**Eddie Price  Initial Coniditions and Existence and Uniquness  Summer 2016**

In the Existence and Uniqueness Theorem for 2nd order linear differential equations, we saw that we had to have initial conditions of the form $y(t_0) = y_0$ and $y'(t_0) = y'_0$.

We show that the initial conditions *must* be of this form. If the initial conditions vary from this, then we may not be guaranteed existence or uniqueness of a solution.

Consider the differential equation

$$y'' + y = 0$$

One can check that the general solution is of the form

$$y(t) = c_1 \cos t + c_2 \sin t$$

and its derivative is

$$y'(t) = -c_1 \sin t + c_2 \cos t$$

Now, we will change up the initial conditions to show that the theorem no longer holds:

$\underline{t_0 \neq t_1}$

Consider the initial conditions $y(0) = 1$, $y'\left(\frac{\pi}{2}\right) = 2$. Using the first initial condition, we get:

$$1 = y(0) = c_1 \cos(0) + c_2 \sin(0) = c_1$$

Thus, the solution is $y(t) = \cos t + c_2 \sin t$. Now, plugging in the second initial condition, we get:

$$2 = y'\left(\frac{\pi}{2}\right) = -\sin\left(\frac{\pi}{2}\right) + c_2 \cos\left(\frac{\pi}{2}\right) = -1$$

So, we get $2 = -1$, a false statement. This shows that *no* solution satisfies the IVP $y''+y = 0$, $y(0) = 1$, $y'\left(\frac{\pi}{2}\right) = 2$. This violates existence of solutions.

Similarly, choosing the initial conditions $y(0) = 1$, $y'\left(\frac{\pi}{2}\right) = -1$, you can see that both initial conditions only give us that $c_1 = 1$. Neither restricts $c_2$ or leads to a contradiction, so there are infinitely many solutions. This violates uniqueness of solutions.

Now, it is *possible* for a unique solution to exist when $t_0 \neq t_1$, but we cannot guarantee it.

(An example where it holds is $y(0) = 1$, $y'(\pi) = 2$. You get the unique solution $y(t) = \cos t - 2 \sin t$, but like was said earlier, we cannot guarantee it.)

Both conditions $y$ or both conditions $y'$.

It is obvious that the we cannot have a solution if we have initial conditions of the form $y(0) = 1$, $y(0) = 2$, or if we have $y'(0) = 1$, $y'(0) = 2$. A function can only have one output for each input, so it is impossible for solutions to exist.

Similarly, choosing initial conditions like $y(0) = 1$, $y(0) = 1$ or $y'(0) = 1$, $y'(0) = 1$ will result in infinitely many solutions. So uniqueness ceases to be.

Going back to our original example, we still can't guarantee a unique solution if we have both $y$ or both $y'$ with $t_0 \neq t_1$. For example, the initial conditions $y(0) = 1$, $y(\pi) = 2$ has no solution, but using the initial conditions $y(0) = 1$, $y(\pi) = -1$ gives infinitely many solutions (and similarly with $y'$).

Of course, like before, it is *possible* that we could have a unique solution anyway, but it is not guaranteed.