Model Categories

I am speaking from Ngunnawal country.

Abstract homotopy theory is about notions of equivalence, specifically about weakening them.

Eg Homeomorphism ⇝ homotopy
Chain Isomorphism ⇝ quasi-isomorphism

In category theory isomorphism is the notion of equivalence.

There is the notion of a homotopical category, where you have a collection of morphisms called weak equivalences which behave like isomorphisms but fail to be invertible.

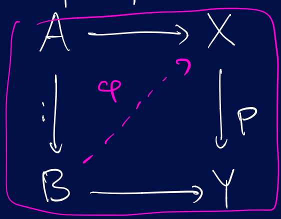
Q: What if we turn all the weak equivalences into isomorphisms?

A: $\mathcal{C}[W^{-1}]$ might not be a (locally small) category. Even if it is, difficult to use

↳ Solution: Model categories.

First a digression about lifting (Do you even?)

Def^n Let $i: A \to B$ and $p: X \to Y$ in $\mathcal{M}$.
If for all commutative squares



there exists $\varphi: B \to X$ such that the diagram still commutes then we say

i has the left lifting property (LLP) w.r.t. p
p has the right lifting property (RLP) w.r.t. i
and we write $i \boxslash p$

Given a collection of maps I we write

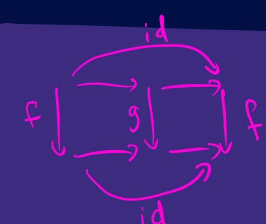
$I^{\boxslash} = \{ g \in Mor(\mathcal{M}) \mid f \boxslash g \text{ for all } f \in I \}$

${}^{\boxslash}I = \{ h \in Mor(\mathcal{M}) \mid h \boxslash f \text{ for all } f \in I \}$

If $L \subseteq {}^{\boxslash}I$ we write $L \boxslash I$

If $R \subseteq I^{\boxslash}$ we write $I \boxslash R$

Def^n A map f is a retract of g if the exists a commutative diagram



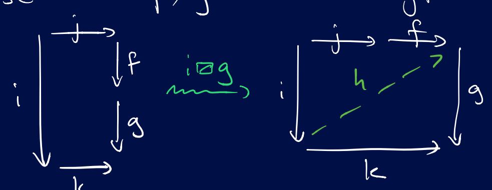
Closure properties:

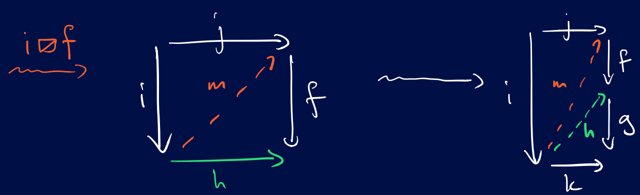
Let $I \subseteq Mor(\mathcal{M})$ then

$I^{\boxslash}$ is closed under (1) composition, (2) retracts, pullbacks

${}^{\boxslash}I$ is closed under // , // , (3) pushouts

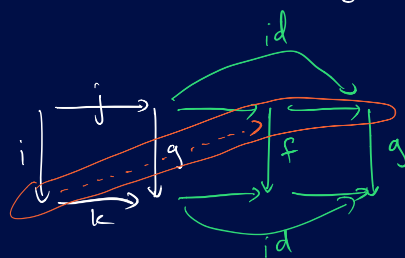
(1) Suppose $i \boxslash f, g$ with gf defined.
Given

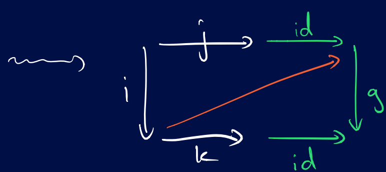




So $i \boxslash gf$

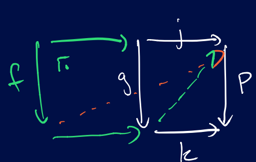
(2) Suppose $i \boxslash f$ and g is a retract of f.
Given





so $i \boxslash g$

(3) Suppose $f \boxslash p$ and g is a pushout of f.
Given

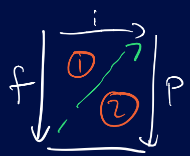


Prop (The retract argument)
Suppose $f = pi$, that is

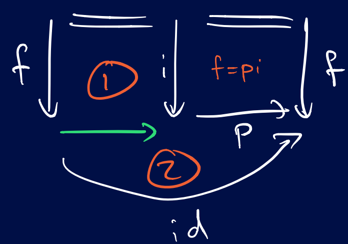
i) If $f \boxslash p$ then f is a retract of i

ii) If $i \boxslash f$ —— // —— p.

Have

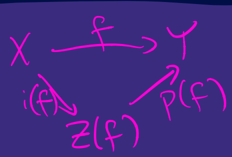


So



Def$^{\underline{n}}$ A weak factorisation system (WFS) on a category $\mathcal{M}$ is a pair of classes of maps (L, R) s.t.

i) For any $f : X \to Y$ in $\mathcal{M}$ we have



s.t. $i(f) \in L$ and $p(f) \in R$

ii) $L = {}^{\boxslash}R$ and $R = L^{\boxslash}$

Prop Suppose (L, R) satisfy i) and $L \boxslash R$. If L and R are closed under retracts then (L, R) is a WFS.

Factor each map + apply retract arg

Def$^{\underline{n}}$ A **model structure** on a bicomplete category, $\mathcal{M}$, consists of three distinguished classes of morphisms $(\mathcal{W}, \mathcal{C}, \mathcal{F})$

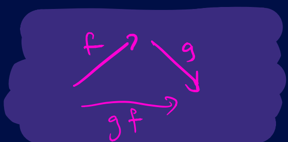
$\mathcal{W}$ = **weak equivalences** denoted $\xrightarrow{\sim}$

$\mathcal{C}$ = **cofibrations** denoted $\rightarrowtail$

$\mathcal{F}$ = **fibrations** denoted $\twoheadrightarrow$

satisfying

i) **2-out-of-3** If two of



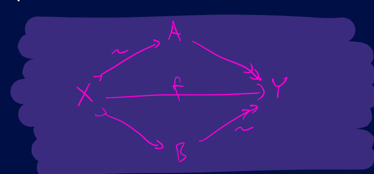
are in $\mathcal{W}$, so is the third

ii) $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$, $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ } are WFS

ii) **Retracts** $\mathcal{W}, \mathcal{F}, \mathcal{C}$ are closed under retracts.

iii) **Lifting** $\mathcal{C} \boxslash (\mathcal{F} \cap \mathcal{W})$ — trivial/acyclic fibrations
$(\mathcal{C} \cap \mathcal{W}) \boxslash \mathcal{F}$ — // cofib

iv) **Factorisation** For any $f : X \longrightarrow Y$ in $\mathcal{M}$ we have factorisations



A **model category** is bicomplete category $\mathcal{M}$ together with a model structure on $\mathcal{M}$.

Prop This is overdetermined in that any two of $\mathcal{W}, \mathcal{C}, \mathcal{F}$ determines the third.

$\mathcal{C} = {}^{\boxslash}(\mathcal{F} \cap \mathcal{W})$

$\mathcal{F} = (\mathcal{C} \cap \mathcal{W})^{\boxslash}$

If $\mathcal{C}, \mathcal{F}$ are known, then we know $(\mathcal{F} \cap \mathcal{W})$ and $(\mathcal{C} \cap \mathcal{W})$ Factorisation & 2-out-of-3.

Examples

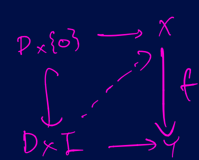
→ Serre model str. on Top

$\mathcal{W}$ = weak homotopy equivalences

↳ iso on path components and $\pi_n(X, x_0) \cong \pi_n(Y, f(x_0))$ for all $x_0 \in X$.

$\mathcal{F}$ = Serre fibrations

↳ Maps with RLP w.r.t. $D^n \times \{0\} \hookrightarrow D^n \times I$ for all n.



$\mathcal{C}$ = relative cell-complexes

→ Hurewicz model str. on Top.

$\mathcal{W}$ = homotopy equivalences

$\mathcal{F}$ = Hurewicz fibrations

↳ maps with RLP w.r.t. $A \times \{0\} \hookrightarrow A \times I$ for all spaces A.

→ sSet

$\mathcal{W}$ = maps which are weak homotopy equivalences under the geometric realisation functor $|-| : sSet \longrightarrow Top$.

$\mathcal{C}$ = maps $f: X \to Y$ such that $f_n : X_n \longrightarrow Y_n$ is a monomorphism for all n.

→ R-Mod for R a frobunnius ring

$\mathcal{C}$ = injections
$\mathcal{F}$ = surjections

→ Projective model str on $Ch(R)_+$ chain complexes of R modules cocentrated in non-negative degrees.

$\mathcal{W}$ = quasi-isomorphisms
$\mathcal{C}$ = $f_\bullet : A_\bullet \longrightarrow B_\bullet$ such that $f_n : A_n \longrightarrow B_n$ is injective with projective cokernel for all n.

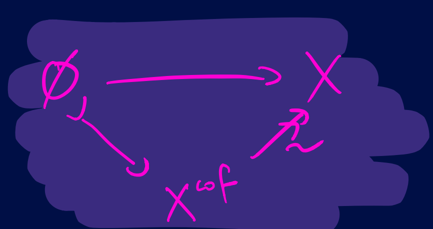
Def$^{\underline{n}}$ X is <u>cofibrant</u> if $\emptyset \hookrightarrow X$ is a cofibration.

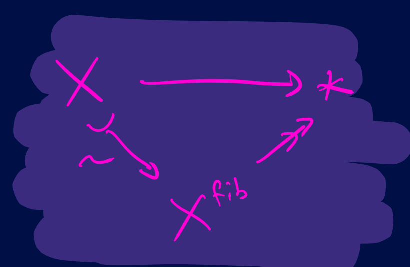
X is <u>fibrant</u> if $X \twoheadrightarrow *$ is a fibration.

Y is a cofibrant replacement for X if Y is cofibrant and there is a weak equivalence $Y \xrightarrow{\sim} X$.

Z is a fibrant replacement for X if Z is fibrant and there is a weak equivalence $X \xrightarrow{\sim} Z$.

We can always find co/fibrant replacements





Eg In Top (Serre model str.) all objects are fibrant and cofibrant replacement is CW-approximation.

In sSet all objects are cofibrant

Def$^{\underline{n}}$ A cylinder object for X is a diagram

$$X \sqcup X \xrightarrow{\ \nabla\ } X, \qquad X \sqcup X \xrightarrow{(i_0, i_1)} Cyl(X) \xrightarrow[\sim]{r} X$$

via the factorisation axiom we can get $(i_0, i_1) \in \mathcal{C}$, $r \in \mathcal{F}$ and $i_0, i_1 \in \mathcal{W}$

Eg In Top (Serre) $X \times I$ is a cylinder object for X.

Def$^{\underline{n}}$ A left homotopy between $f, g : X \longrightarrow Y$ is a map $H : Cyl(X) \longrightarrow Y$ such that $H \circ i_0 = f$ and $H \circ i_1 = g$.

Similarly there is a notion of path object

$$Y \xrightarrow{\ \Delta\ } Y \times Y, \qquad Y \xrightarrow[\sim]{s} PY \xrightarrow{(e_0, e_1)} Y \times Y$$

and a right homotopy between $f, g : X \longrightarrow Y$ is a map $H : X \longrightarrow PY$ such that $e_0 \circ H = f$ and $e_1 \circ H = g$

Think about the adjunction

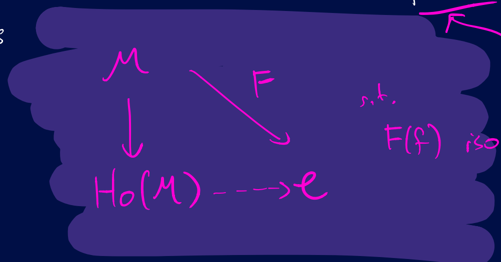
$$Top(X \times I, Y) \cong Top(X, Map(I, Y))$$

We call $f, g : X \longrightarrow Y$ homotopic, denoted $f \simeq g$ if they are both left homotopic and right homotopic.

Prop If X is cofibrant and Y is fibrant then $f \simeq g$ is an equivalence relation on $\mathcal{M}(X,Y)$.

We call an equivalence class a **homotopy class** and denote them $[f]$

Given a category with a "good" class of weak equivalences $\mathcal{W}$, "the" homotopy category is initial among functors which send all $f \in \mathcal{W}$ to isomorphisms



Def $\mathcal{M}$ a model category.
The homotopy category, $Ho(\mathcal{M})$, of $\mathcal{M}$ has the same objects as $\mathcal{M}$ and the morphisms given by

$$Ho(\mathcal{M})(X,Y) = \mathcal{M}\left(\left(X^{cof}\right)^{fib}, \left(Y^{cof}\right)^{fib}\right) / \simeq$$

Def Let $\mathcal{M}, \mathcal{N}$ be model categories.
$F: \mathcal{M} \longrightarrow \mathcal{N}$ is a left Quillen functor if it preserves $\mathcal{C}$ and $\mathcal{C} \cap \mathcal{W}$
$G: \mathcal{N} \longrightarrow \mathcal{M}$ is a right Quillen functor if it preserves $\mathcal{F}$ and $\mathcal{F} \cap \mathcal{W}$
An adjunction $F: \mathcal{M} \rightleftarrows \mathcal{N} : G$ is a Quillen adjunction if F is left Quillen and G is right Quillen

If $F : \mathcal{M} \to \mathcal{N}$ is left Quillen the <u>left derived functor</u> of F is

$$\mathbb{L}F : Ho(\mathcal{M}) \longrightarrow Ho(\mathcal{N})$$

and is given by $\mathbb{L}F(X) = F(X^{cof})$

$G : \mathcal{N} \to \mathcal{M}$ right Quillen

<u>right derived functor</u> of G is

$$\mathbb{R}G : Ho(\mathcal{N}) \longrightarrow Ho(\mathcal{M})$$

and is given by $\mathbb{R}G(X) = G(X^{fib})$

A Quillen adjunction $F : \mathcal{M} \rightleftarrows \mathcal{N} : G$ is a <u>Quillen equivalence</u> if

$$\mathbb{L}F : Ho(\mathcal{M}) \rightleftarrows Ho(\mathcal{N}) : \mathbb{R}G$$

is an adjoint equivalence of category

Eg

$$Top \underset{|-|}{\overset{S_\bullet}{\rightleftarrows}} sSet \overset{\mathbb{Z}}{\dashrightarrow} sAb \underset{\Gamma}{\overset{N}{\rightleftarrows}} Ch(\mathbb{Z})_+ \xrightarrow{H_*} grAb$$

<u>Def$^{\underline{n}}$</u> A model structure is <u>cofibrantly generated</u> if there exists <u>sets</u> I and J of morphisms such that

I: gen. cofib

J: gen acyclic cofib

$$({}^{\boxtimes}(I^{\boxtimes}), I^{\boxtimes}) = (\mathcal{C}, \mathcal{F} \cap \mathcal{W})$$

and

$$({}^{\boxtimes}(J^{\boxtimes}), J^{\boxtimes}) = (\mathcal{C} \cap \mathcal{W}, \mathcal{F})$$

Eg Top:
$$I = \{ S^{n-1} \hookrightarrow D^n \}$$
$$J = \{ D^n \times \{0\} \hookrightarrow D^n \times [0,1] \}$$

Thrm (The baby small object argument)
If $\mathcal{M}$ is a cocomplete category and I is a set of maps whose domains are sequentially small then $({}^{\boxslash}(I^{\boxslash}), I^{\boxslash})$ is a WFS on $\mathcal{M}$.