This document describes the Method of Variation of Parameters to solve a system of 2 first order linear ODEs.

The Fundamental Matrix.

Given a system of equations $\vec{x}' = A\vec{x}$ where A is a 2×2 constant matrix, one can find eigenvectors $\vec{\xi}^{(1)}$ (associated to λ_1) and $\vec{\xi}^{(2)}$ (associated to λ_2). Then a Fundamental Matrix $\Psi(t)$ for the system is given by taking $e^{\lambda_1 t}\vec{\xi}^{(1)}$ as the first column and $e^{\lambda_2 t}\vec{\xi}^{(2)}$ as the second column.

For the constant vector $\vec{c} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$, we have that $\Psi\vec{c}$ is the general solution to $\vec{x}' = A\vec{x}$, and notice that $\Psi' = A\Psi$.

Inverting a 2×2 Matrix.

Given a 2×2 matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, if $\det M \neq 0$, then M is said to be invertible (or non-singular). If M is invertible, then there exists a matrix M^{-1} such that $MM^{-1} = M^{-1}M = I$, where I is the 2×2 identity matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

If M is the 2×2 matrix defined above, then $M^{-1} = \frac{1}{\det M}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$, and $\det M = ad - bc$.

For any homogeneous system $\vec{x}' = A\vec{x}$, fundamental matrices $\Psi(t)$ are *always* invertible.

Variation of Parameters.

Now, suppose you have a nonhomogeneous system $\vec{x}' = A\vec{x} + \vec{g}(t)$.

Much like what we did with variation of parameters for 2nd order linear equations, since $\Psi(t)\,\vec{c}$ is the general solution to the homogeneous system $\vec{x}' = A\vec{x}$, we assume that $\vec{x}(t) = \Psi(t)\,\vec{u}(t)$ is the general solution for the nonhomogeneous system $\vec{x}' = A\vec{x} + \vec{g}(t)$ where $\vec{u}(t)$ is some vector with functions of t as its components (we are letting the parameters vary).

Using the product rule, we get $\vec{x}' = \Psi\vec{u}' + \Psi'\vec{u}$. Then, plugging $\vec{x} = \Psi\vec{u}$ into the system $\vec{x}' = A\vec{x} + \vec{g}$, we get

$$\Psi\vec{u}' + \Psi'\vec{u} = A\Psi\vec{u} + \vec{g}$$

Recall that $\Psi\vec{c}$ is a solution to the homogeneous system $\vec{x}' = A\vec{x}$ for any constant vector $\vec{c}$. Thus, we have that $\Psi' = A\Psi$. This gives that the above equation is

$$\Psi\vec{u}' + \Psi'\vec{u} = \Psi'\vec{u} + \vec{g}$$

Subtracting $\Psi'\vec{u}$ from both sides, we obtain

$$\Psi\vec{u}' = \vec{g}$$

Since Ψ is an invertible matrix, Ψ^{-1} exists, so we get

$$\vec{u}' = I\vec{u}' = \Psi^{-1}\Psi\vec{u}' = \Psi^{-1}\vec{g}$$

We can then integrate both sides of the equation $\vec{u}' = \Psi^{-1}\vec{g}$ to obtain the vector $\vec{u}(t)$, and then have the general solution, which was assumed to be $\Psi(t)\,\vec{u}(t)$.

Steps for Variation of Parameters.

Given a nonhomogeneous system $\vec{x}' = A\vec{x} + \vec{g}(t)$,

1. Find the eigenvalues and corresponding eigenvectors of A.
2. Create a fundamental matrix $\Psi(t)$ from the eigenvalues and eigenvectors.
3. Assume the general solution is $\Psi(t)\,\vec{u}(t)$
4. Find Ψ^{-1}
5. Develop the equation $\vec{u}' = \Psi^{-1}\vec{g}$ and integrate both sides to obtain $\vec{u}$.
6. The general solution is $\Psi(t)\,\vec{u}(t)$.

An Example.

Solve the system $\vec{x}' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \vec{x} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^t$ using Variation of Parameters.

One can check that $\lambda_1 = 1$, $\vec{\xi}^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\lambda_2 = -1$, $\vec{\xi}^{(2)} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ are eigenvalues and eigenvectors of A.

We then form the fundamental matrix $\Psi(t)$ where the first column is $e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and the second column is $e^{-t} \begin{pmatrix} 1 \\ 3 \end{pmatrix}$. This gives

$$\Psi(t) = \begin{pmatrix} e^t & e^{-t} \\ e^t & 3e^{-t} \end{pmatrix}$$

$\det \Psi(t) = e^t \cdot 3e^{-t} - e^t e^{-t} = 3 - 1 = 2$. Hence,

$$\Psi^{-1} = \frac{1}{2} \begin{pmatrix} 3e^{-t} & -e^{-t} \\ -e^t & e^t \end{pmatrix}$$

Notice that $\vec{g}(t) = \begin{pmatrix} e^t \\ -e^t \end{pmatrix}$, so

$$\Psi^{-1}\vec{g} = \frac{1}{2}\begin{pmatrix} 3e^{-t} & -e^{-t} \\ -e^t & e^t \end{pmatrix}\begin{pmatrix} e^t \\ -e^t \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 3+1 \\ -e^{2t} - e^{2t} \end{pmatrix} = \begin{pmatrix} 2 \\ -e^{2t} \end{pmatrix}$$

We know that $\vec{u}' = \Psi^{-1}\vec{g}$, so integrating both sides, we have

$$\vec{u}(t) = \begin{pmatrix} 2t + c_1 \\ -\frac{1}{2}e^{2t} + c_2 \end{pmatrix}$$

Since we assumed the general solution is $\Psi(t)\,\vec{u}(t)$, we have that the general solution is

$$\begin{pmatrix} e^t & e^{-t} \\ e^t & 3e^{-t} \end{pmatrix}\begin{pmatrix} 2t + c_1 \\ -\frac{1}{2}e^{2t} + c_2 \end{pmatrix} = \begin{pmatrix} 2te^t + c_1e^t - \frac{1}{2}e^t + c_2e^{-t} \\ 2te^t + c_1e^t - \frac{3}{2}e^t + 3c_2e^{-t} \end{pmatrix}$$

Separating this out, we get the following

$$te^t\begin{pmatrix} 2 \\ 2 \end{pmatrix} + c_1e^t\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{1}{2}e^t\begin{pmatrix} 1 \\ 3 \end{pmatrix} + c_2e^{-t}\begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

Rearranging we get

$$\vec{x}(t) = c_1e^t\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2e^{-t}\begin{pmatrix} 1 \\ 3 \end{pmatrix} + te^t\begin{pmatrix} 2 \\ 2 \end{pmatrix} - \frac{1}{2}e^t\begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

Notice!: This is the same problem as example 3 in my lecture notes for Lesson 31. We get *almost* the same answer. The only difference is the term $-\frac{1}{2}e^t\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ in this solution while we got $e^t\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ using the Method of Undetermined Coefficients. Notice that in the Method of Undetermined Coefficients, we had a choice of b_2 which affected b_1. We chose $b_2 = 0$, which gave $b_1 = 1$. If we want to see that these two answers really match, go back to example 3 in my Lesson 31 notes and choose $b_2 = -\frac{3}{2}$. You will then get $b_1 = -\frac{1}{2}$, and the answers will match.

Alternatively, we could see that

$$-\frac{1}{2}e^t\begin{pmatrix} 1 \\ 3 \end{pmatrix} = -\frac{1}{2}e^t\left[\begin{pmatrix} 3 \\ 3 \end{pmatrix} + \begin{pmatrix} -2 \\ 0 \end{pmatrix}\right] = -\frac{3}{2}e^t\begin{pmatrix} 1 \\ 1 \end{pmatrix} + e^t\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and the term $-\frac{3}{2}e^t\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ can be absorbed into the part of the complementary solution $c_1e^t\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, and the answers will match.