(2 slugs)

4. Pete stands on the top of a 20 foot train and throws a hammer upward with a speed of 10 ft/s. Suppose there is a force due to air resistance acting on the hammer in the opposite direction of velocity with a magnitude of $\frac{v^2}{2000}$ ft/s. Assuming the hammer misses the train, how long will it take the hammer to hit the ground? (Use g = 32 ft/s^2 as the magnitude of the acceleration due to gravity.)

Same basic set up as 3.

up: $m\frac{dv}{dt} = -mg - \frac{v^2}{2000}$

down: $m\frac{dv}{dt} = -mg + \frac{v^2}{2000}$

(need to separate equations so that when $v<0$, air resistance >0 and when $v>0$, air resistance <0)

First, going up: $\frac{dv}{dt} = -32 - \frac{v^2}{4000} = \frac{-v^2 - 128{,}000}{4000}$

$$\frac{dv}{-128{,}000 - v^2} = \frac{dt}{4000}$$

$$\frac{-1}{128{,}000}\frac{dv}{1+\left(\frac{v}{\sqrt{128{,}000}}\right)^2} = \frac{dt}{4000}$$

(Recall $\int \frac{1}{1+u^2}du = \tan^{-1}(u) + C$)

$$-\frac{\sqrt{128{,}000}}{128{,}000}\tan^{-1}\left(\frac{v}{\sqrt{128{,}000}}\right) = \frac{t}{4000} + C$$

Doing algebra and using the initial value $v(0) = 10$, we get $v(t) = \sqrt{128{,}000}\tan\left(\tan^{-1}\left(\frac{10}{\sqrt{128{,}000}}\right) - \frac{\sqrt{128{,}000}}{4000}t\right)$

Notice that the maximum height occurs when $v(T) = 0$.

Solving this for T, we obtain $T \approx 0.312418658$ seconds

Now, if $h(t)$ is height, $v(t) = \frac{dh}{dt}$

(Recall $\int \tan u\, du = \ln|\sec u| + C$), so

$$h(t) = -4000\ln\left|\sec\left(\tan^{-1}\left(\frac{10}{\sqrt{128{,}000}}\right) - \frac{\sqrt{128{,}000}}{4000}t\right)\right| + C$$

Using the initial value $h(0) = 20$, we obtain

$$h(t) = -4000\ln\left|\sec\left(\tan^{-1}\left(\frac{10}{\sqrt{128{,}000}}\right) - \frac{\sqrt{128{,}000}}{4000}t\right)\right| + 4000\ln\left|\sec\left(\tan^{-1}\left(\frac{10}{\sqrt{128{,}000}}\right)\right)\right| + 20$$

Max height is $h(T) \approx 21.561889966$ feet going down: Reset $t=0$ (for simplicity)

We now have the initial conditions

$v(0)=0, \quad h(0) \approx 21.561889966$

$m\frac{dv}{dt} = -mg + \frac{v^2}{2000}$ so $\frac{dv}{dt} = -32 + \frac{v^2}{4000}$

get $\frac{dv}{-128{,}000+v^2} = \frac{dt}{4{,}000}$ (Recall $\int \frac{1}{1-u^2}\,du = \tanh^{-1}u + C$ when $-1<u<1$)

$\frac{-1}{128{,}000} \cdot \frac{dv}{1-\left(\frac{v}{\sqrt{128{,}000}}\right)^2} = \frac{dt}{4{,}000}$

$\frac{-\sqrt{128{,}000}}{128{,}000}\tanh^{-1}\left(\frac{v}{\sqrt{128{,}000}}\right) = \frac{t}{4000} + C$

Doing algebra and using the initial value $v(0)=0$, we get

$v(t) = \sqrt{128{,}000}\,\tanh\left(-\frac{\sqrt{128{,}000}}{4000}t\right)$ (Recall $\int \tanh u\,du = \ln|\cosh u| + C$)

$\frac{dh}{dt} = v(t)$, so $h(t) = -4000\ln\left|\cosh\left(-\frac{\sqrt{128{,}000}}{4000}t\right)\right| + C$

Using the initial value $h(0) = 21.561889966$, we get

$h(t) = -4000\ln\left|\cosh\left(-\frac{\sqrt{128{,}000}}{4000}t\right)\right| + 21.561889966$

The ball hits the ground when $h(S)=0$.

Solving, we get $S \approx 1.16191277$

Time to go up + Time to go down

$T + S$

$\approx 0.312418658 + 1.16191277$

$\approx \boxed{1.47 \text{ seconds}}$