- Hurewicz theorem: if X is 0-connective ($\pi_* X = 0$ in $* < 0$) then $\pi_0 X = H_0(X; \mathbb{Z})$.
- Whitehead theorem: if $X \to Y$ is a map of connective spectra inducing an iso $H_*(X;\mathbb{Z}) \xrightarrow{\sim} H_*(Y;\mathbb{Z})$ then $X \xrightarrow{\sim} Y$.
- Künneth & Universal coefficient theorems:

$$E_* X,\ E_* Y \rightsquigarrow E_*(X \wedge Y)?$$

$$E_* X \rightsquigarrow F_* X, \quad \text{for } F \text{ a module over } E.$$

$$\pi_* HR = \begin{cases} R & * = 0 \\ 0 & * \neq 0. \end{cases}$$

$$H_*(X; R) = \pi_*(X \wedge HR).$$

<u>Generalization of Hurewicz:</u>

If X, Y are 0-connective, then

$$\pi_0(X \wedge Y) = \pi_0 X \otimes \pi_0 Y.$$

$Y = H\mathbb{Z}$:

$$\pi_0(X \wedge H\mathbb{Z}) = \pi_0 X \otimes \pi_0 H\mathbb{Z} = \pi_0 X.$$

<u>Proof</u>. Let $\mathcal{C}$ be the class of spectra X for which the statement is true for all 0-connective Y.

(1) $S^0 \in \mathcal{C}$.

(2) $\mathcal{C}$ closed under $\bigvee$.

$X = \bigvee X_i$

$$\pi_0(X \wedge Y) = \pi_0\left(\left(\bigvee X_i\right) \wedge Y\right) = \pi_0\left(\bigvee (X_i \wedge Y)\right)$$

$$= [S^0, \bigvee(X_i \wedge Y)] = \bigoplus [S^0, X_i \wedge Y]$$

$$= \bigoplus \pi_0(X_i) \otimes \pi_0 Y = \left(\bigoplus \pi_0 X_i\right) \otimes \pi_0 Y$$

$$= \pi_0 X \otimes \pi_0 Y.$$

③ $\mathcal{C}$ is closed under cofibers.

$X_1, X_2 \in \mathcal{C}$

$$X_1 \longrightarrow X_2 \longrightarrow Z \longrightarrow \Sigma X_1$$

$$X_1 \wedge Y \longrightarrow X_2 \wedge Y \longrightarrow Z \wedge Y \longrightarrow \Sigma X_1 \wedge Y$$

$$\begin{array}{ccccccc} \pi_0(X_1 \wedge Y) & \longrightarrow & \pi_0(X_2 \wedge Y) & \twoheadrightarrow & \pi_0(Z \wedge Y) & \longrightarrow & \pi_0(\Sigma X_1 \wedge Y) \\ \| & & \| & & & & \| \\ \pi_0 X_1 \otimes \pi_0 Y & \longrightarrow & \pi_0 X_2 \otimes \pi_0 Y & \rightarrow & \dfrac{\pi_0 X_2 \otimes \pi_0 Y}{\pi_0 X_1 \otimes \pi_0 Y} & & 0. \\ & & & & \| & & \\ & & & & \pi_0 Z \otimes \pi_0 Y. & & \end{array}$$

$\therefore$ $\mathcal{C}$ contains all 0-connective spectra ∎

<u>Whitehead Thm</u>. $X \longrightarrow Y$ map of n-connective spectra inducing an iso on $H_*(\,\cdot\,; \mathbb{Z})$.

$$F \longrightarrow X \longrightarrow Y \quad \text{fiber sequence.}$$

Want to show $F \simeq *$.

WLOG, $n = 0$.

$$\begin{array}{ccccccc} H_* F & \longrightarrow & H_* X & \xrightarrow{\sim} & H_* Y & \longrightarrow & H_{*-1} F \\ \| & & & & & & \\ 0. & & & & & & \end{array}$$

By Hurewicz, $\pi_* F = 0$, so $F \simeq *$. ∎.

<u>ex</u>. $K \xrightarrow{p} K \longrightarrow K/p$

$$H\mathbb{Z}_*(K/p) = 0.$$

Ring spectra + module spectra.

A <u>ring spectrum</u> is a spectrum E equipped with a multiplication $\mu : E \wedge E \to E$ & a unit $\eta : S^0 \to E$.

$$\begin{tikzcd} E\wedge E\wedge E \arrow[r, "1\wedge\mu"] \arrow[d, "\mu\wedge 1"'] & E\wedge E \arrow[d, "\mu"] \\ E\wedge E \arrow[r, "\mu"'] & E \end{tikzcd} \qquad \begin{tikzcd} E\wedge S^0 \arrow[r, "1\wedge\eta"] & E\wedge E \\ E \arrow[u, "\wr"] \arrow[r, "1"] \arrow[d, "\wr"'] & E \arrow[u, leftarrow] \\ S^0\wedge E \arrow[r, "\eta\wedge 1"'] & E\wedge E \end{tikzcd}$$

These diagrams commute up to homotopy.

E is <u>homotopy commutative</u> if

$$\begin{tikzcd} E\wedge E \arrow[dr, "\mu"] \arrow[d, "\text{swap}"'] & \\ E\wedge E \arrow[r, "\mu"'] & E \end{tikzcd}$$

commutes up to homotopy.

M is a (left) <u>module</u> over E if there's $\nu : E\wedge M \to M$ making similar diagrams commute.

$$E_k(X) \otimes E_\ell(Y) \longrightarrow E_{k+\ell}(X\wedge Y)$$

$$E^k(X) \otimes E^\ell(Y) \xrightarrow{m} E^{k+\ell}(X\wedge Y)$$

$$E^k(X) \otimes E_{k+\ell}(X\wedge Y) \longrightarrow E_\ell(Y).$$

$$[\Sigma^{-k}X, E] \otimes [\Sigma^{-\ell}Y, E]$$

$$\downarrow$$

$$[\Sigma^{-k}X\wedge\Sigma^{-\ell}Y, E\wedge E] \xrightarrow{\mu} [\Sigma^{-k}X\wedge\Sigma^{-\ell}Y, E]$$

$$\| $$

$$[\Sigma^{-k-\ell}(X\wedge Y), E]$$

$$\|$$

$$E^{k+\ell}(X\wedge Y).$$

If E is homotopy commutative ring spectrum, then these products are graded commutative

$$\begin{tikzcd} E^k(X) \otimes E^\ell(Y) \arrow[r] \arrow[d, "s"'] \arrow[dr, phantom, "\circlearrowright (-1)^{k\ell}"] & E^{k+\ell}(X \wedge Y) \arrow[d, "s"] \\ E^\ell(Y) \otimes E^k(X) \arrow[r] & E^{k+\ell}(Y \wedge X) \end{tikzcd}$$

$S^k \wedge S^\ell \xrightarrow{\sim} S^\ell \wedge S^k$ has degree $(-1)^{k\ell}$.

$$\underbrace{E_k(S^0)}_{\pi_k E} \otimes E_\ell(X) \longrightarrow E_{k+\ell}(S^0 \wedge X) = E_{k+\ell}(X)$$

$E_* X$ is a graded module over $\pi_* E$.

$$\underbrace{E^k(S^0)}_{\pi_{-k} E} \otimes E^\ell(X) \longrightarrow E^{k+\ell}(S^0 \wedge X) = E^{k+\ell}(X)$$

$E^* X$ is a graded module over $\pi_{-*} E$.

ex. ① S^0 is a homotopy commutative ring spectrum.

② If R is a ring, HR is a ring spectrum (commutative if R is).

ex symmetric spectrum

$n \longmapsto R\{S^n\}$, free simplicial R-module on simplicial n-sphere

③ If X is a spectrum, $F(X, X)$ is a ring spectrum

④ KU, KO are commutative.

⑤ E a ring spectrum, X a space.

Then $F(\Sigma^\infty_+ X, E)$ is a ring spectrum

$$\begin{aligned} F(\Sigma^\infty_+ X, E) \wedge F(\Sigma^\infty_+ Y, E) \longrightarrow\ & F(\Sigma^\infty_+(X) \wedge \Sigma^\infty_+(X), E \wedge E) \\ & \qquad \| \\ & F(\Sigma^\infty_+(X \times X), E \wedge E) \\ & \qquad \downarrow \Delta^* \mu_* \\ & F(\Sigma^\infty_+ X, E) \end{aligned}$$

$$E^*X \otimes E^*X \longrightarrow E^*X.$$

UCT & Künneth

E = ring spectrum, M = module over E.

Then there are spectral sequences.

$$\mathrm{Tor}^{p,q}_{E_*}(E_*X, M_*) \Longrightarrow M_*X$$

$$\mathrm{Ext}^{p,q}_{E_*}(E_*X, M^*) \Longrightarrow M^*X.$$

$E_* = \pi_* E$, $M_* = \pi_* M$, $M^* = \pi_{-*} M$.

ex. E_*X is projective over E_*.

Both spectral sequences concentrated in $p=0$, & collapse at E_2 page.

$$M^*X = \mathrm{Hom}_{E_*}(E_*X, M^*)$$
$$M_*X = E_*X \otimes_{E_*} M_*.$$

ex. $E = H\mathbb{F}_p$. Then $H_*(X;\mathbb{F}_p)$ is always flat and projective over $E_* = \mathbb{F}_p$.

ex. $M = E \wedge Y$ for some spectrum Y.

$$E \wedge E \wedge Y \xrightarrow{\mu \wedge 1} E \wedge Y.$$

$$\mathrm{Tor}_{E_*}(E_*X, E_*Y) \Longrightarrow E_*(X \wedge Y).$$

$$H_*(X;\mathbb{F}_p) \otimes H_*(Y;\mathbb{F}_p) \cong H_*(X \wedge Y;\mathbb{F}_p).$$

Adams condition. (cf. §13. of blue book)

or ch. 2–3 of Ravenel's <u>Complex Cobordism & π_*S</u>.

E = filtered hocolim of finite spectra E_α such that, for any E-module M,

$$\underset{\substack{\shortparallel \\ [S,\, M \wedge E_\alpha]}}{[DE_\alpha, M]} \xrightarrow{\sim} \mathrm{Hom}_{E_*}(\underset{\substack{\shortparallel \\ E^*DE_\alpha}}{E_*DE_\alpha}, M_*),$$

and such that E_*DE_α is a finite projective E_*-module.

This holds for

$E = S^0, H\mathbb{F}_p, KU, KO, MU, MO, \ldots$

For any X, $\exists$ $W \longrightarrow X$

$$W = \bigvee \Sigma^{n_i} DE_{\alpha_i} \longrightarrow X$$

inducing $E_* W \longrightarrow E_* X$.

$$e \in E_* X \rightsquigarrow e \in E_{\alpha,*} X = [S^n, E_\alpha \wedge X]$$

$$\cong [\Sigma^n DE_\alpha, X]$$