$, $y_{01} = x_1$, etc. An elementary diagram $\mathcal{D}$ of ordered groups is a contravariant functor from an elementary cover $\mathcal{C}$ to the category of countable ordered groups with the property that the image $\mathcal{D}(\iota)$ of any morphism ι between any two objects of the same colour is the identity map. We regard the group $\mathcal{D}(X)$ as having the some colour as X. Equivalently, if for each pair of adjacent intervals Y_i and Z_j, we set $E_i = \mathcal{D}(Y_i)$, $F_j = \mathcal{D}(Z_j)$, then we must have $\mathcal{D}(\{y_{ij}\}) = E_i$ and $\mathcal{D}$ simplifies to the diagram

$$E_i =\!=\!= E_i \xleftarrow{\psi_{ji}} F_j \tag{5.1}$$

We say that an elementary cover $\mathcal{C}'$ refines $\mathcal{C}$ if the intervals of $\mathcal{C}'$ are obtained by dividing some of the intervals of $\mathcal{C}$ into three subintervals of positive length and changing the colour of the interval in the middle of each trisection. If $\mathcal{D}$ is an elementary diagram defined on $\mathcal{C}$, we can extend $\mathcal{D}$ canonically to a diagram $\mathcal{D}^{\mathcal{C}'}$ on $\mathcal{C}'$ by setting $\mathcal{D}^{\mathcal{C}'}(X') = \mathcal{D}(X)$ for all new intervals X' in $\mathcal{C}'$ (including those of length zero) contained in an interval X of $\mathcal{C}$. The new morphisms added to $\mathcal{D}^{\mathcal{C}'}$ are all equalities. If $\mathcal{C}$ has just one element $[0,1]$ and $\mathcal{D}[0,1] = G$ we will let $G^{\mathcal{C}'}$ or even G stand for $\mathcal{D}^{\mathcal{C}'}$.

Let $\mathcal{D}_1$, $\mathcal{D}_2$ be elementary diagrams defined on covers $\mathcal{C}_1$, $\mathcal{C}_2$ such that $\mathcal{C}_1$ refines $\mathcal{C}_2$. A morphism of diagrams $\mathcal{D}_1 \to \mathcal{D}_2$ is by definition a morphism of functors $\mathcal{D}_1^{\mathcal{C}_2} \to \mathcal{D}_2$. Note that both functors $\mathcal{D}_1^{\mathcal{C}_2}$ and $\mathcal{D}_2$ are defined on the same category $\mathcal{C}_2$.

One can replace $[0,1]$ by any interval $U = [a,b]$ in the above setting. If $U = [a,b] \subset [0,1]$, the restriction of $\mathcal{D}$ to U, denoted by $\mathcal{D}_U$, is defined as follows. If X is an object of $\mathcal{C}$, then $X \cap U$ is an object of $\mathcal{C}_U$, the category on which $\mathcal{D}_U$ is defined and $\mathcal{D}_U(X \cap U) = \mathcal{D}(X)$ as coloured groups.

If $\mathcal{D}$ is a diagram of ordered groups on $[a,b]$, we denote by $P(\mathcal{D})$ its pullback. With the above notation, $P(\mathcal{D})$ consists of elements $((e_i)_{i\in I}, (f_j)_{j\in J})$ such that $e_i \in E_i$, $f_j \in F_j$, and $\psi_{ji}(f_j) = e_i$ for all pairs of adjacent intervals Y_i and Z_j.

Via the operations of restrictions and pullbacks, each diagram $\mathcal{D}$ defines a continuous sheaf $\widehat{\mathcal{D}}$ of countable ordered groups on $[0,1]$. Indeed, for each $U = [a,b] \subset [0,1]$ we set $\widehat{\mathcal{D}}(U) = P(\mathcal{D}_U)$. Then one verifies immediately that the restriction maps $P(\mathcal{D}_U) \to P(\mathcal{D}_V)$, defined for $V \subset U$ by dropping from the collection $((e_i)_{i\in I}, (f_j)_{j\in J})$ those elements e_i and f_j for which Y_i and respectively Z_j does not intersect V, satisfy the required properties.

If $\mathcal{S}$ is a sheaf of ordered groups on $[0,1]$, we denote by $\mathcal{S}|_{\mathcal{C}}$ its restriction to $\mathcal{C}$. Let $\alpha : \mathcal{D} \to \mathcal{S}|_{\mathcal{C}}$ be a morphism of functors. In other words we have a commutative diagram

$$\begin{array}{ccccc}
E_i & =\!=\!= & E_i & \xleftarrow{\psi_{ji}} & F_j \\
\downarrow{\scriptstyle \alpha_i} & & \downarrow{\scriptstyle \alpha_{ij}} & & \downarrow{\scriptstyle \alpha_i} \\
\mathcal{S}(Y_i) & \xrightarrow[\pi_{x_{ij}}]{} & \mathcal{S}_{x_{ij}} & \xleftarrow[\pi_{x_{ij}}]{} & \mathcal{S}(Z_j).
\end{array}$$

Then α induces a morphism of sheaves $\widehat{\alpha} : \widehat{\mathcal{D}} \to \mathcal{S}$. Indeed for each $U \subset [0,1]$ as above α induces a morphism $\alpha_U : \mathcal{D}_U \to \mathcal{S}|_{\mathcal{C}_\mathcal{U}}$. Recall that the objects of $\mathcal{C}_U$ are of the form $X \cap U$ where X is an object of $\mathcal{C}$. The component of α_U corresponding to the object $X \cap U$ is

$$\mathcal{D}_U(X\cap U) = \mathcal{D}(X) \xrightarrow{\alpha|_X} \mathcal{S}(X) \xrightarrow{\pi_{X\cap U}} \mathcal{S}(X\cap U)$$

By passing to pullbacks α_U gives a morphism of groups $\widehat{\alpha}_U : \widehat{D}(U) \to \mathcal{S}(U)$.

5.2. **Inductive limit representations of sheaves.** A morphism of ordered groups $\alpha : G \to H$ is called strictly positive if $\alpha(g) > 0$ whenever $g > 0$.

Definition 5.1. A sheaf $\mathcal{S}$ of ordered groups is called *strictly positive* if for any closed interval Z and any point $z \in Z$, the morphism $\pi_z : \mathcal{S}(Z) \to \mathcal{S}_z$ is strictly positive. An elementary diagram of finitely generated dimension groups is called strictly positive if all its morphisms are strictly positive. With notation as in (5.1), this means precisely that all the morphisms $\psi_{ji} : F_j \to E_i$ are strictly positive.

Throughout the remainder of this section we shall assume that $\mathcal{S}$ is a continuous and strictly positive sheaf of countable ordered groups on $[0,1]$ and that its stalks are dimension groups. Moreover we shall assume that $\mathcal{S}$ is pointed in the sense that a positive nonzero element $\nu \in \mathcal{S}[0,1]$ has been chosen.

Let us recall that any finitely generated dimension group is isomorphic to $\mathbb{Z}^k$ for some $k \geq 0$.

Definition 5.2. Let G be a finitely generated dimension group and let $Z = [a,b] \subset [0,1]$. Let $\iota : G \to \mathcal{S}(Z)$ be a strictly positive homomorphism. A *simple interpolant* of ι consists of a point $z \in Z$, a finitely generated dimension group F, and strictly positive homomorphisms $\eta : F \to \mathcal{S}(Z)$ and $\theta : G \to F$, such that the diagram

$$\begin{array}{ccc} G & \xrightarrow{\quad\iota\quad} & \mathcal{S}(Z) \\ & {\scriptstyle\theta}\searrow \quad \nearrow{\scriptstyle\eta} & \\ & F & \end{array}$$

commutes and $\mathrm{Ker}(\pi_z\iota) = \mathrm{Ker}(\theta)$. A simple interpolant for $\iota : G \to \mathcal{S}(Z)$ as above is denoted by $(G, Z, z, \theta, \eta, F)$. We shall say that a simple interpolant is *open* provided that $a < z$ if $a \neq 0$ and $z < b$ if $b \neq 1$.

An *interpolant* for a strictly positive morphism $\iota : G \to \mathcal{S}[a,b]$ consists of a strictly positive elementary diagram $\mathcal{D}$ of finitely generated dimension groups, with cover $\mathcal{C}$ of $[a,b]$, together with morphisms $\theta : G^{\mathcal{C}} \to \mathcal{D}$ and $\eta : \mathcal{D} \to \mathcal{S}|_{\mathcal{C}}$ such that the diagram

$$\begin{array}{ccc} G^{\mathcal{C}} & \xrightarrow{\quad\iota\quad} & \mathcal{S}|_{\mathcal{C}} \\ & {\scriptstyle\theta}\searrow \quad \nearrow{\scriptstyle\eta} & \\ & \mathcal{D} & \end{array}$$

commutes, and for each $X \in \mathcal{C}$ which does not reduce to a point,

$$\begin{array}{ccc} G & \xrightarrow{\pi_X \iota} & \mathcal{S}(X) \\ & {\scriptstyle\theta_X}\searrow \quad \nearrow{\scriptstyle\eta_X} & \\ & \mathcal{D}(X) & \end{array}$$

is a simple interpolant of $\pi_X \iota$ for some point $x \in X$.

We say that an interpolant is *open* if the simple interpolants that correspond to the initial component $[a, c]$ and the final component $[d, b]$ of $\mathcal{C}$ are both open.

The property of an interpolant of being open will be used in the process of gluing interpolants described in Lemma 5.5.

Lemma 5.3. *Let G be a finitely generated dimension group. For any strictly positive homomorphism $\iota : G \to \mathcal{S}[a, b]$, any $x \in [a, b]$, there exist a closed interval $Z \subset [a, b]$ with $x \in \mathring{Z}$ and a simple interpolant*

$$\begin{array}{ccc} G & \xrightarrow{\pi_Z \iota} & \mathcal{S}(Z) \\ & {\scriptstyle\theta}\searrow \quad \nearrow{\scriptstyle\eta} & \\ & F & \end{array}$$

of $\pi_Z \iota$ such that $\mathrm{Ker}(\pi_x \iota) = \mathrm{Ker}(\theta)$.

Proof. Since G is finitely generated, we may assume that $G = \mathbb{Z}^n$, for some n, with the usual order. Since $\mathcal{S}$ and ι are strictly positive, the homomorphism $\pi_x \iota : G \to \mathcal{S}_x$ is also strictly positive. Since $\mathcal{S}_x$ is a dimension group, by Theorem 3.1 of [7], there are a finitely generated dimension group F and strictly positive homomorphisms $\theta : G \to F$, $\eta' : F \to \mathcal{S}_x$ such that the diagram

$$\begin{array}{ccc} G & \xrightarrow{\pi_x \iota} & \mathcal{S}_x \\ & {\scriptstyle\theta}\searrow \quad \nearrow{\scriptstyle\eta'} & \\ & F & \end{array}$$

commutes and $\mathrm{Ker}(\pi_x \iota) = \mathrm{Ker}(\theta)$.

Using the definition of $\mathcal{S}_x$, we find a closed neighbourhood V of x in $[a, b]$ such that the map η' lifts to a positive homomorphism $\eta : F \to \mathcal{S}(V)$. Moreover, since G is finitely generated, there is closed interval $Z \subset V$ with $x \in \mathring{Z}$ such that the restriction of $\eta\theta - \pi_V \iota$ to Z vanishes. We conclude the proof by replacing η by $\pi_Z \eta$ and observing that this map must be strictly positive since it is a lifting of a strictly positive map η' and since $\mathcal{S}$ is strictly positive. □

Remark 5.4. Let $(G, Z, z, \theta, \eta, F)$ be a simple interpolant for $\iota : G \to \mathcal{S}(Z)$ as in Definition 5.2. Arguing as in the proof of Lemma 5.3 and using the continuity of $\mathcal{S}$ one verifies immediately that there is a closed neighbourhood Z^ε of Z and strictly positive liftings $\iota^\varepsilon : G \to \mathcal{S}(Z^\varepsilon)$ and $\eta^\varepsilon : F \to \mathcal{S}(Z^\varepsilon)$ of ι and η such that $(G, Z^\varepsilon, z, \theta, \eta^\varepsilon, F)$ is a simple interpolant of ι^ε. This is called a local extension of the given interpolant.

Lemma 5.5. *Let $\mathcal{C}^\ell$ be an elementary cover of $[x^\ell, x]$ and let $\mathcal{C}^r$ be an elementary cover of $[x, x^r]$. Let*

(5.2)
$$\begin{array}{ccc} G^\ell & \xrightarrow{\iota^\ell} & \mathcal{S}|_{\mathcal{C}^\ell} \\ & {\scriptstyle\theta^\ell}\searrow \quad \nearrow{\scriptstyle\eta^\ell} & \\ & \mathcal{D}^\ell & \end{array}$$

and

(5.3)
$$\begin{array}{ccc} G^r & \xrightarrow{\iota^r} & \mathcal{S}|_{\mathcal{C}^r} \\ & {\scriptstyle\theta^r}\searrow \quad \nearrow{\scriptstyle\eta^r} & \\ & \mathcal{D}^r & \end{array}$$

be two open interpolants, and let G be a finitely generated dimension group with strictly positive homomorphisms $\alpha^\ell : G \to G^\ell$ and $\alpha^r : G \to G^r$ such that $\pi_x \iota^\ell \alpha^\ell = \pi_x \iota^r \alpha^r$.

There exists an open interpolant

(5.4)
$$G^{\mathcal{C}} \xrightarrow{\underline{\theta}} \mathcal{D} \xrightarrow{\underline{\eta}} \mathcal{S}|_{\mathcal{C}}$$

for the morphism $\alpha : G \to \mathcal{S}[x^\ell, x^r]$ induced by the pair $\iota^\ell\alpha^\ell, \iota^r\alpha^r$ such that the restrictions of $\mathcal{C}$ to $[x^\ell, x]$ and $[x, x^r]$ refine $\mathcal{C}^\ell$ and $\mathcal{C}^r$ respectively, and if $\mathcal{C}^\ell_{new} := \mathcal{C}|_{[x^\ell,x]}$ and $\mathcal{C}^r_{new} := \mathcal{C}|_{[x,x^r]}$, then there are morphisms $\mathcal{D}^\ell|_{\mathcal{C}^\ell_{new}} \to \mathcal{D}|_{\mathcal{C}^\ell_{new}}$, $\mathcal{D}^r|_{\mathcal{C}^r_{new}} \to \mathcal{D}|_{\mathcal{C}^r_{new}}$, such that the diagrams

$$\begin{array}{ccccc} & & \mathcal{D}|_{\mathcal{C}^\ell_{new}} & & \\ & {\scriptstyle\underline{\theta}|_{\mathcal{C}^\ell_{new}}}\nearrow & \uparrow & \searrow{\scriptstyle\underline{\eta}|_{\mathcal{C}^\ell_{new}}} & \\ G^\ell & \xrightarrow{\theta^\ell} & \mathcal{D}^\ell|_{\mathcal{C}^\ell_{new}} & \xrightarrow{\eta^\ell} & \mathcal{S}|_{\mathcal{C}^\ell_{new}} \end{array}$$

and

$$\begin{array}{ccccc} & & \mathcal{D}|_{\mathcal{C}^r_{new}} & & \\ & {\scriptstyle\underline{\theta}|_{\mathcal{C}^r_{new}}}\nearrow & \uparrow & \searrow{\scriptstyle\underline{\eta}|_{\mathcal{C}^r_{new}}} & \\ G^r & \xrightarrow{\theta^r} & \mathcal{D}^r|_{\mathcal{C}^r_{new}} & \xrightarrow{\eta^r} & \mathcal{S}|_{\mathcal{C}^r_{new}} \end{array}$$

commute.

Moreover, suppose that $\mathcal{D}_1$ is a strictly positive elementary diagram on the interval $[x^\ell, x^r]$ with cover $\mathcal{C}_1$: $[x^\ell, x] \cup \{x\} \cup [x, x^r]$ and objects and maps

$$K^\ell = K^\ell \xleftarrow{\psi} K^r \ ,$$

and suppose that there are morphisms $\mu^\ell : K^\ell \to G^\ell$, $\mu^r : K^r \to G^r$ *such that* $\pi_x \iota^\ell \mu^\ell \psi = \pi_x \iota^r \mu^r$, *so that the pair* $\iota^\ell \mu^\ell, \iota^r \mu^r$ *induces a morphism of functors* $\underline{\eta}_1 : \mathcal{D}_1 \to \mathcal{S}|_{\mathcal{C}_1}$. *Then the interpolant* (5.4) *can be chosen such that there is a morphism* $\underline{\mu}_1 : \mathcal{D}_1 \to \mathcal{D}$ *satisfying* $\underline{\eta}_1 = \underline{\eta} \circ \underline{\mu}_1$ *and the diagram*

$$\begin{array}{ccc} \mathcal{D}_1^{\mathcal{C}}(X) & \xrightarrow{\quad \underline{\eta}_1 \quad} & \mathcal{S}|_X \\ & {\scriptstyle \underline{\mu}_1} \searrow \qquad \nearrow {\scriptstyle \underline{\eta}} & \\ & \mathcal{D}(X) & \end{array} \tag{5.5}$$

is an interpolant for all nondegenerate intervals X *of* $\mathcal{C}$.

Proof. We are going to describe a procedure for gluing the two given open interpolants. Since this procedure is local, we may assume without loss of generality that the interpolants (5.2) and (5.3) are simple. Denote by $V^\ell = [x^\ell, x]$ and $V^r = [x, x^r]$. Using our earlier notation, the two simple interpolants for ι^ℓ and ι^r are

$$(G^\ell, V^\ell, z^\ell, \theta^\ell, \eta^\ell, F^\ell) \quad \text{and} \quad (G^r, V^r, z^r, \theta^r, \eta^r, F^r),$$

where $z^\ell < x < z^r$ by the openness assumption. These interpolants admit local extensions across x corresponding to a common $\varepsilon > 0$, as noted in Remark 5.4. Let us set $V = (V^\ell)^\varepsilon \cap (V^r)^\varepsilon$. Consider the map $\iota^\varepsilon : F^\ell \oplus F^r \to \mathcal{S}(V)$, $\iota^\varepsilon(a, b) = \pi_V(\eta^\ell)^\varepsilon(a) + \pi_V(\eta^r)^\varepsilon(b)$. By Lemma 5.3, after shrinking ε if necessary, there is a simple interpolant $(F^\ell \oplus F^r, V, x, \gamma, \eta_H, H)$ for ι^ε, where $V = [s^\ell, s^r]$ and $x^\ell < z^\ell < s^\ell < x < s^r < z^r < x^r$. Denoting by γ^ℓ and γ^r the restrictions of γ to F^ℓ and F^r respectively, we have the following commutative diagram:

$$\begin{array}{ccccc} & & H & & \\ & {\scriptstyle \gamma^\ell} \nearrow & \downarrow {\scriptstyle \eta_H} & \nwarrow {\scriptstyle \gamma^r} & \\ F^\ell & \xrightarrow[\pi_V(\eta^\ell)^\varepsilon]{} & \mathcal{S}(V) & \xleftarrow[\pi_V(\eta^r)^\varepsilon]{} & F^r \end{array}$$

We define a cover $\mathcal{C}$ of $[x^\ell, x^r]$ by setting $Y = V = [s^\ell, s^r]$ and $Z^\ell = [x^\ell, s^\ell]$, $Z^r = [s^r, x^r]$. Then we define a strictly positive elementary diagram $\mathcal{D}$ on $\mathcal{C}$ as follows. Its objects are $\mathcal{D}(Y) = \mathcal{D}(Y \cap Z^\ell) = \mathcal{D}(Y \cap Z^r) = H$, $\mathcal{D}(Z^\ell) = F^\ell$ and $\mathcal{D}(Z^r) = F^r$. Its morphisms are $\gamma^\ell : F^\ell \to H$, $\gamma^r : F^r \to H$ and the identity maps; see the diagram following (5.11).

Next we define a morphism of functors $\underline{\eta} : \mathcal{D} \to \mathcal{S}|_{\mathcal{C}}$ by

$$\underline{\eta}_{Z^\ell} = \pi_{Z^\ell}\eta^\ell, \quad \underline{\eta}_{Z^r} = \pi_{Z^r}\eta^r, \quad \underline{\eta}_Y = \eta_H, \quad \underline{\eta}_{Y \cap Z^\ell} = \pi_{s^\ell}\eta_H, \quad \underline{\eta}_{Y \cap Z^r} = \pi_{s^r}\eta_H.$$

We need to check that $\underline{\eta}$ is a morphism of functors. That reduces to the following equalities:

$$\pi_{s^\ell}\eta_H\gamma^\ell(f) = \pi_{s^\ell}\pi_{Z^\ell}\eta^\ell(f), \quad \text{for all } f \in F^\ell,$$

$$\pi_{s^r}\eta_H\gamma^r(f) = \pi_{s^r}\pi_{Z^r}\eta^r(f), \quad \text{for all } f \in F^r.$$

Let us only verify the first equality, since the second equality can be verified in a similar way. For any $f \in F^\ell$, we have

$$\pi_{s^\ell}\eta_H\gamma^\ell(f) = \pi_{s^\ell}\pi_V(\eta^\ell)^\varepsilon(f) = \pi_{s^\ell}\eta^\ell(f) = \pi_{s^\ell}\pi_{Z^\ell}\eta^\ell(f).$$

Define a morphism of functors $\underline{\theta} : G^{\mathcal{C}} \to \mathcal{D}$ by

$$\underline{\theta}_{Z^\ell} = \theta^\ell\alpha^\ell, \quad \underline{\theta}_{Z^r} = \theta^r\alpha^r, \quad \underline{\theta}_Y = \gamma^\ell\theta^\ell\alpha^\ell, \quad \underline{\theta}_{Y\cap Z^\ell} = \gamma^\ell\theta^\ell\alpha^\ell, \quad \underline{\theta}_{Y\cap Z^r} = \gamma^r\theta^r\alpha^r.$$

Let us verify that $\underline{\theta}$ is a morphism of functors. It suffices to check

$$\gamma^\ell\theta^\ell\alpha^\ell(g) = \gamma^r\theta^r\alpha^r(g) \quad \text{for any } g \in G.$$

Indeed, since $\pi_x\iota^\ell\alpha^\ell(g) = \pi_x\iota^r\alpha^r(g)$, one has that $\pi_x\eta^\ell\theta^\ell\alpha^\ell(g) = \pi_x\eta^r\theta^r\alpha^r(g)$, and hence $\pi_x\iota^\varepsilon(\theta^\ell\alpha^\ell(g), -\theta^r\alpha^r(g)) = 0$. Since $(F^\ell \oplus F^r, V, x, \gamma, \eta_H, H)$ is a simple interpolant, $\ker(\pi_x\iota^\varepsilon) = \ker(\gamma)$. Thus from $\gamma(\theta^\ell\alpha^\ell(g), -\theta^r\alpha^r(g)) = 0$ we obtain that $\gamma^\ell\theta^\ell\alpha^\ell(g) = \gamma^r\theta^r\alpha^r(g)$.

Let us verify that

$$G^{\mathcal{C}} \xrightarrow{\underline{\theta}} \mathcal{D} \xrightarrow{\underline{\eta}} \mathcal{S}|_{\mathcal{C}} \tag{5.6}$$

is an interpolant of G. First we verify

$$\eta^H\gamma^\ell\theta^\ell\alpha^\ell(g) = \pi_V\alpha(g)$$

where $\alpha : G \to \mathcal{S}[x^\ell, x^r]$ is the homomorphism induced by $\iota^\ell\alpha^\ell$ and $\iota^r\alpha^r$. Let us verify it pointwise. If $z \in [s^\ell, x]$, then

$$\pi_z\pi_V\alpha(g) = \pi_z\iota^\ell\alpha^\ell(g) = \pi_z\eta^\ell\theta^\ell\alpha^\ell(g) = \pi_z\pi_V(\eta^\ell)^\varepsilon\theta^\ell\alpha^\ell(g) = \pi_z\eta^H\gamma^\ell\theta^\ell\alpha^\ell(g).$$

Using $\gamma^\ell\theta^\ell\alpha^\ell(g) = \gamma^r\theta^r\alpha^r(g)$, a similar argument also shows that

$$\pi_z\pi_V\alpha(g) = \pi_z\eta^H\gamma^\ell\theta^\ell\alpha^\ell(g)$$

for any $z \in [x, s^r]$. Next we verify the kernel conditions corresponding to $Z^\ell = [x^\ell, s^\ell]$, $Z^r = [s^r, x^r]$ and $Y = [s^\ell, s^r]$:

$$\ker(\theta^\ell\alpha^\ell) = \ker(\pi_{z^\ell}\pi_{Z^\ell}\iota^\ell\alpha^\ell) = \ker(\pi_{z^\ell}\iota^\ell\alpha^\ell) \tag{5.7}$$

$$\ker(\theta^r\alpha^r) = \ker(\pi_{z^r}\pi_{Z^r}\iota^r\alpha^r) = \ker(\pi_{z^r}\iota^r\alpha^r) \tag{5.8}$$

$$\ker(\gamma^\ell\theta^\ell\alpha^\ell) = \ker(\pi_x\iota^\ell\alpha^\ell) = \ker(\pi_x\iota^r\alpha^r) = \ker(\gamma^r\theta^r\alpha^r) \tag{5.9}$$

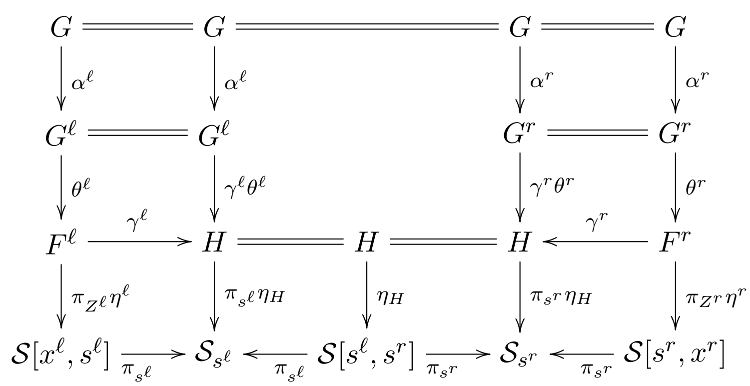
The kernel condition for the interpolant (5.2) amounts to $\ker(\theta^\ell) = \ker(\pi_{z^\ell}\iota^\ell)$ and this clearly implies (5.7). Similarly, (5.8) follows from (5.3). Next we observe that the first half of (5.9) is equivalent to $\ker(\gamma^\ell\theta^\ell\alpha^\ell) = \ker(\pi_x\eta^\ell\theta^\ell\alpha^\ell)$, since $\iota^\ell = \eta^\ell\theta^\ell$. The desired equality follows since $\ker(\gamma^\ell) = \ker(\pi_x\eta^\ell)$ from the construction of the simple interpolant for ι^ε. One argues in a similar way to justify the second half of (5.9).

In the same way, one verifies that the restrictions of $\underline{\eta}$ to $\mathcal{C}^{\ell}_{\mathrm{new}} := \mathcal{C} \cap [x^{\ell}, x]$ and $\mathcal{C}^{r}_{\mathrm{new}} := \mathcal{C} \cap [x, x^{r}]$ give two interpolants

$$(G^{\ell})^{\mathcal{C}^{\ell}_{\mathrm{new}}} \xrightarrow{\underline{\theta}} \mathcal{D}|_{\mathcal{C}^{\ell}_{\mathrm{new}}} \xrightarrow{\underline{\eta}} \mathcal{S}|_{\mathcal{C}^{\ell}_{\mathrm{new}}}\ , \tag{5.10}$$

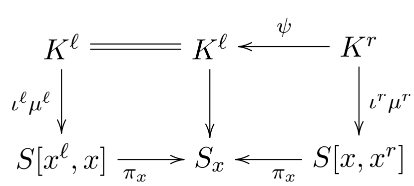
$$(G^{r})^{\mathcal{C}^{r}_{\mathrm{new}}} \xrightarrow{\underline{\theta}} \mathcal{D}|_{\mathcal{C}^{r}_{\mathrm{new}}} \xrightarrow{\underline{\eta}} \mathcal{S}|_{\mathcal{C}^{r}_{\mathrm{new}}}\ . \tag{5.11}$$

These two interpolants are assembled together in the following diagram:



This proves the first part of the lemma. For the second part, because of the locality of our construction, we may also assume that the interpolants (5.2) and (5.3) are simple.

Suppose now that $\mathcal{D}_1$ is a diagram as in the statement. The morphism of functors $\underline{\eta}_1 : \mathcal{D}_1 \to \mathcal{S}|_{\mathcal{C}_1}$ is illustrated by the commutative diagram



Since $\pi_x \iota^{\ell} \mu^{\ell} \psi(g^r) = \pi_x \iota^r \mu^r (g^r)$ for any $g^r \in K^r$, we obtain that $\pi_x \eta^{\ell} \theta^{\ell} \mu^{\ell} \psi(g^r) = \pi_x \eta^r \theta^r \mu^r (g^r)$ for any $g^r \in K^r$, and hence $\pi_x((\eta^{\ell})^{\varepsilon} \oplus (\eta^r)^{\varepsilon})(\theta^{\ell} \mu^{\ell} \psi(g^r), -\theta^r \mu^r (g^r)) = 0$. Since $(F^{\ell} \oplus F^r, V, x, \gamma, \eta_H, H)$ is a simple interpolant, we have that $\gamma(\theta^{\ell} \mu^{\ell} \psi(g^r), -\theta^r \mu^r (g^r)) = 0$, and hence $\gamma^{\ell} \theta^{\ell} \mu^{\ell} \psi = \gamma^r \theta^r \mu^r$. Therefore, we have the following commutative diagram:

$$
\begin{array}{ccccccccc}
K^\ell & = & K^\ell & \xleftarrow{\quad\psi\quad} & & & K^r & = & K^r \\
\downarrow{\scriptstyle \theta^\ell\mu^\ell} & & \downarrow{\scriptstyle \gamma^\ell\theta^\ell\mu^\ell} & & & & \downarrow{\scriptstyle \gamma^r\theta^r\mu^r} & & \downarrow{\scriptstyle \theta^r\mu^r} \\
F^\ell & \xrightarrow{\gamma^\ell} & H & = & H & = & H & \xleftarrow{\gamma^r} & F^r \\
\downarrow{\scriptstyle \pi_{Z^\ell}\eta^\ell} & & \downarrow{\scriptstyle \pi_{s^\ell}\eta_H} & & \downarrow{\scriptstyle \eta_H} & & \downarrow{\scriptstyle \pi_{s^r}\eta_H} & & \downarrow{\scriptstyle \pi_{Z^r}\eta^r} \\
\mathcal{S}[x^\ell, s^\ell] & \xrightarrow[\pi_{s^\ell}]{} & \mathcal{S}_{s^\ell} & \xleftarrow[\pi_{s^\ell}]{} & \mathcal{S}[s^\ell, s^r] & \xrightarrow[\pi_{s^r}]{} & \mathcal{S}_{s^r} & \xleftarrow[\pi_{s^r}]{} & \mathcal{S}[s^r, x^r]
\end{array}
$$

In fact, the diagram expands to the larger diagram

$$
\begin{array}{ccccccccccccc}
K^\ell & = & K^\ell & = & K^\ell & = & K^\ell & \xleftarrow{\psi} & K^r & = & K^r & = & K^r \\
\downarrow{\scriptstyle \theta^\ell\mu^\ell} & & \downarrow{\scriptstyle \gamma^\ell\theta^\ell\mu^\ell} & & \downarrow{\scriptstyle \gamma^\ell\theta^\ell\mu^\ell} & & \downarrow{\scriptstyle \gamma^\ell\theta^\ell\mu^\ell} & & \downarrow{\scriptstyle \gamma^r\theta^r\mu^r} & & \downarrow{\scriptstyle \gamma^r\theta^r\mu^r} & & \downarrow{\scriptstyle \theta^r\mu^r} \\
F^\ell & \xrightarrow{\psi^\ell} & H & = & H & = & H & = & H & = & H & \xleftarrow{\psi^r} & F^r \\
\downarrow{\scriptstyle \pi_{Z^\ell}\eta^\ell} & & \downarrow{\scriptstyle \pi_{s^\ell}\eta_H} & & \downarrow{\scriptstyle \pi_{[s^\ell,x]}\eta_H} & & \downarrow{\scriptstyle \pi_x\eta_H} & & \downarrow{\scriptstyle \pi_{[x,s^r]}\eta_H} & & \downarrow{\scriptstyle \pi_{s^r}\eta_H} & & \downarrow{\scriptstyle \pi_{Z^r}\eta^r} \\
\mathcal{S}[x^\ell, s^\ell] & \xrightarrow[\pi_{s^\ell}]{} & \mathcal{S}_{s^\ell} & \xleftarrow[\pi_{s^\ell}]{} & \mathcal{S}[s^\ell, x] & \xrightarrow[\pi_x]{} & \mathcal{S}_x & \xleftarrow[\pi_x]{} & \mathcal{S}[x, s^r] & \xrightarrow[\pi_{s^r}]{} & \mathcal{S}_{s^r} & \xleftarrow[\pi_{s^r}]{} & \mathcal{S}[s^r, x^r]
\end{array}
$$

Thus we have a morphism $\mu_1 : \mathcal{D}_1 \to \mathcal{D}$ such that $\underline{\eta}_1 = \underline{\eta} \circ \underline{\mu}_1$. Arguing as in the first part of the proof one checks that the columns of the previous diagram corresponding to nondegenerate intervals are interpolants. Here one works with the elementary cover $\mathcal{C}$ that consists of $[x^\ell, s^\ell], [s^\ell, x], [x, s^r]$ and $[s^r, x^r]$. This proves the second part of the lemma. $\square$

Lemma 5.6. *Let G be a finitely generated dimension group with a strictly positive morphism $\iota : G \to \mathcal{S}[a, b]$. Then there is an open interpolant*

$$
\begin{array}{ccc}
G^{\mathcal{C}} & \xrightarrow{\quad\iota\quad} & S|_{\mathcal{C}} \\
& {\scriptstyle \theta}\searrow \quad \nearrow{\scriptstyle \eta} & \\
& \mathcal{D} & .
\end{array}
$$

Proof. By applying Lemma 5.3 and using the compactness of $[a, b]$ we find points $a = y_0 < \cdots < y_m = b$ and open simple interpolants

$$I(k) = (G, Z^k, z^k, \theta^k, \eta^k, F^k)$$

for $\pi_{Z^k}\iota$, where $Z^k = [y_{k-1}, y_k]$, $k = 1, ..., m$. Then we glue together the simple interpolants $I(k)$, $k = 1, ..., m$ by applying the first part of Lemma 5.5. $\square$

Theorem 5.7. *Let $\mathcal{S}$ be a continuous and strictly positive sheaf of countable ordered groups on $[0, 1]$. If the stalks of $\mathcal{S}$ are dimension groups, then $\mathcal{S}$ is the K_0-sheaf of a continuous field of AF-algebras over $[0, 1]$ with $[1_A] = \nu$ for a given nonzero element $\nu \in \mathcal{S}[0, 1]^+$.*

Proof. We shall construct inductively a sequence of elementary diagrams $(\mathcal{D}_n)$ with covers $(\mathcal{C}_n)$ of $[0,1]$ and morphisms $\Psi_{n,n+1} : \mathcal{D}_n \to \mathcal{D}_{n+1}$ and $\Psi_{n,\infty} : \mathcal{D}_n \to \mathcal{S}|_{\mathcal{C}_n}$ $\Psi_{n+1,\infty} \circ \Psi_{n,n+1} = \Psi_{n,\infty}$ such that the induced map

$$\Lambda : \varinjlim (\widehat{\mathcal{D}}_n, \widehat{\Psi}_{n,n+1}) \to S$$

is an isomorphism of sheaves. Then one lifts each $\mathcal{D}_n$ to a diagram $\mathcal{D}'_n$ of finite dimensional C*-algebras and each morphism $\Psi_{n,n+1}$ to $\Psi'_{n,n+1} \in \mathrm{Hom}_{\mathcal{D}'_n}(\mathcal{D}'_n, \mathcal{D}'_{n+1})$, as in Subsection 3.1. In other words, $\mathrm{K}_0(\mathcal{D}'_n) = \mathcal{D}_n$ and $\mathrm{K}_0(\Psi'_{n,n+1}) = \Psi_{n,n+1}$. The unital morphisms $\gamma_{i,j}$ (of $\mathcal{D}'_n$) are liftings of strictly positive maps and hence they are injective. Therefore the pullbacks A_n of $\mathcal{D}'_n$ are unital continuous fields. By continuity of K-theory it follows that the K_0-sheaf of $\varinjlim(A_n, \widehat{\Psi}'_{n,n+1})$ is isomorphic to $\mathcal{S}$.

Let us now turn to the construction of $\mathcal{D}_n$, $\Psi_{n,n+1}$ and $\Psi_{n,\infty}$. By an elementary set theoretic argument, there is a sequence of rational intervals $[a_n, b_n]$ in $[0,1]$ and for each $n \geq 1$ a positive element $s_n \in \mathcal{S}[a_n, b_n]$ such that for each interval $[a,b]$ with rational endpoints and each positive element $s \in \mathcal{S}[a,b]$, there is $n \geq 1$ such that $[a,b] = [a_n, b_n]$, and $s = s_n$.

The diagrams $\mathcal{D}_n$ and the various morphisms are constructed inductively such that $[a_n, b_n]$ is a union of components of $\mathcal{C}_n$, all the components of $\mathcal{C}_n$ have length $\leq 1/n$, and for each n and each nondegenerate interval $X \in \mathcal{C}_{n+1}$, the commutative diagram

(5.12)
$$\begin{array}{ccc} \mathcal{D}_n^{\mathcal{C}_{n+1}}(X) & \xrightarrow{\ \Psi_{n,\infty}\ } & \mathcal{S}|_X \\ & \searrow^{\Psi_{n,n+1}} \quad {}^{\Psi_{n+1,\infty}}\nearrow & \\ & \mathcal{D}_{n+1}(X) & \end{array}$$

is an interpolant. Moreover we arrange that s_{n+1} is in the image of $\widehat{\Psi}_{n+1,\infty}$.

To construct $\mathcal{D}_1$ we apply Lemma 5.6 to $\iota_1 : G = \mathbb{Z}\, s_1 \hookrightarrow \mathcal{S}[a_1, b_1]$ and find an interpolant of ι_1 on $[a_1, b_1]$. Then we extend the morphism $\underline{\eta}$ that appears in this interpolant to a morphism $\Psi_{1,\infty} : \mathcal{D}_1 \to \mathcal{S}$ on $[0,1]$ by gluing of simple interpolants. It is then clear that $s_1 \in \mathrm{Image}(\widehat{\Psi}_{1,\infty})$ by the first part of Lemma 5.5.

Suppose now that $\mathcal{D}_1$,...,$\mathcal{D}_n$ and the morphisms $\Psi_{j,\infty}$, $1 \leq j \leq n$ and $\Psi_{j,j+1}$, $1 \leq j < n$ were constructed. We must construct $\mathcal{D}_{n+1}$ and the morphisms $\Psi_{n,n+1}$ and $\Psi_{n+1,\infty}$.

Let $\mathcal{C}'_n$ be a refinement of $\mathcal{C}_n$ such that $[a_{n+1}, b_{n+1}]$ is a union of components of $\mathcal{C}'_n$. Let us denote by X^k the components of $\mathcal{C}'_n$ which are not points and set $F^k = \mathcal{D}_n^{\mathcal{C}'_n}(X^k)$. For each X^k, denote by $\eta^k : F^k \to \mathcal{S}(X^k)$ the corresponding component of $\Psi_{n,\infty}$. For each k, define elements $t^k \in \mathcal{S}(X^k)$ by $t^k = s_{n+1}|_{X^k}$ if $[a_{n+1}, b_{n+1}] \cap X^k = X^k$ and $t^k = 0$ otherwise. For each k consider the map $\iota^k : G^k := \mathbb{Z}t^k \oplus F^k \to \mathcal{S}(X^k)$, $\iota^k(mt^k, f^k) = mt^k + \eta^k(f^k)$.

Applying Lemma 5.6, we obtain an interpolant

$$\begin{array}{ccc} (G^k)^{\mathcal{C}_{n+1,k}} & \xrightarrow{\quad \iota^k \quad} & S|_{\mathcal{C}_{n+1,k}} \\ & {\scriptstyle \underline{\theta}_{n+1,k}} \searrow \quad \nearrow {\scriptstyle \underline{\eta}_{n+1,k}} & \\ & \mathcal{D}_{n+1,k} & \end{array} \quad .$$

By Lemma 5.5, the above interpolants based on the diagrams $\{\mathcal{D}_{n+1,k}\}$ can be glued together to get an elementary diagram $\mathcal{D}_{n+1}$ on $[0,1]$ and a morphism

$$\Psi_{n+1,\infty} : \mathcal{D}_{n+1} \to S|_{\mathcal{C}_{n+1}}. \tag{5.13}$$

Moreover, by the second part of Lemma 5.5, there is a morphism $\Psi_{n,n+1} : \mathcal{D}_n^{\mathcal{C}_{n+1}} \to \mathcal{D}_{n+1}$ such that $\Psi_{n+1,\infty} \circ \Psi_{n,n+1} = \Psi_{n,\infty}$ and such that (5.12) is an interpolant.

We must show that the map Λ is bijective. The surjectivity is verified as follows. It suffices to show that for any n, one has that $s_n \in \mathrm{Im}(\Lambda)$. But this is clear since $s_n \in \mathrm{Im}(\widehat{\Phi}_{n,\infty})$ by construction.

Let us now verify that Λ is injective. Let $s \in \mathcal{D}_n[a,b]$ be such that $\Lambda(s) = 0$. We are going to show that for each $x \in [a,b]$ there is a neighbourhood W of x and there is $m \geq n$ such that $\widehat{\Psi}_{n,m}(s)$ vanishes on W in $\widehat{\mathcal{D}}_m(W)$. From the compactness of $[a,b]$ this will eventually imply that $\widehat{\Psi}_{n,m}(s) = 0$ in $\widehat{\mathcal{D}}_m[a,b]$ for some $m \geq n$. If s is as above, by continuity of $\mathcal{S}$ we may assume that there is $\delta > 0$ such that s extends to $V = [a-\delta, b+\delta] \cap [0,1]$ and that $\Lambda(s)$ vanishes on V. Let $x \in [a,b]$. Since the intervals of $\mathcal{C}_n$ have length $\leq 1/n$, after increasing n and replacing s by $\widehat{\Psi}_{n,m}(s)$ if necessary, we may arrange that $[x-\varepsilon, x+\varepsilon] \subset X^k \cup X^{k+1} \subset [x-\delta, x+\delta]$, for some $0 < \varepsilon < \delta$, where X^k and X^{k+1} are consecutive intervals in the elementary diagram $\mathcal{C}_n$ of $\mathcal{D}_n$.

The restriction of s to $X^k \cup X^{k+1}$ is of the form (f^k, f^{k+1}) , where $f^i \in F^i = \mathcal{D}_m(X^i)$, $i = k, k+1$. Let $\eta^i : F^i \to S(X^i)$, $i = k, k+1$, denote the corresponding components of $\Psi_{n,\infty}$. Then $\widehat{\Psi}_{n,\infty}(s)$ is equal to the element of $\mathcal{S}(X^k \cup X^{k+1})$ given by the pair $(\eta^k(f^k), \eta^{k+1}(f^{k+1})) \in \mathcal{S}(X^k) \oplus \mathcal{S}(X^{k+1})$, and hence $\eta^i(f^i) = 0$, $i = k, k+1$. Since (5.12) is an interpolant, so also is

$$\begin{array}{ccc} \mathcal{D}_n(X^i) & \xrightarrow{\qquad \eta^i \qquad} & S|_{X^i} \\ & {\scriptstyle \Psi_{n,n+1}} \searrow \quad \nearrow {\scriptstyle \Psi_{n+1,\infty}} & \\ & \mathcal{D}_{n+1}|_{\mathcal{C}_{n+1} \cap X^i} & \end{array}$$

$i = k, k+1$, since $\mathcal{C}_{n+1}$ is finer than $\mathcal{C}_n$. Therefore $\Psi_{n,n+1}(f^i) = 0$ for $i = k, k+1$ and hence $\Psi_{n,n+1}(s)$ is zero on the interval $X^k \cup X^{k+1}$ which contains $[x-\varepsilon, x+\varepsilon]$.

Therefore, the map

$$\Lambda : \varinjlim\, (\widehat{\mathcal{D}}_n, \widehat{\Psi}_{n,n+1}) \to S$$

is an isomorphism. For a given nonzero element $\nu \in \mathcal{S}[0,1]^+$, we can assume that there are nonzero elements $\nu_n \in \widehat{\mathcal{D}}_n[0,1]^+$ such that the sequence (ν_n) is sent to ν. Then, one lifts each diagram $\mathcal{D}_n$ to an elementary diagram $\mathcal{D}'_n$ of finite dimensional C*-algebras with $[1_{A_n}] = \nu_n$ where A_n is the pullback of $\mathcal{D}'_n$, and lifts each morphism $\Psi_{n,n+1}$ to $\Psi'_{n,n+1} \in \mathrm{Hom}_{\mathcal{D}'_n}(\mathcal{D}'_n, \mathcal{D}'_{n+1})$. Then, $\varinjlim(A_n, \widehat{\Psi}'_{n,n+1})$ is a continuous field of AF-algebras, and satisfies the requirements of the theorem. □

5.3. **K_0-sheaves of one-parameter continuous fields of certain Kirchberg algebras.** Let $\mathcal{C}$ denote the class of Kirchberg algebras satisfying the UCT with torsion free K_0-group and trivial K_1-group. The separable unital continuous fields of C*-algebras on $[0,1]$ with fibres in $\mathcal{C}$ are shown to be classified by their K_0-sheaves pointed by the class of the unit; see [2].

In analogy with Theorem 5.7, we have the following Effros-Handelman-Shen type theorem for this class of continuous fields:

Theorem 5.8. *A pointed sheaf $\mathcal{S}$ of countable abelian groups over $[0,1]$ is isomorphic to the K_0-sheaf of a continuous field over $[0,1]$ of Kirchberg algebras with trivial K_1-group if and only if $\mathcal{S}$ is continuous. If, moreover, the stalks of $\mathcal{S}$ are torsion free abelian groups, then the fibres of A can be chosen to be in the class $\mathcal{C}$.*

Proof. The proof of the first part of the theorem is contained implicitly in the proof of Theorem 5.7. The finite dimensional algebras are replaced by Kirchberg algebras and the ordered abelian groups are replaced by abelian groups. The second part of the theorem follows from the classification theorem of Kirchberg and Phillips. □

6. Examples

6.1. **Fields whose fibres are matrix algebras.** Here, we use Theorem 5.7 to give a concise classification of the one-parameter unital separable continuous fields of matrix algebras.

Let us call a function $f : [0,1] \to \mathbb{N}^* = \{1, 2, 3, ...\}$ *d-continuous* if for each $x \in [0,1]$ the set $\{y \in [0,1] : f(y) \text{ is divisible by } f(x)\}$ is open. In other words f is d-continuous if and only if it is continuous with respect to the (non-Hausdorff) topology of $\mathbb{N}^*$ with basis $\{n\mathbb{N}^* : n \in \mathbb{N}^*\}$. For a continuous field A of matrix algebras let us denote by f_A the dimension function $f(x) = \dim(A(x))$, where $\dim(A(x))$ denotes the size of the matrix algebra $A(x)$, i.e., $\dim(M_n(\mathbb{C})) = n$ for any natural number. We shall use the same notation in the rest of the paper.

Theorem 6.1. *The map $A \mapsto f_A$ is a bijection from the isomorphism classes of one-parameter unital separable continuous fields of matrix algebras to the set of all d-continuous functions $[0,1] \to \mathbb{N}^*$.*

Proof. Let $\mathcal{V}$ be the family of non-degenerate subintervals of $[0,1]$ and let $(f_U)_{U\in\mathcal{V}}$ be a family of functions $f_U : U \to \mathbb{N}^*$ satisfying the following conditions:

(i) If $V \subset U$, then $f_U|_V = nf_V$ for some integer $n > 0$.

(ii) Each $x \in [0,1]$ has a neighbourhood U such that $f_U(x) = 1$.

(iii) If $U = [a,b]$, $V = [b,c]$ and $pf_U(b) = qf_V(b)$ for some positive integers p, q and the function g on $[a,c]$ is defined by

$$g(x) = \begin{cases} pf_U(x) & \text{if } x \in [a,b] \\ qf_V(x) & \text{if } x \in [b,c] \end{cases}$$

then $g = nf_{U\cup V}$ for some integer $n > 0$. Using Proposition 2.4 one verifies immediately that the ordered groups $\mathcal{S}(U) = \mathbb{Z}f_U$, and the maps $\phi^U_V : \mathcal{S}(U) \to \mathcal{S}(V)$, $V \subset U$, $\phi^U_V(kf_U) = kf_U|_V$, form a strictly positive continuous sheaf. Using Proposition 4.5 one shows that if A is a continuous field as in the statement of the theorem, then its K_0-sheaf is isomorphic to a sheaf $\mathcal{S}$ as above with $[1_A]$ corresponding to $nf_{[0,1]}$ for some $n \in \mathbb{N}^*$. This isomorphism is obtained by considering the images of $\mathrm{K}_0(A(U))$ in $\prod_{x\in U} \mathrm{K}_0(A(x)) = \prod_{x\in U} \mathbb{Z}$. Conversely, if $\mathcal{S}$ is as above and $n \geq 1$ then by Theorem 5.7, $\mathcal{S}$ is isomorphic to the K_0-sheaf associated to a unital separable one-parameter continuous field A of matrix algebras with $[1_A] = nf_{[0,1]}$.

In the second part of the proof we show that a sheaf $\mathcal{S}$ defined by a family $(f_U)_{U\in\mathcal{V}}$ satisfying the conditions (i), (ii) and (iii) is uniquely determined by the d-continuous function $\alpha := f_{[0,1]}$. For $x \in [0,1]$ let U be a neighbourhood of x given by (ii) such that $f_U(x) = 1$. By (i) we have that $\alpha|_U = nf_U$ for some integer n and hence that $\alpha|_U = \alpha(x)f_U$. It follows that $\alpha(x)$ divides all numbers $\alpha(y)$ for $y \in U$ and so α is d-continuous. For $U \in \mathcal{V}$ let d_U denote the greatest common divisor of the elements of the set $\alpha(U)$, $d_U = \gcd(\alpha(U))$. We assert that

$$f_U = \frac{\alpha|_U}{d_U}. \tag{6.1}$$

First we show that $d_{[0,1]} = 1$. Set $d_{[0,1]} = d$. Arguing as above we find $0 = a_1 < a_2 < ... < a_m = 1$ and points $x_i \in U_i = [a_{i-1}, a_i]$ such that $\alpha|_{U_i} = \alpha(x_i)f_{U_i} = dk_if_{U_i}$ for some $k_i \in \mathbb{N}^*$. Since $k_if_{U_i}(a_i) = k_{i+1}f_{U_{i+1}}(a_i) = \frac{1}{d}\alpha(a_i)$, it follows by the condition (iii) that the sections $k_if_{U_i}$ glue together to a global section g which satisfies $\alpha = dg$. Since α is the generator of $\mathcal{S}[0,1] = \mathbb{Z}\alpha$ we must have $d = 1$. Arguing in a similar way one shows that $\gcd(f_U(U)) = 1$ for all $U \in \mathcal{V}$. By the condition (i) for each U there is n such that $\alpha|_U = nf_U$. Therefore $d_U = \gcd(\alpha(U)) = \gcd(nf_U(U)) = n$ which proves (6.1).

In the last part of the proof we show that each d-continuous function $f : [0,1] \to \mathbb{N}^*$ defines a sheaf $\mathcal{S}$ as above given by a family of functions $(f_U)_{U\in\mathcal{V}}$ satisfying the conditions (i), (i) and (iii). To that purpose we set $\alpha = \frac{f}{\gcd(f[0,1]}$ and let the functions f_U be defined by (6.1). Equivalently, $f_U = \frac{f}{\gcd(f(U)}$. If $V \subset U$ then d_U divides d_V and hence (i) holds with $n = \frac{d_V}{d_U}$. To verify (ii) let us note that for $x \in [0,1]$, by the d-continuity of α there is a neighbourhood U of x such that $\alpha(x)$ divides all the elements in $\alpha(U)$ and hence $\alpha(x) = d_U$.

Therefore $f_U(x) = 1$. To verify (iii), suppose that $pf_U(b) = qf_V(b)$ for some $p, q \in \mathbb{N}^*$. By (6.1) this implies that $pd_V = qd_U$ and hence $pr = qs$ where $d_V = rd_{U\cup V}$ and $d_U = sd_{U\cup V}$, since $d_{U\cup V} = \gcd\{d_U, d_V\}$. It follows that s divides p, since r and s are relatively prime and hence $n = \frac{p}{s} \in \mathbb{N}^*$. Therefore if g is given by the same formula as in (iii), then for $x \in U$

$$g(x) = p\frac{\alpha|_U(x)}{d_U} = \frac{p}{s}\frac{\alpha|_{U\cup V}(x)}{d_{U\cup V}} = nf_{U\cup V}(x)$$

and similarly $g(x) = nf_{U\cup V}(x)$ for $x \in V$.

Given f_A as in the statement we set $f_{[0,1]} = \frac{f_A}{\gcd f_A[0,1]}$ and then construct the corresponding sheaf. Let us note that $[1_A]$ corresponds to the function f_A in $\mathrm{K}_0(A) = \mathbb{Z}f_{[0,1]}$. □

Example 6.2. Here we construct explicitly a continuous field A corresponding to a given d-continuous function $f : [0,1] \to \mathbb{N}^*$. The values assumed by f form a sequence $n_1, n_2, ...$ such that n_k divides n_{k+1}. Each set $E_k := \{x \in X : \ f(x) \leq n_k\}$ is closed in $[0,1]$ and $E_k \subset E_{k+1}$. Fix unital embeddings $M_{n_k} \subset M_{n_{k+1}}$ and define an increasing sequence (A_k) of unital continuous fields on $[0,1]$ as follows. Set $A_1 = C([0,1], M_{n_1})$, $A_2 = \{f \in C([0,1], M_{n_2}) : f(x) \in M_{n_1}, \forall x \in E_1\}$,...,

$$A_k = \{f \in C([0,1], M_{n_k}) : f(x) \in M_{n_i}, \forall x \in E_i \setminus E_{i-1}, i = 1, 2, ..., k\},$$

with the convention that $E_0 = \varnothing$. Then the completion of $\bigcup_{k=1}^{\infty} A_k$ is a unital continuous field of matrix algebras with dimension function $f_A = f$. It is also clear that the isomorphism classes of unital separable continuous field of matrix algebras over $[0,1]$ are in bijection with pairs of sequences $(E_k)_k$, $(n_k)_k$ of the same length (finite or infinite) where $E_1 \subset E_2 \subset \dots$ are closed sets whose union is equal to $[0,1]$ and each number $n_k \in \mathbb{N}^*$ divides its successor n_{k+1}.

A dimension function f_A is not necessarily bounded. Indeed if we set $I_0 = [\frac{1}{2}, 1]$ and $I_n = [\frac{1}{2^{n+1}}, \frac{1}{2^n})$ for $n \geq 1$, then the function f_A defined by

$$f_A(x) = \begin{cases} 2^n & \text{if } x \in I_n, \\ 1 & \text{if } x = 0 \end{cases}$$

is d-continuous and unbounded. With the notation from above, $E_k = I_0 \cup ... \cup I_{k-1}$. Nevertheless, f_A must be constant on some open set as is shown by the following proposition.

Proposition 6.3. *For any unital separable continuous field of matrix algebras over a compact metrizable space X, there is a closed subspace U of X with nonempty interior such that the restriction of the field to U is the trivial field.*

Proof. Let A be a field as in the statement with unit e. Define $f : X \to \mathbb{N}^*$ by $f(x) = \mathrm{rank}(e(x)) = \dim(A(x))$. We assert that this map is d-continuous. Indeed, fix x and set $n = \mathrm{rank}(e(x))$. If q_x is a minimal projection of $A(x)$, then q_x lifts to a projection $q \in A(U)$ for some closed neighbourhood U of x such that $\pi_U(e)$ is equivalent to $n \cdot q$ in matrices over $A(U)$ and so $f(y)$ is divisible by $n = f(x)$ for all $x \in U$.

For any natural number n, define $E_n := \{x \in X : f(x) \le n\}$. The set E_n is closed by the d-continuity of f and $X = \bigcup_{n=1}^{\infty} E_n$. It follows by the Baire category theorem that there is n such that $(E_n)^\circ \neq \varnothing$. Consequently, after restricting A to a closed subspace of X with nonempty interior, we may assume that the function f is bounded. If $n = \max f(X)$, then the set $F_n := \{x \in X : f(x) = n\}$ is nonempty and open since we also have $F_n = \{x \in X : f(x) > n-1\}$. Let Y be a closed subspace of F_n with nonempty interior. Then $A(Y)$ is a separable unital continuous field with all fibres isomorphic to M_n and therefore it is locally trivial by [9]. □

6.2. **Fields whose fibres are unital hereditary sub-C*-algebras of $\mathcal{O}_\infty$.** As a counterpart to continuous fields of matrix algebras in the stably finite case, we consider a special class of unital continuous fields of unital hereditary sub-C*-algebras of $\mathcal{O}_\infty$, where $\mathcal{O}_\infty$ is the Cuntz algebra with $\mathrm{K}_0(\mathcal{O}_\infty) \cong \mathbb{Z}$. For any integer n, let p_n be a nonzero projection in $\mathcal{O}_\infty$ such that $[p_n] = n$, and denote by $\mathcal{M}_n$ the unital hereditary sub-C*-algebra $p_n \mathcal{O}_\infty p_n$. Note that the $(\mathrm{K}_0(\mathcal{M}_n), [1]_0)$ is then isomorphic to $(\mathbb{Z}, n)$. Up to isomorphism, these C*-algebras are the only nonzero unital hereditary sub-C*-algebras of $\mathcal{O}_\infty$. Since $\mathrm{K}_0(\mathcal{O}_\infty) \cong \mathbb{Z}$, we can again represent the K_0-sheaf of a continuous field C*-algebra as integer valued functions, and have the following result of Effros-Handelman-Shen type. For each U, let $\mathcal{S}(U)$ be a set of maps from U to $\mathbb{Z}$ satisfying the following conditions:

(1) If $V \subset U$ and $f \in \mathcal{F}(U)$, then $f|_V \in \mathcal{F}(V)$;
(2) For any $x \in [0,1]$ there is a neighbourhood $U \in \mathcal{U}$ of x and there is $f \in \mathcal{F}(U)$ such that $f(x) = 1$;
(3) For any $U \in \mathcal{U}$ and $f \in \mathcal{F}(U)$, the null set of f, $\mathrm{null}(f) = \{x \in U : f(x) = 0\}$ is open in U;
(4) $\mathcal{F}[a,b] \cong \{(f,g) \in \mathcal{F}[a,c] \oplus \mathcal{F}[c,b] : f(c) = g(c)\}$, for $a < c < b$.

Corollary 6.4. *A sheaf $\mathcal{F}$ on $[0,1]$ of countable abelian groups consisting of integer valued functions is isomorphic to the K_0-sheaf of a unital continuous field of hereditary sub-C*-algebras of $\mathcal{O}_\infty$ if and only if it satisfies the conditions (1) through (4) from above.*

Proof. We have seen earlier that $\mathcal{F}$ is a continuous sheaf and that all continuous sheaves with stalk $\mathbb{Z}$ are of this form, up to isomorphism; see Proposition 2.4. Thus the result follows from Theorem 5.8. □

Comparing the corollary above with Theorem 5.8, we see that the nonzero integer valued functions associated with a continuous field of unital hereditary sub-C*-algebras of $\mathcal{O}_\infty$ may vanish at certain points. For example, denote by ϕ the unital *-homomorphism $\mathcal{M}_1 \to \mathcal{M}_0$ which induces the K_0-map $(\mathbb{Z}, 1) \to (\mathbb{Z}, 0), n \mapsto 0$. Then, the continuous field C*-algebra

$$A = \{f \in \mathrm{C}([0,1], \mathcal{M}_0); f(x) \in \mathrm{Image}(\phi) \text{ if } x \in [0,1/2]\}$$

is simple on $[0,1/2]$ with fibre $\mathcal{M}_1$ and simple on $(1/2,1]$ with fibre $\mathcal{M}_0$, and the function $\Phi([1])$ of $A[0,1]$ is 1 on $[0,1/2]$ and 0 on $(1/2,1]$.

7. The null set of sheaves with integer fibres

Let $\mathcal{S}$ be a continuous sheaf of abelian groups on $X = [0,1]$ with all stalks isomorphic to $\mathbb{Z}$. In the following, we shall study the set of null sets of $\mathcal{S}$ in detail.

For any $p \in \mathcal{S}[0,1]$, by Theorem 6.4, $\mathrm{null}(p) = \{x \in [0,1] : p(x) = 0\}$ is an open subset of $[0,1]$. Denote by $\mathrm{null}(\mathcal{S})$ the set of the points where all the elements of $\mathcal{S}[0,1]$ vanish:

$$\mathrm{null}(\mathcal{S}) := \bigcap_{p \in \mathcal{S}[0,1]} \mathrm{null}(p).$$

Lemma 7.1. *Let $p \neq 0$ be an element of $\mathcal{S}[0,1]$ and let $U = (a,b)$ be a maximal open subinterval of* $\mathrm{null}(p)$*. Then, for any q in $\mathcal{S}[0,1]$, there are $c, d \in (a,b)$ such that q vanishes on $(a,c) \cup (d,b)$. A similar statement holds if $U = (a,1]$ of $U = [0,1)$. Thus $\partial(\mathrm{null}(p)) \subset \overline{\mathrm{null}(q)}$.*

Proof. For the first part we shall prove the existence of c. The existence of d is proved in a similar way. Set $m = p(a) \in \mathbb{Z}$ and $n = q(a) \in \mathbb{Z}$ and note that $m \neq 0$ by maximality of (a,b). Then $(np - mq)(a) = 0$ and hence $np - mq$ vanishes on a neighbourhood V of a since $\mathrm{null}(np - mq)$ is open. If $c \in (a,b) \cap V$ and $x \in (a,c)$, then $-mq(x) = (np - mq)(x) = 0$ and hence $q(x) = 0$ since $m \neq 0$. The second part of the statement follows from the first part, since any point in $\partial(\mathrm{null}(p))$ is either equal to a boundary point of some maximal open subinterval of $\mathrm{null}(p)$ or it is a limit point of the set of all such boundary points. □

Since $\mathcal{S}[0,1]$ is at most countable, we can write $\mathcal{S}[0,1] = \{0, p_1, p_2, \cdots, p_n, \cdots\}$. For $p \in \mathcal{S}[0,1]$, let us set $\mathrm{supp}(p) = \{x \in [0,1] : p(x) \neq 0\}$. It is a closed subset of $[0,1]$. Note that

$$[0,1] = \mathrm{null}(p) \cup \partial(\mathrm{null}(p)) \cup \mathrm{supp}(p)^\circ = \overline{\mathrm{null}(p)} \cup \mathrm{supp}(p)^\circ$$

are partitions of $[0,1]$.

Suppose that there is a nonzero $p \in \mathcal{S}[0,1]$ such that $\mathrm{null}(p) \neq \varnothing$. Then $\partial(\mathrm{null}(p)) \neq \varnothing$ since $[0,1]$ is connected. The set

$$E = \bigcap_n \overline{\mathrm{null}(p_n)} = [0,1] \setminus \bigcup_n \mathrm{supp}(p_n)^\circ$$

is closed and nonempty since $\partial(\mathrm{null}(p)) \subset \overline{\mathrm{null}(p_n)}$ for all n by Lemma 7.1 and therefore $\partial(\mathrm{null}(p)) \subset E$.

Lemma 7.2. *For any p_n, the (relatively) open set* $\mathrm{null}(p_n) \bigcap E$ *is dense in E.*

Proof. Fix n. For any $x \in E$ and V an open interval containing x we shall show that $\mathrm{null}(p_n) \bigcap E \cap V \neq \varnothing$. If $x \in E^\circ$, then there is an open interval (a,b) containing x such that $(a,b) \subset V \cap E$. On the other hand, $x \in \overline{\mathrm{null}(p_n)}$ by the definition of E and hence $\mathrm{null}(p_n) \cap (a,b) \neq \varnothing$. Thus $\mathrm{null}(p_n) \bigcap E \cap V \neq \varnothing$ since $(a,b) \subset E \cap V$.

If $x \notin E^\circ$, then V intersects nontrivially the complement of E and hence $V \cap \mathrm{supp}(p_m)^\circ \neq \varnothing$ for some m. Since $x \in E$, $x \in \overline{\mathrm{null}(p_m)}$ and hence $V \cap \mathrm{null}(p_m) \neq \varnothing$. Since V is connected, we cannot have $V \subset \mathrm{null}(p_m) \cup \mathrm{supp}(p_m)^\circ$ and so $V \cap \partial(\mathrm{null}(p_m)) \neq \varnothing$. Therefore, in order to show that x is in the closure of $\mathrm{null}(p_n) \bigcap E$, it is enough to show that any point y in any boundary set $\partial(\mathrm{null}(p_m))$ can be approximated by points in $\mathrm{null}(p_n) \bigcap E$. We distinguish two cases in this situation.

Case 1. $p_n(y) = 0$. Then $y \in \mathrm{null}(p_n) \cap \partial(\mathrm{null}(p_m)) \subset \mathrm{null}(p_n) \bigcap E$ since $\partial(\mathrm{null}(p_m)) \subset E$ by Lemma 7.1.

Case 2. $p_n(y) \neq 0$. Since $y \in E \subset \overline{\mathrm{null}}(p_n)$ and hence $x \in \partial(\mathrm{null}(p_n))$, there is a sequence (c_k, d_k) of maximal open subintervals of $\mathrm{null}(p_n)$ such that either the sequence c_k is nonincreasing and converges to y or the sequence d_k is nondecreasing and converges to y. Let us assume that we are in the first situation with $c_k \searrow y$. The other situation when $d_k \nearrow y$ is treated similarly. Set $e_k = c_k + (d_k - c_k)/k$. Suppose that (c_k, e_k) in contained in E for infinitely many indices k. Then the midpoints y_k of the corresponding intervals (c_k, e_k) form a sequence of points in $\mathrm{null}(p_n) \cap E$ which converges to y. Therefore we may assume that each (c_k, e_k) intersects nontrivially the complement of E. Thus there is a sequence $n(k)$ such that $(c_k, e_k) \bigcap \mathrm{supp}(p_{n(k)})^\circ \neq \varnothing$. We also have $(c_k, e_k) \bigcap \mathrm{null}(p_{n(k)}) \neq \varnothing$ by Lemma 7.1. Since (c_k, e_k) is connected there is $y_k \in (c_k, e_k) \bigcap \partial(\mathrm{null}(p_{n(k)})) \subset \mathrm{null}(p) \bigcap E$. Since each of the sequences c_k and e_k converges to y so does y_k. □

The following two results apply to the K-theory sheaf of a separable continuous field of C*-algebras over $[0,1]$ with all fibres stably isomorphic to $\mathcal{O}_\infty$.

Theorem 7.3. *The set* $\mathrm{null}(\mathcal{S})$ *is nonempty if and only if the set* $\mathrm{null}(p)$ *is nonempty for some nonzero* $p \in \mathcal{S}[0,1]$. *For each* $p \in \mathcal{S}[0,1]$, *the boundary points of* $\mathrm{null}(p)$ *are in the closure of* $\mathrm{null}(\mathcal{S})$.

Proof. The lemma above, together with the Baire Category Theorem, shows that the set

$$\bigcap_n (\mathrm{null}(p_n) \cap E) = (\bigcap_n \mathrm{null}(p_n)) \cap E = \mathrm{null}(\mathcal{S}) \cap E$$

is dense in E and hence nonempty. In particular, $\mathrm{null}(\mathcal{S})$ is nonempty and for each p, $\partial(\mathrm{null}(p)) \subset E = \overline{\mathrm{null}(\mathcal{S}) \cap E} \subset \overline{\mathrm{null}(\mathcal{S})}$. □

A subset Z of $[0,1]$ is called *half-open* if for each $x \in Z$, there is $\varepsilon > 0$ such that either $\varnothing \neq (x - \varepsilon, x] \subset Z$ or $\varnothing \neq [x, x + \varepsilon) \subset Z$. Consequently if $Z \neq \varnothing$, then Z is a countable disjoint union of subintervals of $[0,1]$ of types $[a,b]$, $[a,b)$, $(a,b]$, or (a,b) with $a < b$. In particular Z is the closure of its interior Z°.

Proposition 7.4. *The set* $\mathrm{null}(\mathcal{S})$ *is half open. For each* $p \in \mathcal{S}[0,1]$, *the boundary points of* $\mathrm{null}(p)$ *are in the closure of* $\mathrm{null}(\mathcal{S})^\circ$.

Proof. Denote the complement set of $\mathrm{null}(\mathcal{S})$ by $\mathrm{supp}(\mathcal{S})$. Note that $x \in \mathrm{supp}(\mathcal{S})$ if and only if there is $p \in \mathcal{S}[0,1]$ such that $p(x) \neq 0$. In order to prove the lemma, it is enough to prove that if $x \in [0,1]$ and if there exist an increasing sequence $\{x_n\}$ and a decreasing sequence $\{y_n\}$ in $\mathrm{supp}(\mathcal{S})$ both convergent to x, then $x \in \mathrm{supp}(\mathcal{S})$. Let us assume that $0 < x < 1$. If $x = 0$ or $x = 1$, only one of these sequences will be considered.

Let $\{p_n\}$ and $\{q_n\}$ be sequences in $\mathcal{S}[0,1]$ such that $p_n(x_n) \neq 0$ and $q_n(y_n) \neq 0$. We are going to construct $p \in \mathcal{S}[0,1]$ such that $p(x) \neq 0$. Since the stalk of S at x is nonzero, there is a closed subinterval $V = [\alpha, \beta]$ with $x \in V^\circ$ and an element $p_V \in S(V)$ such that $p_V(x) \neq 0$.

On the other hand, we may assume that $p_n(x) = 0$ for any p_n. Otherwise, one of the $\{p_n\}$ will be the desired element. By Corollary 6.4, for each p_n, the set $\mathrm{null}(p_n)$ is open. Denote by U_n the maximal open subinterval of $\mathrm{null}(p_n)$ containing x. Then $U_n = (a_n, b_n)$ or $U_n = (a_n, 1]$. Since $p_n(x_n) \neq 0$, we have that $x_n \notin U_n$ and hence $x_n \leq a_n$. Since x_n converges to x, there is n such that $x_n \in V$. Therefore, $[a_n, \beta] \subset V$. By Lemma 7.1, there is $c_n \in (a_n, x)$ such that p_V is zero on $(a_n, c_n]$. By Corollary 6.4, we can glue the restriction of p_V to $[\frac{a_n+c_n}{2}, \beta]$ with the restriction of p_n to $[0, c_n]$, since both these elements vanish on $[\frac{a_n+c_n}{2}, c_n]$. The outcome is an element $p' \in S[0, \beta]$ such that $p'(x) = p_V(x) \neq 0$. Arguing in a similar way, one shows that there is $\beta' \in (x, \beta]$ such that the restriction of p' to $[0, \beta']$ extends to some $p \in \mathcal{S}[0,1]$. In particular $p(x) \neq 0$.

The cases $x = 0$ and $x = 1$ are treated in a similar way. $\square$

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