Why ∞-categories?

$\mathcal{M}$ - simplicial (or topological) model category
$\mathrm{Ho}\,\mathcal{M} = \mathcal{M}^{cf}/\sim$
$\underline{\mathrm{Maps}}_{\mathcal{M}}(X,Y)$ — a simplicial set or space

Can $\mathrm{Ho}\,\mathcal{M}$ as enriched over $\mathrm{Ho}(\mathrm{Spaces})$.

Approach 1: work directly with simplicial/topological categories.

Can think of a category enriched in htpy types as a htpy type in sSet Cat / Top Cat.

Thm (Bergner). There's a model structure on sSetCat where:

$\mathcal{C} \xrightarrow[F]{\sim} \mathcal{D}$ $F: \mathcal{C}(x,y) \xrightarrow{\sim} \mathcal{D}(Fx, Fy)$
$F: \mathrm{Ho}\,\mathcal{C} \xrightarrow{\sim} \mathrm{Ho}\,\mathcal{D}$ equiv of 1-categories

$\mathcal{C} \overset{F}{\twoheadrightarrow} \mathcal{D}$ $F: \mathcal{C}(x,y) \twoheadrightarrow \mathcal{D}(Fx, Fy)$
F is an "isofibration", i.e.,
given $x \in \mathcal{C}$ and $Fx \xrightarrow{\sim} y$ in $\mathcal{D}$,
this lifts to $x \xrightarrow{\sim} y'$ in $\mathcal{C}$.

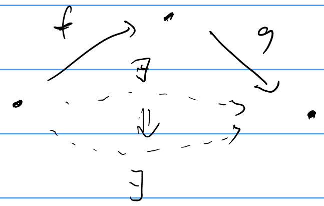
$\mathcal{C}$ is a fibrant sSet Cat $\iff$ it's enriched over fibrant sSets, i.e. Kan complexes.

Drawback: Easier to talk about commutative diagrams in a sSet Cat than to talk about homotopy coherent diagrams.

- homotopy colimits
- Don't want Sp to have a strict symmetric monoidal structure — more like a coherent symmetric monoidal structure

Approach 2: Quasicategories (Joyal, Lurie)

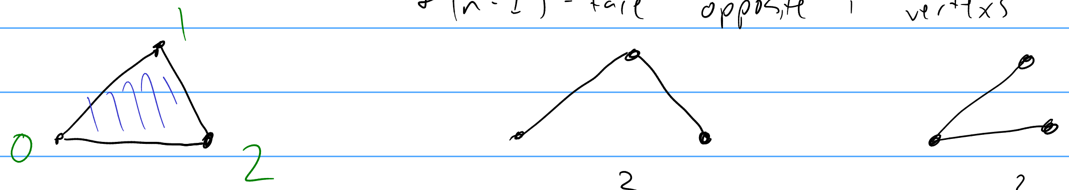
Idea:



Def. Δ^n = n-simplex

$\cup|$

$\Lambda^n_i = \Delta^n - \{$central n-simplex $+ (n-1)$-face opposite i^{th} vertex$\}$ $(0 \le i \le n)$



Δ^2 Λ^2_1 Λ^2_0.

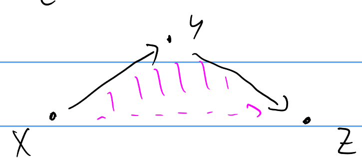
An inner horn is Λ^n_i where $0 < i < n$

Def. A quasicategory (henceforth ∞-category) is a simplicial set $\mathcal{C}$ such that

$$\begin{tikzcd} \text{inner horn } \Lambda^n_i \arrow[r] \arrow[d] & \mathcal{C} \\ \Delta^n \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

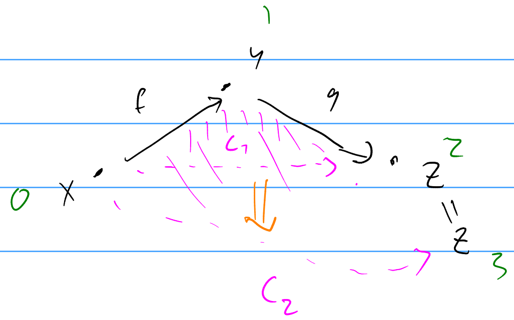
ex. ob $\mathcal{C} = \mathcal{C}_0$

$\mathrm{Maps}_{\mathcal{C}}(x,y)$ = 1-simplices from x to y.



$$\begin{tikzcd} \Lambda^2_1 \arrow[r] \arrow[d] & \mathcal{C} \\ \Delta^2 \arrow[ur, dashed] & \end{tikzcd}$$

ex.



$$\begin{tikzcd} \Lambda^3_1 \arrow[r] \arrow[d] & \mathcal{C}. \\ \Delta^3 \arrow[ur, dashed] & \end{tikzcd}$$

3 major examples:

① If $\mathcal{C}$ is a 1-cat, $N(\mathcal{C}) \in sSet$

In $N(\mathcal{C})$, every inner horn has a unique filler

So $N(\mathcal{C})$ is an ∞-cat.

② If X is a space, can regard it as a Kan complex ($Sing(X)$)

In a Kan complex, every horn has a filler.

So X is an ∞-category. (∞-groupoid.)

③ If $\mathcal{C}$ is a cofibrant topologically or fibrant simplicially enriched cat, we can form its homotopy coherent nerve $N(\mathcal{C})$.

Idea: $N(\mathcal{C})_0 = Ob\,\mathcal{C}$

$N(\mathcal{C})_1 =$ morphisms in $\mathcal{C}$ (0-simplices / points in each $\mathcal{C}(x,y)$)

$N(\mathcal{C})_2 = \{$
$$\begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, "h"'] & & z \end{tikzcd}$$
, w/ a path in $\mathcal{C}(x,z)$ from gf to $h\}$

etc.

$$\mathcal{M} \longmapsto N(\mathcal{M}^{cf})$$

1 additional construction: Given a $K \in sSet$ and an ∞-category $\mathcal{C}$,

define $Fun(K, \mathcal{C}) = \underline{Maps}_{sSet}(K, \mathcal{C})$.

$(Fun(K,\mathcal{C})_n = Maps_{sSet}(K \times \Delta^n, \mathcal{C}))$.

This is also an ∞-category.

ex. $Fun(\Delta^2, \mathcal{C}) =$
$$\begin{tikzcd} & \cdot \arrow[dr, "g"] & \\ \cdot \arrow[ur, "f"] \arrow[rr] \arrow[rr, phantom, shift left=3, "\Downarrow"] & & \cdot \end{tikzcd}$$

<u>Thm</u> (Joyal). There's a model structure on sSet in which:

$\hookrightarrow$ monomorphisms of sSets

$A \xrightarrow{\sim} B$ if for every ∞-cat $\mathcal{C}$,

$\mathrm{hoFun}(B, \mathcal{C}) \xrightarrow{\sim} \mathrm{hoFun}(A, \mathcal{C})$ equiv of 1-categories

$X \twoheadrightarrow *$ iff X is an ∞-category.

$$\mathrm{sSetCat}_{\mathrm{Bergner}} \rightleftarrows \mathrm{sSet}_{\mathrm{Joyal}} : N$$

Quillen equiv.

An <u>initial object</u> in an ∞-cat $\mathcal{C}$ is $x \in \mathcal{C}$ such that $\mathrm{Maps}_{\mathcal{C}}(x, y) \simeq *$.

Can define colimits: given $p : K \to \mathcal{C}$, $\mathrm{colim}\, p$ is an initial object in an ∞-category of cones on p.

<u>ex</u>.

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \\ \bullet & \end{tikzcd} \qquad \begin{tikzcd} \bullet \arrow[r] \arrow[d] \arrow[dr] & \bullet \arrow[d] \\ \bullet \arrow[r] \arrow[ur, phantom, "\Rightarrow"] & \bullet \end{tikzcd}$$

p cone on p

<u>Stable ∞-categories</u>.

<u>Def</u>. $\mathcal{C}$ is <u>stable</u> if

1. $\mathcal{C}$ is <u>pointed</u>
   ($\mathcal{C}$ has an initial & final object, & they're equivalent — call this $*$)
2. $\mathcal{C}$ has finite lims & colims
3. A square
$$\begin{tikzcd} \cdot \arrow[r] \arrow[d] \arrow[dr] & \cdot \arrow[d] \\ \cdot \arrow[r] \arrow[ur, phantom, "\Rightarrow"] & \cdot \end{tikzcd}$$
in $\mathcal{C}$ is a pushout square iff it's a pullback square.

Equivalent to (3): Σ and Ω are inverse equivalences

If M is a stable simplicial model category, $N(M^{cf})$ is a stable ∞-cat.

Stabilization.

If $\mathcal{C}$ is a pointed ∞-cat,

$$Sp(\mathcal{C}) = \lim(\cdots \xrightarrow{\Omega} \mathcal{C} \xrightarrow{\Omega} \mathcal{C} \xrightarrow{\Omega} \mathcal{C})$$

the ∞-category of spectrum objects in $\mathcal{C}$.

(limit in the ∞-category of ∞-categories)

objects: $X_0 \xrightarrow{\simeq} \Omega X_1 \xrightarrow{\simeq} \Omega^2 X_2 \xrightarrow{\simeq} \cdots$

(If $\mathcal{C}$ is presentable) $\Sigma^\infty : \mathcal{C} \rightleftarrows Sp(\mathcal{C}) : \Omega^\infty$

ex. Given a top. space Z, there's an ∞-cat of sheaves of spaces on Z.

($\{U_i\}$ open cover of $U \subseteq Z$,

$$\mathcal{F}(U) \longrightarrow \left(\prod \mathcal{F}(U_i) \rightrightarrows \prod \mathcal{F}(U_i \cap U_j) \rightrightarrows \prod \mathcal{F}(U_i \cap U_j \cap U_k) \cdots\right)$$

homotopy limit diagram.)

$Sh(Z, \text{Spaces}) \longrightarrow Sh(Z, Sp) = Sp(Sh(Z, \text{Spaces}))$.

ex. {smooth schemes over some base}

$\Rightarrow$ motivic homotopy theory.

Gepner - Spitzweck.

$$\mathrm{Fun}^{\text{colim-preserving}}(Sp, \mathcal{D}) \xrightarrow[\mathrm{ev}(S^0)]{\sim} \mathcal{D}$$

stable.

"Abstract Goerss–Hopkins theory"
Pstrągowski – VanKoughnett.

$\underline{S}^n$ (constant sheaf on S^n)

$\mathbb{G}_m^{\wedge n}$

KO

KU

$=$

$(\Sigma_+^\infty \mathbb{C}P^\infty)[\beta^{-1}]$

$$TAQ(\Sigma_+^\infty \mathbb{C}P^\infty / \mathbb{S}) = (\Sigma_+^\infty \mathbb{C}P^\infty) \wedge \mathbb{C}P^\infty$$