Eddie Price **Proof that we have all the solutions** **Summer 2016**

Here, we give a detailed proof that if y_1 and y_2 are solutions to $L[y] = y'' + py' + qy = 0$ and if the Wronskian of y_1 and y_2 is nonzero, then all solutions of $L[y] = 0$ are of the form $c_1y_1 + c_2y_2$.

Proof.

Showing the Solutions form a Vector Space.

First, we know from Linear Algebra that continuous functions form a real vector space. Next, we show that solutions to $L[y] = 0$ form a subspace of this vector space. By the existence and uniqueness theorem, there exist solutions to this differential equation (also, since the equation is homogeneous, $y = 0$ is a solution). Hence, the set of solutions is nonempty. Each solution is also guaranteed to be twice differentiable on some interval I, so the solutions must then also be continuous on that same interval I (for if they were not continuous at some point in I, they would also not be differentiable there). Thus, the set of solutions of $L[y] = 0$ is a nonempty subset of $C(I)$, the vector space of continuous functions on the interval I (where the coefficient polynomials p and q are continuous on I).

It remains to show that solutions are closed under addition and closed under scalar multiplication. Both of these are proven by the Principle of Superposition. Let ϕ_1 and ϕ_2 be solutions. Then, by the Principle of Superposition, $c_1\phi_1 + c_2\phi_2$ is a solution for any real numbers c_1 and c_2. Choosing $c_1 = c_2 = 1$ shows that $\phi_1 + \phi_2$ is a solution, so the set of solutions is closed under addition. Moreover, choosing $c_2 = 0$ and $c_1 = c$ for any real number c, we see that $c\phi_1$ is a solution, so the set of solutions is closed under scalar multiplication. Thus, the set of solutions to $L[y] = 0$ is a subspace of $C(I)$, and hence, is in and of itself a vector space.

Showing the Vector Space of Solutions to $L[y] = 0$ has dimension 2.

The Existence and Uniqueness Theorem states that given two initial conditions, we can find a unique solution to the initial value problem. The number of initial conditions correlates to the number of arbitrary constants included in the general solution.

In more detail: If the dimension were only 1, all solutions would be of the form c_1y_1 for some function y_1. We only need one initial condition to determine the value of c_1. In other words, $y(t_0) = y_0$ is enough to determine the value of c_1. We could then choose our second initial condition as follows: choose any real number a except do not allow $\frac{a}{c_1} = y_1'(t_0)$. We choose our second initial condition to be $y'(t_0) = a$. It is then the case that c_1y_1 does not satisfy this second initial condition, and so it does not satisfy the initial value problem. This means that there are some initial value problems which do not have solutions. This contradicts the existence part of the existence and uniquess theorem. Hence, our assumption that the dimension is 1 leads to a contradiction. This means that the dimension is greater than 1.

If the dimension were $n \geq 3$, then all solutions would be of the form $c_1y_1 + ... + c_ny_n$ for some functions $y_1, ..., y_n$. Given two initial conditions, we would be able to remove two of the arbitrary constants. We would then be left with $n-2$ arbitrary constants. Since $n \geq 3$, we would have at least 1 arbitrary constant left in our solution. Of course, we could choose that to be any value. Thus, we would have infinitely many solutions to the initival value problem, which contradicts the uniqueness part of the existence and uniqueness theorem. This means that the dimension is less than 3.

Thus, the only possibility is that the dimension is exactly 2.

Showing that a nonzero Wronskian implies that the solutions are linearly independent.

Recall that if a vector space has dimension 2 and a set of 2 vectors is shown to be linearly independent, then it must be the case that the set of vectors is a basis for the vector space.

If y_1 and y_2 are solutions to $L[y] = 0$ and have nonzero Wronskian, it must be the case that y_1 and y_2 are linearly independent. To show this, we show that if y_1 and y_2 are linearly dependent, then they have Wronskian equal to 0.

If y_1 and y_2 are linearly dependent, then there exist nonzero scalars c_1 and c_2 such that $c_1y_1 + c_2y_2 = 0$. We can solve for y_2, giving us then that $y_2 = -\frac{c_1}{c_2}y_1$.

Then $y_2' = -\frac{c_1}{c_2}y_1'$. In particular, the Wronskian of y_1 and y_2 is

$$y_1y_2' - y_1'y_2 = y_1\left(-\frac{c_1}{c_2}y_1'\right) - y_1'\left(-\frac{c_1}{c_2}y_1\right) = 0$$

Therefore, if two solutions are linearly dependent, then they must have Wronskian equal to 0, so if the Wronskian is not equal to 0, then they are linearly independent.

This shows that if y_1 and y_2 are solutions to $L[y] = y'' + py' + qy = 0$ and if the Wronskian of y_1 and y_2 is nonzero, then y_1 and y_2 form a basis of the solution space for the differential equation. In particular, y_1 and y_2 form a spanning set of the solution space, so all solutions of $L[y] = 0$ are linear combinations of y_1 and y_2; i.e., they are of the form $c_1y_1 + c_2y_2$. $\square$