References:

- Barnes & Roitzheim, *Foundations*;
- Adams, *Stable Homotopy & Generalized Homology* (part III)

Suspension

Spaces (= CGWH or sSets)

$\mathrm{Spaces}_* =$ pointed spaces

$[X, Y]$ = pointed homotopy classes of maps.

$X \wedge Y = X \times Y / X \vee Y$

S^0 is the monoidal unit.

The suspension of X is

$$\Sigma X = S^1 \wedge X = [0,1] \times X \Big/ \begin{matrix} \{0\} \times X \\ \{1\} \times X \\ [0,1] \times * \end{matrix}$$

ex. $\Sigma S^n = S^{n+1}$.

$$[X, Y] \longrightarrow [\Sigma X, \Sigma Y].$$

Observation 1 ~ these preserve some information.

Freudenthal suspension thm.

If X is a CW-complex of $\dim \le 2n$, if Y is n-connected ($\pi_k Y = 0$ for $k \le n$), then $[X, Y] \xrightarrow{\sim} [\Sigma X, \Sigma Y]$.

Cor. If Y is n-connected,

$$\pi_k Y \xrightarrow{\sim} \pi_{k+1} \Sigma Y \quad \text{for } k \le 2n.$$

Cor. $\pi_{n+k} S^n \xrightarrow{\sim} \pi_{n+k+1} S^{n+1}$ for $n \geq k+2$.

ex. $$\begin{array}{ccccccccccc} \pi_1 S^0 & \to & \pi_2 S^1 & \to & \pi_3 S^2 & \to & \pi_4 S^3 & \to & \pi_5 S^4 & \to & \cdots \\ \| & & \| & & \| & & \| & & \| & & \\ 0 & & 0 & & \mathbb{Z} & & \mathbb{Z}/2 & & \mathbb{Z}/2 & & \end{array}$$

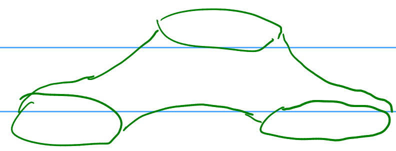
$$\pi_k^{st} S = \operatorname*{colim}_{n \to \infty} \pi_{n+k} S^n$$

Observation 2. Many other things are stable.

ex. $\tilde{H}^* X = \tilde{H}^{*+1} \Sigma X$

This preserves the action of the Steenrod operations. (not the cup product)

ex. Consider the group of n-manifolds with (complex structure, orientation, spin structure, ...) up to bordism.



$\mathrm{Bord}\, G_n$ = bordism group of n-manifolds with G-structure.

Thom defined spaces $MG(n)$

$$\mathrm{Bord}\, G_n = \operatorname*{colim}_{k \to \infty} \pi_{n+k} MG(k).$$

The Spanier-Whitehead category.

Objects: pairs (X, n) pointed

X = finite CW-complex, $n \in \mathbb{Z}$.

$$\mathrm{Maps}_{SW}((X,n),(Y,m)) = \operatorname*{colim}_{k \to \infty} [\Sigma^{n+k} X, \Sigma^{m+k} Y].$$

$$(X, n+m) \cong (\Sigma^n X, m) \text{ in } SW.$$

Think of (X, n) as a formal suspension/desuspension.

In Spaces, $\pi_a S^b \times \pi_b S^c \xrightarrow{\circ} \pi_a S^c$.

In SW, $\pi_* S = \bigoplus_{n \in \mathbb{Z}} \mathrm{Maps}_{SW}((S^0, n), (S^0, 0))$

is a graded ring. $(S^n, m) \simeq (S^0, n+m)$

Metaphor 1 - Spectra are supposed to contain "infinite suspensions" $\Sigma^\infty X$ of any pointed space X. Suspension is invertible.

Generalized cohomology theories

Def. These are functors $\tilde{E}^*: \mathrm{Spaces}_*^{op} \longrightarrow \mathrm{GrAb}$ such that:

1. $\tilde{E}^*$ is homotopy invariant.
2. $\tilde{E}^* X \cong \tilde{E}^{*+1} \Sigma X$, natural in X.
3. If $A \hookrightarrow X$ is an inclusion of pointed spaces, there's an exact sequence
$$\tilde{E}^*(X/A) \longrightarrow \tilde{E}^*(X) \longrightarrow \tilde{E}^*(A).$$
4. $\tilde{E}^*$ takes coproducts to products.

Unreduced: $E^*(X) = \tilde{E}^*(X \amalg *)$.

Can use (3) and (4) to reduce computation of $\tilde{E}^* X$ to $\tilde{E}^* S^n$, and (2) to $\tilde{E}^*(S^0) = E^*(*)$.

ex. $\tilde{H}^*(X; R)$. $\tilde{H}^*(S^0; R) = R$.

ex. $KU^0 X$ = group completion of $\{\text{iso classes of complex VBs} / X\}$

$\widetilde{KU}^0 X = \ker(KU^0 X \longrightarrow KU^0(*))$.

$$\begin{tikzcd}
A \arrow[r, hook, "i"] \arrow[dr, bend right] & X \arrow[r] \arrow[dr] & X/A \arrow[d, "\cong" sloped] \\
& & X \cup_i CA \arrow[r] & \Sigma A \arrow[r] & \Sigma X
\end{tikzcd}$$

$$\widetilde{KU}^{-n} X = \widetilde{KU}^0(\Sigma^n X) \quad \text{for} \quad n > 0.$$

Bott periodicity:

$$\widetilde{KU}^0(X) \cong \widetilde{KU}^0(\Sigma^2 X).$$

Define $\widetilde{KU}^n(X) = \widetilde{KU}^n(\Sigma^{2k} X) \cong \widetilde{KU}^{n-2k} X.$

$$\widetilde{KU}^*(S^0) = \mathbb{Z}[\beta^{\pm 1}], \quad |\beta| = 2.$$

<u>Metaphor 2</u>. Since cohomology theories are stable invariants, they should be representable as spectra.

$$\begin{array}{ccc} X & \longmapsto & \Sigma^\infty X \\ \cap\!\!\!\!\!\! \text{\small\textbar} & & \cap\!\!\!\!\!\! \text{\small\textbar} \\ \text{Spaces}_* & & \text{Spectra.} \end{array}$$

<u>Brown representability</u>. For any $\tilde{E}^*$, there's a spectrum E such that $\tilde{E}^n(X) = [\Sigma^{-n}\Sigma^\infty X, E]$.

<u>Infinite loop spaces</u>.

Brown showed: for any $\tilde{E}^*$, there are spaces E^n such that

$$\tilde{E}^n(X) = [X, E^n].$$

<u>ex</u>. $\tilde{H}^n(X; R) = [X, K(R, n)]$ this has $\pi_n K(R,n) = R$, $\pi_k K(R,0) = 0$ for $k \neq n$.

<u>ex</u>. $$\widetilde{KU}^n(X) = \begin{cases} [X, \mathbb{Z} \times BU] & (n \text{ even}) \\ [X, \Omega(\mathbb{Z} \times BU)] & (n \text{ odd}). \end{cases}$$

$$[X, E^n] \cong [\Sigma X, E^{n+1}] \cong [X, \Omega E^{n+1}].$$

$$E^n \simeq \Omega E^{n+1} \simeq \Omega^2 E^{n+2} \simeq \cdots$$

E^n is an infinite loop space.

ex. $K(R, n) \simeq \Omega K(R, n+1) \simeq \Omega^2 K(R, n+2)$.

ex.

$$\Omega^2 (\mathbb{Z} \times BU) = \Omega^2 BU \simeq \mathbb{Z} \times BU.$$

So $\mathbb{Z} \times BU$ is an infinite loop space.

If X is a loop space, $X = \Omega X'$, there's a multiplication map

$$X \times X \longrightarrow X.$$

$[A, X]$ is a group.

If X is a double loop space, $[A, X]$ is an abelian group.

An infinite loop space has an E_∞ algebra structure — a multiplication that's associative & commutative up to a homotopy that's as "coherent as possible."

Thm (May). Suppose X is an E_∞-algebra, and $\pi_0 X$ is a group. Then X is equivalent (as E_∞-algebras) to an infinite loop space.

Metaphor 3 Every infinite loop space is associated to a spectrum.

$$E \longleftrightarrow \Omega^\infty E$$

$$[\Sigma^\infty X, E]_{Sp} \cong [X, \Omega^\infty E]_{Spaces_*}$$

In Spaces,

cofiber sequence:
$$A \longrightarrow X \longrightarrow X \cup CA \longrightarrow \Sigma A \longrightarrow \Sigma X$$

induce LES on cohomology (& on homology).

fiber sequence.
$$\Omega E \longrightarrow \Omega B \longrightarrow F \longrightarrow E \longrightarrow B$$

induce LES on homotopy groups.

Spectra is a category "like Spaces" in which fiber sequences are cofiber sequences & vice versa.

Other examples: derived category of a ring R, $\mathcal{D}(R)$

Abstract framework: stable model categories / stable ∞-categories.

Thm (Lurie) / Metaphor 4.
Spectra is the free presentable stable ∞-category on one object.
It's also the initial presentably symmetric monoidal stable ∞-category.

ex. $\mathcal{D}(R) \simeq \{$Modules over HR inside spectra$\}$

HR represents $\tilde{H}^*(; R)$.

$$[\Sigma^\infty X, \Sigma^\infty Y] = \operatorname{colim} [\Sigma^k X, \Sigma^k Y].$$