Why $\wedge$? (on Spaces$_*$, also have $\times$).

$$\Sigma X = S^1 \wedge X$$
$$\Sigma(X \wedge Y) = \Sigma X \wedge Y.$$
$X \wedge \cdot$ is left adjoint to $\mathrm{Maps}_*(X, \cdot)$

What do we want $\wedge$ on Sp to do?

- symmetric monoidal structure on Sp.
- This makes Sp a monoidal model category.

  (enriched over itself as a model category.)

  ↳ If $A \overset{i}{\hookrightarrow} B$, $X \overset{j}{\hookrightarrow} Y$ in Sp,

$$A \wedge Y \coprod_{A \wedge X} B \wedge X \longrightarrow B \wedge Y$$

is a cofibration which is acyclic if i or j is.

$$F(B, X) \longrightarrow F(A, X) \times_{F(A,Y)} F(B, Y)$$

If i $(A \to B)$ is a cofibration and $j: X \to Y$ is a fibration.

$$\begin{tikzcd} A \arrow[r] \arrow[d] & X \arrow[d] \\ B \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

- $X \wedge \cdot$ has a right adjoint $F(X, \cdot)$: "function spectrum"

If X is cofibrant, get a Quillen adjunction:

$$X \wedge \cdot : Sp \rightleftarrows Sp : F(X, \cdot).$$

- Compatible with enrichment by Spaces$_*$:

$$\Sigma^\infty K \wedge X \simeq K \wedge X \quad \text{(if } K, X \text{ cofibrant)}$$

(using $\wedge$ on Sp) (using Spaces$_* \times Sp \xrightarrow{\wedge} Sp$).

$$\Omega^\infty F(X, Y) = Sp(X, Y).$$ (if X is cofibrant, Y fibrant)

- $\mathbb{S} = \Sigma^\infty S^0$ is monoidal unit.

$$\Sigma X = \Sigma^\infty S^1 \wedge X$$

$$\Omega X = F(\Sigma^\infty S^1, X).$$

- $\Sigma^\infty (X \wedge Y) \xrightarrow{\sim} \Sigma^\infty X \wedge \Sigma^\infty Y$

$$\tilde{E}_n(X) = [S^n, \Sigma^\infty X \wedge E]$$
$$= \pi_n(\Sigma^\infty X \wedge E).$$

<u>Thm (Lewis)</u>. Consider these axioms on a category Sp of spectra:

1. Sp has a symmetric monoidal product, $\wedge$.
2. $\Sigma^\infty : \mathrm{Spaces}_* \rightleftarrows Sp : \Omega^\infty$
3. $\Sigma^\infty S^0$ is the monoidal unit.
4. Σ^∞ is an oplax monoidal functor
$$\Sigma^\infty(X \wedge Y) \longrightarrow \Sigma^\infty X \wedge \Sigma^\infty Y$$
<u>OR</u> Ω^∞ is a lax monoidal functor
$$\Omega^\infty X \wedge \Omega^\infty Y \longrightarrow \Omega^\infty(X \wedge Y).$$
5. $\Omega^\infty \Sigma^\infty X \simeq \operatorname{colim}_n \Omega^n \Sigma^n X.$

No category satisfies these axioms!

<u>Proof</u>. $\Sigma^\infty S^0$ is a commutative monoid in Sp.

$\Omega^\infty \Sigma^\infty S^0$ is a commutative monoid in Spaces_*.

$(\Omega^\infty \Sigma^\infty S^0)_{\text{basept}}$ is a connected comm. monoid in Spaces_*.

Thm (Moore): any such space is a product of $K(\pi, n)$'s.

Strickland:
$$\begin{array}{ccc} \pi_2 \Omega^\infty \Sigma^\infty S^0 & \xrightarrow[\neq 0]{\eta} & \pi_3 \Omega^\infty \Sigma^\infty S^0 \\ \| & & \| \\ \pi_2 \mathbb{S} & & \pi_3 \mathbb{S} \end{array}$$

∎

Adams's "handcrafted" smash product

(Boardman)

Def. $X \wedge Y$ has terms

$X_0 \wedge Y_0, \quad X_1 \wedge Y_0, \quad X_1 \wedge Y_1, \quad X_2 \wedge Y_1, \quad \ldots$

$\Sigma X_0 \wedge Y_0 \to X_1 \wedge Y_0$

$X_1 \wedge \Sigma Y_0 \longrightarrow X_1 \wedge Y_1 \quad \ldots$

Problem. Not associative.

$$[(X \wedge Y) \wedge Z]_2 = (X_1 \wedge Y_0) \wedge Z_1 \neq X_1 \wedge (Y_1 \wedge Z_0) = [X \wedge (Y \wedge Z)].$$

Adams says: given any $\mathbb{N} = B \amalg C$,

$(X \underset{BC}{\wedge} Y)_0 = X_0 \wedge Y_0$

Obtain $(X \underset{BC}{\wedge} Y)_{n+1}$ from $(X \underset{BC}{\wedge} Y)_n$ by raising the index of X if $n \in B$, and of Y if $n \in C$.

These are equivalent to each other as long as B, C both infinite.

Some ideas for fixing it:

$(X \wedge Y)_1$ wants to be both $X_1 \wedge Y_0$ and $X_0 \wedge Y_1$.

Can it contain both?

Spectra are indexed by $\mathbb{N}$ so that their spaces can be related to each other by suspensions:

$S^k \wedge X_n = \Sigma^k X_n \longrightarrow X_{k+n}$

$S^k = \mathbb{R}^k \amalg \{\infty\} \circlearrowleft O(k)$

Can replace the poset $\mathbb{N}$ with the category of finite-dimensional $\mathbb{R}$-vector spaces & isometries.

Symmetric spectra.

Def. A <u>symmetric space</u> is a collection of spaces$_*$ $X_n \circlearrowleft \Sigma_n$.

Equivalently: a functor $\mathrm{FinSet}^{\simeq} \to \mathrm{Spaces}_*$.

Σ-Spaces_* = cat of symm. spaces.

$$(X \wedge Y)_n = \bigvee_{i+j=n} (\Sigma_n)_+ \wedge_{\Sigma_i \times \Sigma_j} (X_i \wedge Y_j).$$

$$(X \wedge Y)(A) = \operatorname*{colim}_{B \amalg C \xrightarrow{\sim} A} X(B) \wedge Y(C)$$

ex. $\mathbb{S} = \{ S^n \circlearrowleft \Sigma_n \}$ is a commutative monoid in Σ-Spaces_*.

$$S^n = \mathbb{R}^n \amalg \{*\}$$

A <u>symmetric spectrum</u> is an $\mathbb{S}$-module in Σ-Spaces_*.

$$\mathbb{S} \wedge X \longrightarrow X$$

$$\bigvee_{i+j=n} (\Sigma_n)_+ \wedge_{\Sigma_i \times \Sigma_j} (S^i \wedge X_j) \longrightarrow X_n \qquad \Sigma_n\text{-equivariant}$$

$$S^i \wedge X_j \longrightarrow X_n \qquad \Sigma_i \times \Sigma_j\text{-equivariant.}$$

$$S^1 \wedge X_j \longrightarrow X_{1+j}$$

Define a levelwise model structure in which fibs & WEs are levelwise fibs & WEs of Σ_n-spaces.

Define a stable model structure by forcing all π_*-isos on fibrant replacements to be WEs.

[Hovey - Shipley - Smith]

Thm Σ-Sp is a symmetric monoidal stable model category Quillen equivalent to Sp.

Orthogonal spectra

Def. An orthogonal space is a functor

{finite-dimensional real inner product spaces, isometries} $\longrightarrow$ Spaces_*

$\simeq$

$\mathbb{N}$ where $\mathrm{Hom}(m,n) = \begin{cases} O(n) & \text{if } m=n \\ \emptyset & \text{otherwise.} \end{cases}$

$O(n)$ acts continuously on the n^{th} space.

O-Spaces_*

$$(X \wedge Y)_n = \bigvee_{i+j=n} O(n)_+ \underset{O(i)\times O(j)}{\wedge} X_i \wedge X_j.$$

$$\mathbb{S}_n = S^n = \mathbb{R}^n \amalg \{*\}.$$

An *orthogonal spectrum* is a module over $\mathbb{S}$ in O-Spaces_*.

Thm There's a stable model category of O-Sp which is a symmetric monoidal model category under $-\wedge-$.

[Lewis - May]

[Mandell - May - Schwede - Shipley]

$$\mathbb{N} \wedge (\mathbb{Q} \wedge \mathbb{Q}) \not\simeq (\mathbb{N} \wedge \mathbb{Q}) \wedge \mathbb{Q}$$

$$X(V)$$

$$H\mathbb{Z}_n = \mathbb{Z}[S^n]$$

$$MO_n = \mathrm{Th}(\underset{\substack{\circlearrowright \\ O(n)}}{V} \longrightarrow BO(n))$$