History of Complex Numbers.

We will look at a very, very brief history of complex numbers in this document. Should the reader wish to further pursue the topic of the history of complex numbers, a highly recommended reference is Paul J. Nahin's book An Imaginary Tale: The Story of $\sqrt{-1}$.

The history of mathematics is rife with controversy. The Pythagoreans supposedly killed someone for showing that $\sqrt{2}$ could not be written as a fraction of whole numbers. The number zero was banned in certain locations; negative numbers were shunned. Over time, though, mathematicians eventually came to accept all of these different types of numbers - but none of this acceptance was immediate.

When confronted with the question, "What is $\sqrt{-1}$?," the answer was obvious: no such number exists. All numbers are either positive, negative, or zero. The square of a positive number is positive, the square of a negative number is positive, and the square of zero is zero. Therefore, there is no number which squares to a negative number.

This was the viewpoint held by many for a long time. If a mathematician applied the quadratic formula to a polynomial such as $x^2 + 1$, they would obtain the square root of a negative number. They simply claimed that the polynomial had no roots - and that was fine! After all, if you graph this function, it never touches the x-axis.

Much like there is a quadratic formula which can be used to find the roots of quadratic polynomials, there is also a cubic formula (and quartic formula too!) to find the roots of cubic (and quartic) polynomials. These formulas involve some heavy-duty square roots. When people would come across the square root of a negative number in these contexts, much like in the quadratic case, they would simply ignore the result - there is no such number!

Italian mathematician Gerolamo Cardano decided to take a different approach with the cubic formula. Cardano was not scared of negative numbers like others were, and when he encountered the square root of a negative number, rather than simply saying such a number was impossible and ending his investigation, he decided to treat it like a variable and continue his investigation. He found that for many cubic polynomials, in using this approach, the square roots of negative numbers would cancel each other out, and he would be left with a real number - one which when plugged back into the original cubic polynomial would yield zero - an actual real number root!

Mathematicians begrudgingly accepted that this technique worked, but still criticized it for using numbers which were "impossible" or "imaginary" (this is how the name came to be - and also how the name of the "real numbers" came to be, even though nowadays we know that the imaginary/complex numbers are just as real as the real numbers!).

Over time, many applications and interpretations for the imaginary numbers developed. Many mathematicians proved some very interesting results with them. We will see one result proven by Leonhard Euler. German mathematician Johann Carl Friedrich Gauss was perhaps the most influential mathematician in terms of the acceptance of complex numbers. He proved a great many theorems, including the Fundamental Theorem of Algebra (which states that any polynomial of degree n has exactly n roots in the complex numbers, counted with multiplicity). He coined the term "complex number," and he developed a great foundation for the study of functions of a complex variable.

In the real world, we cannot measure with complex numbers. Our measurements are at best real numbers. This would make it seem like complex numbers would not be very useful in the physical sciences. Throughout history, however, we have seen that using complex numbers often allows us to obtain real number solutions to problems that were previously unanswerable with only a theory of real numbers. This is a great power of the complex numbers, and it is the way we will use complex numbers in this course. When the roots of the characteristic polynomial are complex conjugates, we can use the theory of complex numbers to help us uncover the real-valued solutions to such differential equations.

Working with Complex Numbers.

We begin with the question "What is an imaginary number?". An imaginary number is a number of the form $\pm\sqrt{-c}$, where c is a positive real number. Often, rather than writing $\pm\sqrt{-c}$, we write it as $\pm i\sqrt{c}$. As such, all imaginary numbers can be written in the form bi where b is a real number (not necessarily positive). Since $\sqrt{-1} = i$, it follows that $i^2 = -1$, $i^3 = -i$, and $i^4 = 1$. Because of this, we then get $i^5 = i$, $i^6 = -1$, etc.

A complex number is a number of the form $a + bi$ where a and b are real numbers. Given a complex number $z = a + bi$, the real part of z is a, and the imaginary part of z is b (note that the imaginary part is the real number b, *not* the imaginary number bi). A complex number $a + bi$ is purely real if $b = 0$ and is purely imaginary if $a = 0$. 0 is the only number which is both purely real and purely imaginary.

Two complex numbers $z = a + bi$ and $w = c + di$ are equal if and only if their real parts are equal and their imaginary parts are equal. i.e., $z = w$ if and only if $a = c$ and $b = d$.

Adding, subtracting, and multiplying two complex numbers together is pretty simple - we treat i like a variable which has the additional property that $i^2 = -1$. To add or subtract, we combine like terms. To multiply, we distribute (FOIL).

Examples:

$$\begin{aligned}(1-2i)+(3+4i) &= (1+3)+(-2+4)\,i \\ &= 4+2i\end{aligned}$$

$$\begin{aligned}(1-2i)(3+4i) &= 1\cdot 3 + 1\cdot 4i + (-2i)\cdot 3 + (-2i)\cdot 4i\\ &= 3+4i-6i-8i^2\\ &= 3+4i-6i-8\cdot(-1)\\ &= 3+4i-6i+8\\ &= 3+8+4i-6i\\ &= 11-2i\end{aligned}$$

Given a complex number $z = a+bi$, there is an associated complex number $\overline{z} = \overline{a+bi} = a-bi$, called the conjugate of z. Notice that for any complex number z, we have that $z\cdot\overline{z}$ is purely real (since $z\cdot\overline{z} = a^2+b^2$). This fact will be very useful in division by complex numbers.

If $z = \overline{z}$, then it must be the case that z is purely real. This is because $z = \overline{z}$ implies that $a+bi = a-bi$. By the equality of complex numbers, this forces $b = -b$, which is only possible if $b = 0$.

If z and w are complex numbers, and $w \neq 0$, then we can compute $\frac{z}{w}$. We do this by multiplying both the numerator and the denominator by $\overline{w}$, which forces a purely real number in the denominator.

Example:

$$\begin{aligned}\frac{1-2i}{3+4i} &= \frac{1-2i}{3+4i}\cdot\frac{3-4i}{3-4i}\\ &= \frac{-5-10i}{25}\\ &= -\frac{5}{25}-\frac{10}{25}i\\ &= -\frac{1}{5}-\frac{2}{5}i\end{aligned}$$

Another fact to notice is that if a polynomial has purely real coefficients and if the polynomial has a non-real complex root z, then the conjugate $\overline{z}$ is also a root of the polynomial. This follows from the following facts, which one can show quite easily by writing $z = a + bi$ and $w = c + di$:

F1. $\overline{z \cdot w} = \overline{z} \cdot \overline{w}$
F2. $\overline{z + w} = \overline{z} + \overline{w}$
F3. If a is purely real, then $\overline{a} = a$.

Let $p(x) = a_n x^n + ... + a_1 x + a_0$ be a polynomial where a_i is purely real for each i. Now, suppose that z is a non-real complex root of $p(x)$. Then:

$$0 = p(z) = a_n z^n + ... + a_1 z + a_0$$

Conjugating both sides, we get

$$\begin{aligned}
\overline{0} &= \overline{a_n z^n + ... + a_1 z + a_0} \\
&\overset{\text{F2}}{=} \overline{a_n z^n} + ... + \overline{a_1 z} + \overline{a_0} \\
&\overset{\text{F1}}{=} \overline{a_n}\left(\overline{z}\right)^n + ... + \overline{a_1}\left(\overline{z}\right) + \overline{a_0} \\
&\overset{\text{F3}}{=} a_n\left(\overline{z}\right)^n + ... + a_1\left(\overline{z}\right) + a_0 \\
&= p(\overline{z})
\end{aligned}$$

Since $\overline{0} = 0$, we see that $p(\overline{z}) = 0$, so $\overline{z}$ is also a root of $p(x)$.

<u>Some Maclaurin Series</u>.

We review the following Maclaurin Series, as they will be used in class to show why Euler's formula should be true.

$$\begin{aligned}
e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + ... \\
\cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ... \\
\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ...
\end{aligned}$$