## Finite spectra.

**Def.** A *finite spectrum* is a spectrum of the form
$$\Sigma^{-N}\Sigma^{\infty} K$$
where $K$ is a finite CW-complex.

These are built out of finitely many cells.

**Thm.** Every spectrum is a filtered hocolim of finite spectra.

**Pf** Can write $X = \operatorname{hocolim}_{n\to\infty} \tau_{\geq -n} X$

$$\pi_* \tau_{\geq -n} X = \begin{cases} 0 & * < -n \\ \pi_* X & * \geq -n. \end{cases}$$

Say $X$ is an $\Omega$-spectrum

$$\begin{array}{ccccccc} X_0, & X_1, & \cdots, & X_n, & \cdots \\ & & & \uparrow & \uparrow \\ *, & \cdots & , & \tau_{\geq 0} X_n, & \tau_{\geq 1} X_{n+1}, & \cdots \end{array}$$

$$\tau_{\geq -n} X \longrightarrow X.$$

Reduce to the case $X$ is $(-n)$-connective
After suspending, reduce to the case $X$ is $0$-connective.

We can write $X \simeq X^{CW}$, where $X^{CW}$ is a CW-spectrum
$X^{CW}$ = hocolim of its finite subcomplexes.

$\blacksquare$

$$\Sigma^{-1} Y \xrightarrow{0} X \longrightarrow X \vee Y$$

## Dévissage

If some property $P$ holds for $\mathbb{S}$, is closed under suspension & desuspension, and is closed under cofibers, then it holds for all finite spectra.

If $P$ is also closed under filtered hocolims, then it holds for all spectra.

## Spanier-Whitehead duality

Recall: $\cdot \wedge \cdot$, $F(\cdot, \cdot)$, $\mathrm{Maps}(\cdot, \cdot) = \Omega^\infty F(\cdot, \cdot)$

These are always secretly derived, e.g. $X \wedge Y = X^c \wedge Y^c$.

**Def.** The Spanier-Whitehead dual of $X$ is
$$DX = F(X, \mathbb{S}).$$

$$[W, DX] = \underline{[W \wedge X, \mathbb{S}]}$$

Say that $X$ is (strongly) dualizable if we have
$$[W, DX \wedge Y] \cong [W \wedge X, Y]$$
(naturally in $W$ and $Y$.)

$$[W, F(X,Y)]$$

$$\Leftrightarrow \quad DX \wedge Y \xrightarrow{\simeq} F(X, Y) \qquad DX \wedge Y \wedge X \xrightarrow{ev_X} Y$$

$$\Leftrightarrow \quad \mathbb{S} \xrightarrow{\eta} DX \wedge X$$

$$DX \wedge X \xrightarrow{\varepsilon} \mathbb{S} \qquad F(X, \mathbb{S}) \wedge X \xrightarrow{\varepsilon = ev} \mathbb{S}$$

$$\underbrace{X \simeq X \wedge \mathbb{S} \xrightarrow{1 \wedge \eta} X \wedge DX \wedge X \xrightarrow{\varepsilon \wedge 1} \mathbb{S} \wedge X \simeq X}_{id}$$

$$\underbrace{DX \xrightarrow{1 \wedge \eta} DX \wedge X \wedge DX \xrightarrow{\varepsilon \wedge 1} DX}_{id}$$

If $- \wedge DX : Sp \rightleftarrows Sp : X \wedge -$

they're also adjoint in the other direction.

In particular, if $X$ is dualizable, $DDX \simeq X$.

ex. $X = S^n$

$$[W, X \wedge Y] = [W, \Sigma^n Y] = [\Sigma^{-n} W, Y]$$
$$= [W \wedge S^{-n}, Y]$$

$DS^n \simeq S^{-n}$

$S^n$ is dualizable.

Def. A spectrum is invertible if it's dualizable and

$\mathbb{S} \xrightarrow{\sim} DX \wedge X \xrightarrow{\sim} \mathbb{S}$

$S^n \wedge S^{-n} \simeq \mathbb{S}$.

ex. The mod $p$ Moore spectrum is the cofiber

$$S^0 \xrightarrow{p} S^0 \longrightarrow M(p) \longrightarrow S^1 \xrightarrow{p} S^1 \longrightarrow \cdots$$

$\pi_0 S^0 = \mathbb{Z}$

$$S^0 \xleftarrow{p} S^0 \longleftarrow DM(p) \longleftarrow S^{-1} \xleftarrow{p} S^{-1}$$

$$DM(p) \simeq \Sigma^{-1} M(p)$$

$$\begin{tikzcd} S^0 \arrow[r, "\Delta"] \arrow[rrr, bend right, "p"'] & (S^0)^{\times p} \simeq (S^0)^{\vee p} \arrow[r, "\nabla"] & S^0 \end{tikzcd}$$

M(p) isn't invertible: $\Sigma M(p) \simeq \Sigma^\infty(\underbrace{\mathrm{cof}(S^1 \xrightarrow{p} S^1)}_{X})$.

$X = S^1 \cup_p D^2$

$\tilde{H}_*(X; \mathbb{F}_p) = \mathbb{F}_p$ in degrees 1 and 2.

$\tilde{H}_*(X \wedge X; \mathbb{F}_p) = \tilde{H}_*(X; \mathbb{F}_p) \otimes \tilde{H}_*(X; \mathbb{F}_p)$

$= \mathbb{F}_p$ in degree 2

$\mathbb{F}_p^{\oplus 2}$ 3

$\mathbb{F}_p$ 4.

This is also $H\mathbb{F}_{p*}(\Sigma^\infty(X \wedge X))$
$$\shortparallel$$
$$H\mathbb{F}_{p*}(\Sigma^2(M(p) \wedge M(p)))$$

So $M(p) \wedge DM(p) \simeq \Sigma^{-1} M(p) \wedge M(p)$ is not a sphere.

ex. $DH\mathbb{F}_p = 0$.

**Theorems.** ① $X$ is dualizable $\iff$ $X$ is finite.
② $X$ is invertible $\iff$ $X$ is a sphere.

Remark. (Invertible objects in $(\mathcal{C}, \otimes, \mathbb{1})$)$/\cong$
$=$ Picard group of $\mathcal{C}$

$\mathrm{Pic}(Sp) = \mathbb{Z}$ gen. by $S^1$

"Picard-graded homotopy groups"
$\pi_\star X = [S^\star, X]$ for $S^\star \in \mathrm{Pic}(\mathcal{C})$.

Proof of ① If $X$ is dualizable, then
$F(X, \cdot) \simeq DX \wedge \cdot$
commutes with hocolims.
In particular, it commutes with filtered hocolims,
i.e. $X$ is compact.

$X = \mathrm{hocolim}\, F_\alpha$, $F_\alpha$ finite

$X \xrightarrow{id_X} X = \mathrm{hocolim}\, F_\alpha$
factors through some $F_\alpha$.
$$\begin{tikzcd} X \arrow[r] \arrow[rr, bend right, "id_X"'] & F_\alpha \arrow[r] & X \end{tikzcd}$$

$X$ is a retract of a finite spectrum.

⚠ A retract of a finite CW-complex in Spaces may not be finite.

$$H_*(X;\mathbb{Z}) \longrightarrow H_*(F_\alpha;\mathbb{Z}) \longrightarrow H_*(X;\mathbb{Z})$$

So $H_*X$ is a retract of a f.g. ab gp, so it's a f.g. ab gp.

$\coprod S^n \longrightarrow X$ inducing a surjection on the lowest nonzero homology group.

If $X$ has one nonzero homology group,

$$\coprod S^n \longrightarrow X.$$

free. $\longrightarrow \mathbb{Z}^N \twoheadrightarrow H_n(X)$

∎

<u>Pf of ②</u>. $H_*(X \wedge Y; k) \cong H_*(X;k) \otimes H_*(Y;k)$ for any field $k$.

$X$ is an invertible spectrum.

$X$ is dualizable, so it's finite.

WLOG, $X$ is 0-connective but not 1-connective.

$$X \wedge DX \simeq S^0$$

$$H_*(X;k) \otimes H_*(DX;k) \cong H_*(S^0;k) = k.$$

So $H_*(X;k) \cong k$

Using UCT, $H_*(X;\mathbb{Z}) \cong \mathbb{Z}$.

$$S^0 \longrightarrow X$$

∎

If $X \longrightarrow Y$ is a map of connective spectra

$$H_*(X;\mathbb{Z}) \xrightarrow{\cong} H_*(Y;\mathbb{Z})$$

then $X \xrightarrow{\simeq} Y$.

# Alexander duality

$X$ = finite CW-complex, w/ basept.

$X \hookrightarrow S^N$ sending $* \longmapsto 0$ and missing $\infty$.

$S^N - X$ has basept $\infty$.

$$\tilde{H}^*(X) \cong \tilde{H}_{N-*-1}(S^N - X)$$

$$X \hookrightarrow S^N$$

$$\Sigma X \hookrightarrow S^{N+1} \qquad S^{N+1} - \Sigma X \simeq \Sigma(S^N - X)$$

$$D\Sigma^\infty X = \Sigma^{-N+1}\Sigma^\infty(S^N - X)$$

$$X \longmapsto \mathrm{Hom}(\pi_{-*}X, \mathbb{Q}/\mathbb{Z})$$
$$\shortparallel$$
$$[X, I_{\mathbb{Q}/\mathbb{Z}}]$$

$$\mathrm{Hom}(\pi_{-*}X, \mathbb{Q})$$
$$\shortparallel$$
$$[X, I_{\mathbb{Q}}].$$

$$I_{\mathbb{Q}} = H\mathbb{Q}.$$

$$I_{\mathbb{Z}} \longrightarrow I_{\mathbb{Q}} \longrightarrow I_{\mathbb{Q}/\mathbb{Z}}$$