Thm. E = ring spectrum s.t.

E = filtered hocolim of finite spectra E_α s.t.
$E_* DE_\alpha$ is finite projective as an E_*-module
and for any E-module M,

$$[DE_\alpha, M] \xrightarrow{\sim} \mathrm{Hom}_{E_*}(E_* DE_\alpha, M_*).$$

Then there's a spectral sequence

$$E_2^{p,q} = \mathrm{Ext}^{p,q}_{E_*}(E_* X, M_*) \Rightarrow M^* X. \quad (M^{p-q} X)$$

Proof. ① $\forall X$, there's a map $W \to X$
where $W = \bigvee_j \Sigma^{\beta_j} DE_{\alpha_j}$

and inducing a surjection $E_* W \twoheadrightarrow E_* X$.

②

$$\begin{tikzcd}
X = X_0 \arrow[r] & X_1 \arrow[r] & X_2 \arrow[r] & \cdots \\
W_0 \arrow[u] & W_1 \arrow[u] & W_2 \arrow[u] &
\end{tikzcd}$$

- Each $W_i = \bigvee \Sigma^{\beta_j} DE_{\alpha_{ij}}$ and $E_* W_i \twoheadrightarrow E_* X_i$
- $W_i \to X_i \to X_{i+1}$ is a cofiber sequence.

$$\begin{tikzcd}
\bigoplus M^* X_i & \bigoplus M^* X_i \arrow[l] \arrow[dr] & \bigoplus M^* X_i \arrow[l] \\
\bigoplus M^* W_i \arrow[ur] \arrow[rr, bend left, "d_1"] & & \bigoplus M^* W_i \arrow[u] \\
\arrow[from=1-1, to=2-1]
\end{tikzcd}$$

$$E_1 = \bigoplus M^* W_i = \bigoplus \mathrm{Hom}_{E_*}(E_* W_i, M_*).$$

$$W_i \longrightarrow X_i \longrightarrow \Sigma W_{i-1}$$

$$\mathrm{Hom}_{E_*}(E_* W_i, M_*) \leftarrow \mathrm{Hom}_{E_*}(E_* X_i, M_*) \leftarrow \mathrm{Hom}_{E_*}(E_* \Sigma W_{i-1}, M_*)$$

$$E_* X \twoheadleftarrow E_* W_0 \leftarrow E_* \Sigma^{-1} W_1 \leftarrow E_* \Sigma^{-2} W_2$$

is a projective resolution.

$$E_2 = \mathrm{Ext}_{E_*}(E_* X, M_*). \qquad \blacksquare$$

<u>Tor version:</u>

$$E_2 = \mathrm{Tor}_{E_*}(E_* X, M_*) \Rightarrow M_* X.$$

$$M_* DE_2 \cong E_* DE_2 \otimes_{E_*} M_* \qquad ?$$

$$\mathrm{Tor}_{E_*}(X_*, M_*) \Rightarrow \pi_* M \wedge_E X.$$

(w/o conditions on E_*)

Adams condition satisfied for
$E = \mathbb{S}, H\mathbb{F}_p, KO, KU, MO, MU, \ldots$

$$[X, M] = \mathrm{Hom}_{\mathbb{F}_p}(H_*(X, \mathbb{F}_p), M_*)$$

<u>Some examples</u>

<u>Def.</u> Let A be an abelian group.
The <u>Moore spectrum</u> for A, SA,
is a connective spectrum whose 0^{th} homology group is A
and whose other homology groups are 0.

$$0 \to \bigoplus_\beta \mathbb{Z} \to \bigoplus_\alpha \mathbb{Z} \to A \to 0$$

$$\bigvee_\beta S \to \bigvee_\alpha S \to SA$$

$$\bigvee_\beta E \to \bigvee_\alpha E \to EA$$

Given E, $EA = E \wedge SA$.

If either E_* or A is flat over $\mathbb{Z}$,

then $EA_* = E_* \otimes A$.

$$\mathrm{Tor}_{E_*}(E_*X, EA_*) \Rightarrow EA_*X$$

$$\| $$

$$\mathrm{Tor}_{\mathbb{Z}}(E_*X, A).$$

If A is flat over $\mathbb{Z}$,

$$E_*X \otimes A \cong EA_*X.$$

ex. $A = \mathbb{Z}[1/p]$.

$A = \mathbb{Z}_{(p)}$

$A = \mathbb{Q}$.

$$\pi_*(X \wedge S\mathbb{Q}) = \pi_*X \otimes \mathbb{Q}.$$

$X = S$: $\pi_* S\mathbb{Q} = \pi_* S \otimes \mathbb{Q} = \mathbb{Q}$ in degree 0.

$\pi_* S$ is a finite abelian group for $* \geq 1$.

$$\therefore S\mathbb{Q} = H\mathbb{Q}.$$

$$H_*(X;\mathbb{Q}) = \pi_*(X \wedge H\mathbb{Q}) = \pi_*X \otimes \mathbb{Q}$$

Rational spectra = graded $\mathbb{Q}$-vector spaces.

$$KO \longrightarrow KU \longrightarrow \Sigma^2 KO \longrightarrow \Sigma^1 KO$$

$$KO(S \longrightarrow C\eta \longrightarrow S^2 \xrightarrow{\Sigma\eta} S^1)$$

$$KO \wedge C\eta = KU.$$

$$2 \cdot \eta = 0.$$

$S^3 \longrightarrow S^2$ does not satisfy $2\eta = 0$

$S^4 \longrightarrow S^3$ does satisfy $2\eta = 0$

$$S[1/2] \to C\eta[1/2] \to S^2_{[1/2]} \xrightarrow{0} S^1[1/2]$$

$$C\eta[1/2] = (S^0 \vee S^2)[1/2]$$

$$KU[1/2] = KO[1/2] \vee \Sigma^2 KO[1/2]$$

$$ko \to ku \to \Sigma^2 ko$$

$$H_* ko = (A /\!/ A(1))_* \qquad H_* ku \overset{?}{=} (A /\!/ E(1))_*$$

<u>Limits</u>. $X = \operatorname{holim} X_n$

$$X \dashv \cdots \xrightarrow{f_3} X_2 \xrightarrow{f_2} X_1 \xrightarrow{f_1} X_0$$

$$X = \operatorname{hoeq}\left(\prod X_n \underset{\prod f_n}{\overset{1}{\rightrightarrows}} \prod X_n \right)$$

$$= \operatorname{hofib}\left(\prod X_n \xrightarrow[1-\prod f_n]{} \prod X_n \right)$$

$$X \longrightarrow \prod X_n \xrightarrow[1-\prod f_n]{} \prod X_n \qquad \prod \pi_* (X_n) = \pi_*\left(\prod X_n\right)$$

$$0 \to \operatorname{coker}\left(1-\prod f_n : \pi_{*+1}\left(\prod X_n\right) \to \pi_{*+1}\left(\prod X_n\right)\right) \longrightarrow \pi_* X \longrightarrow \ker\left(1-\prod f_n : \pi_*\left(\prod X_n\right) \to \pi_*\left(\prod X_n\right)\right) \to 0$$

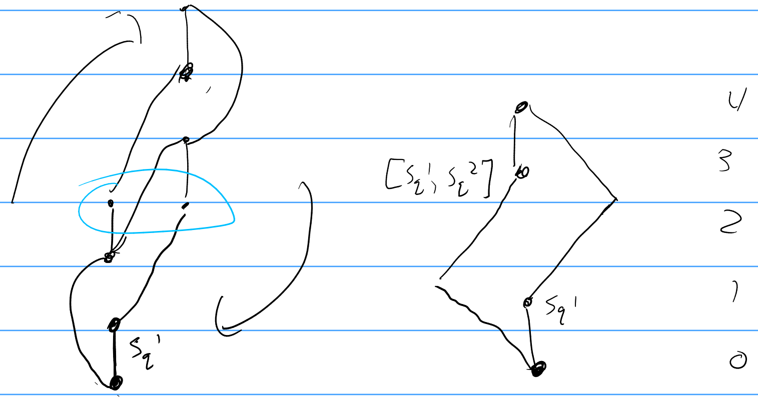
$$\operatorname{coker}(\ldots) = \lim{}^1 \pi_{*+1} X_n$$

$$\ker(\ldots) = \{(c_n \in \pi_* X_n) : f_n(c_n) = c_{n-1}\} = \lim \pi_* X_n$$

$$0 \to \lim{}^1 \pi_{*+1} X_n \longrightarrow \pi_* X \longrightarrow \lim \pi_* X_n \longrightarrow 0.$$

$$ko \to ku \to \Sigma^2 ko$$

$H_* ko = (A/\!/A(1))_*$ $\qquad$ $H_* ku \overset{?}{=} (A/\!/E(1))_*$



$$\boxed{\Sigma^\infty \mathbb{C}P^\infty [\beta^{-1}]} = KU.$$

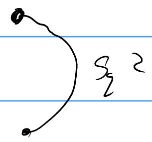
$$\mathrm{Ext}_A\left(A/\!/E(1), H^*(ko \wedge C\eta)\right) \Rightarrow [ko \wedge C\eta, ku]$$

$$\mathrm{Ext}_{E(1)}\left(\mathbb{F}_2, \, H^*(ko \wedge C\eta)\right).$$

$$H^*(ko \wedge C\eta) = H^* ko \otimes H^* C\eta, \qquad H^* ko = A/\!/A(1)$$

$A/\!/E(1) \cong$?

$H^* C\eta =$



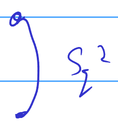
$$S^1 \to S^0 \to C\eta \to S^2$$

$$C\eta = \Sigma^{-2} \Sigma^\infty \mathbb{C}P^2 \qquad H^* \mathbb{C}P^2$$

$$S^3 \to S^2 \to \mathbb{C}P^2$$

$A/\!/A(1) = A \otimes_{A(1)} \mathbb{F}_2$

$$H^*(ko \wedge C\eta) \cong A/\!/A(1) \otimes$$



Prop. $\cong A/\!/E(1)$

$$A/\!/E(1) \to A/\!/E(1)$$

$$H\mathbb{F}_{p,*} KU = 0$$

$H\mathbb{Z}_* KU$ is rational

$$\mathbb{Z} \times BU \to \Omega^2(\mathbb{Z} \times BU) \qquad \Sigma^2(\mathbb{Z} \times BU) \to \mathbb{Z} \times BU.$$

If $\pi_* X_n$ satisfies the <u>Mittag-Leffler condition</u>, $\lim^1 \pi_* X_n = 0$.

$\operatorname{im}\left(\pi_* X_{n+k} \longrightarrow \pi_* X_n\right)$ stabilize (for each $*$, n) as $k \to \infty$.

<u>ex</u>. If $\pi_* X_{n+1} \twoheadrightarrow \pi_* X_n$, then $\lim^1$ vanishes.

<u>ex</u>. If for all $*$, n $\exists k$ s.t. $\pi_* X_{n+k} \longrightarrow \pi_* X_n$ is 0, then both $\lim$ and $\lim^1$ vanish.

$$X = \operatorname{hocolim} X^{(n)}$$

$$\underline{F(\Sigma^\infty_+ X, H) \simeq \operatorname{holim} F(\Sigma^\infty_+ X^{(n)}, H)}$$

$$0 \to \lim{}^1 H^{*-1} X^{(n)} \to H^* X \to \lim H^* X^{(n)} \to 0$$

<u>ex</u>. $H\mathbb{Z} \xleftarrow{p} H\mathbb{Z} \xleftarrow{p} H\mathbb{Z} \xleftarrow{p}$

$$\operatorname{holim} \simeq \Sigma^{-1} H(\mathbb{Z}_p/\mathbb{Z})$$

<u>Postnikov towers</u>.

In Spaces:

$$\begin{tikzcd}
 & X \arrow[d] & \\
 & \vdots & \\
K(\pi_{n+1} X, n+1) \arrow[r] & X_{\leq n+1} \arrow[d] & \\
 & X_{\leq n} \arrow[r] \arrow[d] & K(\pi_{n+1} X, n+2) \\
 & \vdots &
\end{tikzcd}$$

$$X_{\geq n+1} \longrightarrow X \longrightarrow X_{\leq n}$$