Stable Homotopy Seminar

3/22/2021

# Homotopy Groups of Spectra

(Thank you Paul!)

# Motivation: Some Impossible Goals

## · Attempt 1: Classify Top

$X = \mathrm{Ob}(\mathrm{Top})$
$Y = \mathrm{Mor}(\mathrm{Top})$ } Fully Understand these as ....

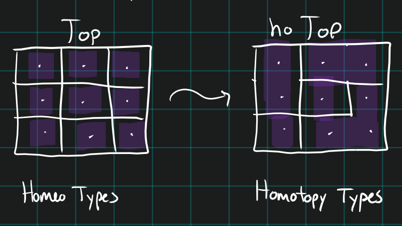
(spaces? stacks? alg structs?)

↳ Hard! Enriched over itself
$\mathrm{Top}(\mathbb{R}^2, \mathbb{R}^2)$ not contractible

$\mathrm{Emb}(X,Y)$, $\mathrm{Homeo}(X)$, $\mathrm{MCG}(X)$, ...

↳ Weird! Space-filling curves

$\mathrm{Top}(A,B) \longrightarrow [A,B]$




$\cong$ $\qquad$ $\simeq$

Eg. $\mathbb{R}^n \not\cong \mathbb{R}^m$ when $m \neq n$
("Invariance of domain", Brouwer 1912)

But $\mathbb{R}^n \simeq \mathbb{R}^m \simeq \mathbb{R}^0 := \{*\}$.

Q: When are hty-equiv spaces homeo?

## · Attempt 2: Classify hoTop

↳ Still hard!

Simplest case is wide open:

$$\pi_* \mathbb{S}^* := \left\{ [S^n, S^m] \;\middle|\; n, m \geq \mathbb{Z}^{\geq 0} \right\}$$ only barely known!

$n, m \geq 2$: Simply connected Compact Smooth mfds (nice)

↳ Recall
Freudenthal suspension thm, corollary:

$$\begin{array}{ccc} [S^{n+k}, S^n] & \xrightarrow{\Sigma^*} & [S^{n+k+1}, S^{n+1}] \\ \wr\| & & \wr\| \\ \pi_{n+k} S^n & \xrightarrow{\sim} & \pi_{n+k+1} S^{n+1} \end{array}$$

for $n \geq k+2$

$n = k+2$

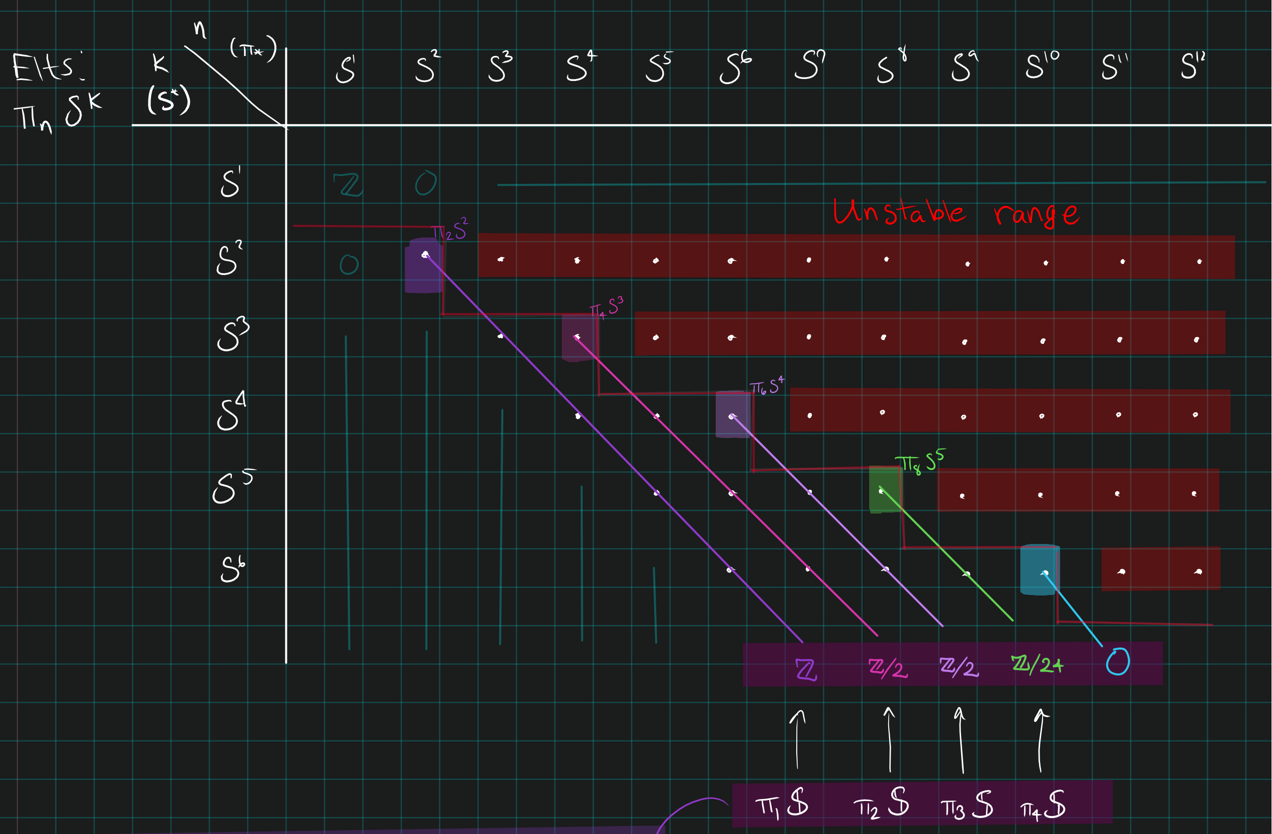
**Def** (Recall): $\pi_k^{st} S^n := \operatorname{colim}_{k} \pi_{n+k} S^{n+k}$

↳ Take $n = k+2$

| $k$ | $\pi_{2k+2} S^{k+2}$ |
|---|---|
| 0 | $\pi_2 S^2$ |
| 1 | $\pi_4 S^3$ |
| 2 | $\pi_6 S^4$ |
| 3 | $\pi_8 S^5$ |
| ⋮ | ⋮ |

bidegree (2, 1)

## Homotopy Groups of Spheres




## · Attempt 3: Classify "Stab(hoTop)" $\approx \mathcal{S}p$.

↳ Make $\Omega, \Sigma$ endofunctors inducing self-equivs

$\bigoplus_{i,j} [S^i, S^j]^{st} \xrightarrow[\text{package up}]{\rightsquigarrow} \boxed{\pi_* \mathbb{S}}$

(hoTop) $\qquad$ ("Spectra")

Many models

**Rmk**: can localize $\mathbb{S} \longrightarrow \underbrace{\mathbb{S}_{(p)}}_{\text{p-local sphere spectrum}}$

Hopkins–Ravenel "Chromatic convergence thm"

$$\operatorname{holim}\left( \cdots \to \underbrace{L_{E(1)}}_{\text{Bousfield loc wrt Morava E}} \mathbb{S}_{(p)} \to L_{E(0)} \mathbb{S}_{(p)} \right) \simeq \mathbb{S}_{(p)}$$

$\rightsquigarrow$ Chromatic hty theory!

Try to compute one prime at a time.

(Modern Stuff)

# Why Care About $\pi_* \mathbb{S}$ ?

## ↳ Building CW complexes

- Hatcher 0.18: For $A \subseteq X$ CW

  $[f]=[g]: A \to Y$ an attaching map,

  $$X \sqcup_f Y \simeq X \sqcup_g Y \quad (\text{rel } Y)$$

$\Rightarrow$ Need gluing data to find the homotopy type!

Attaching maps: $\left[ \{ \phi_n^i \in Top(\partial B^n, X^{n-1}) \}_{i \in I} \right]$

$S^{n-1}$    $\leq n$ cells $\simeq S^1, S^2, \dots$

Iterated pushouts:

$$\begin{tikzcd} S^{n-1} \arrow[r, "\phi_n^i"] \arrow[d, hook, "\partial"'] & X^{n-1} \arrow[d] \\ B^n \arrow[r] & X^n \arrow[ul, phantom, "\lrcorner", very near start] \end{tikzcd}$$

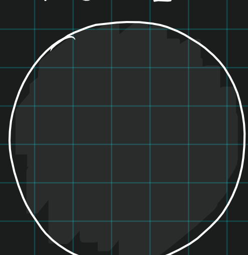
## Ex: Where do $\pi_k S^n$ show up?

Art break!

$B^1$

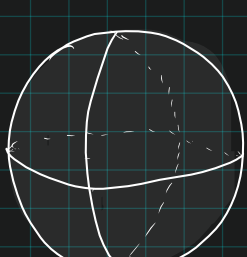
$\partial B^1 = S^0 = * \sqcup *$

$B^2$



$\partial B^2 = S^1$

$B^3$

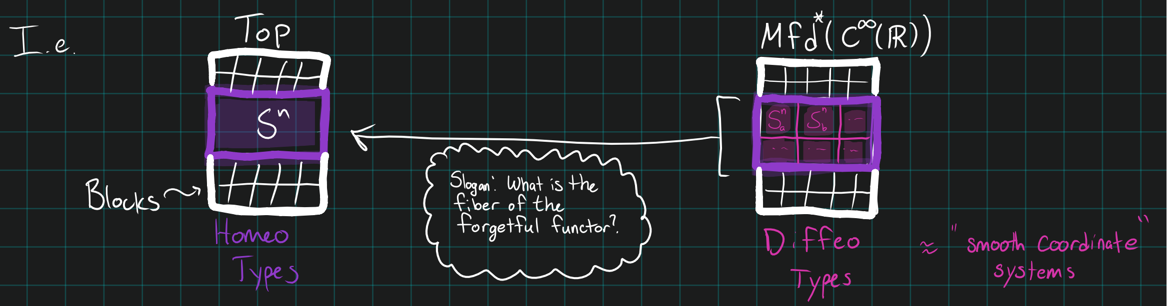


$\partial B^3 = S^2$

# Why Care About $\pi_* \mathbb{S}$?

**Question**: For which $n$ are there exotic $S^n$?

I.e.




- (Milnor, '56) $\exists$ exotic $S^7$'s
- $\{\text{Smooth structures on } S^n\} \rightleftarrows \underbrace{\overbrace{\{\text{Oriented hty } S^n\}}^{\Theta_n} / \text{h-cobordism}}$

  Finite abelian group under # for $n \neq 4$.

  For $n=4$, very open (Poincaré!!!)

- There is a map

$$\Theta_n / bP_{n+1} \hookrightarrow \pi_n \mathbb{S} / J$$

J-homomorphism

The image is either

- Index 2
- Index 1, an iso if $\exists M \in Mfd^n_{Fr}(C^\infty)$ s.t. $\Phi(M) = 1$

Kervaire invariant one problem

$\Phi$: Kervaire Invariant

**Thm** (Hill-Hopkins-Ravenel): These can only exist in dimensions

$n = 2, 6, 14, 30, 62, \Big]\ 2^k - 2$, $\Phi \neq 0$

$\boxed{\text{Only open case: } n = 126!}$

(Atiyah had some conjectures!)

# Tools from Top that we want in Sp

(CGWH, Classical model struct)

## • Homotopy gps ☆

We'll try to get analogs for ☆ in Spectra today.

$$\pi_\bullet : \mathrm{Top} \to \mathrm{Gr}^{\mathbb{Z}}\mathrm{Ab}$$

$$X \mapsto \pi_\bullet X := \bigoplus_{i \geq 1} [S^i, X]$$

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## • Suspension - Loop Adjunction

$$[\Sigma A, B] \cong [A, \Omega B]$$

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## • Puppe (Cofibre) ☆

BTW: $C_f = Y \cup_f CX$

$$\left(X \xrightarrow{f} Y\right) \rightsquigarrow \left(X \xrightarrow{f} Y \to C_f\right) \overset{\text{Resolve}}{\rightsquigarrow}$$

cof.b  Cofib. replacement(?)

Extract hocofibers

$$\begin{array}{l} X \xrightarrow{f} Y \to C_f \\ \to \Sigma X \to \Sigma Y \to \Sigma C_f \\ \to \Sigma^2 X \to \cdots\cdots \end{array}$$

$\mathrm{LES}^{\mathrm{cof}}$

Apply a functor

$F(\cdot) = [\cdot, K(G,1)]$

Uses:

$$[\Sigma X, K(G,1)]$$
$$\parallel$$
$$[X, \Omega K(G,1)]$$
$$\parallel \text{ adjunction}$$
$$[X, K(G,2)]$$
$$\parallel$$
$$H^{n-1}(X;G)$$

$H^j(\Sigma X) \cong H^{j-1}(X)$

$$\begin{array}{l} H^n X \xrightarrow{f} H_n Y \to H_n C_f \\ \to H_{n-1} X \to H_{n-1} Y \to H_{n-1} C_f \\ \to \cdots\cdots \end{array}$$

$\rightsquigarrow$ LES in homology.

$$\begin{aligned} H_n C_f &= H_n(Y \sqcup_f CX) \\ &= H_n(Y/X) \\ &= H_n(Y, X) \end{aligned}$$

(LES of a pair.)

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## • Puppe (fiber) ☆

$$X \xrightarrow{f} Y \rightsquigarrow \left(M_f \to X \hookrightarrow Y\right) \overset{\text{resolve}}{\rightsquigarrow}$$

fibration  Cofib replacement

Extract hofibers

$$\begin{array}{l} \cdots\cdots \to \Omega^2 Y \\ \to \Omega M_f \to \Omega X \to \Omega Y \\ \to M_f \to X \to Y \end{array}$$

$\mathrm{LES}^{\mathrm{fib}}$

Apply a functor

$G(\cdot) = [S^0, \cdot]$

Uses

$$[S^0, \Omega^k X]$$
$$\parallel$$
$$[\Sigma^k S^0, X]$$
$$\parallel$$
$$[S^k, X]$$
$$\parallel$$
$$\pi_k X$$

$$\begin{array}{l} \cdots\cdots \\ \to \pi_1 M_f \to \pi_1 X \to \pi_1 Y \\ \to \pi_0 M_f \to \pi_0 X \to \pi_0 Y \end{array}$$

LES in homotopy

(Relative hty LES, using

$\pi_0 \Omega^k M_f := [S^0, \Omega^k M_f] = [S^k \wedge S^0, M_f] = \pi_k M_f \xrightarrow{\cong} \pi_k(Y, X)$.)

See Eckmann-Hilton duality!

# Tools from Top that we want in Sp:

- Weak equivalences ☆

$W = $ Hty equivs

Apply the functor

$\mathrm{Top}^{W}(\cdot,\cdot) \xrightarrow{\pi_*(\cdot)} \mathrm{GrAb}^{\cong}(\cdot,\cdot)$

Induce isos on $\pi_*$

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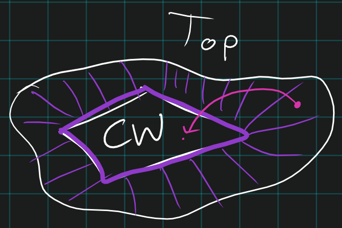
- Cofibrant objects & replacement

$\mathrm{Top}^{CW} \hookrightarrow \mathrm{Top}$, every $X \in \mathrm{Top}$ weakly equiv to some $X' \in \mathrm{Top}^{CW}$

by CW approximation

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- A functor $\mathrm{Top} \to \mathrm{hoTop}$ where $\mathrm{ho}(W)$ are isomorphisms

$\hookrightarrow \cong \mathrm{Top}[W^{-1}]$

and hoTop is triangulated

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- Spectral Sequences

Ex A Serre fibration induces

$$\boxed{\begin{array}{ccc} F & \to & E \\ & & \downarrow \\ & & B \end{array}}$$

$$E_2^{p,q} = H^p(B, H^q F) \Rightarrow H^{p+q} E$$

(Serre Spectral Sequence)

In Sp

Ex $\boxed{Sp(A,B)}$ induces:

$$E_2^{p,q} = \underbrace{\mathrm{Ext}^{p,q}_{A}}_{\text{Steenrod}}(\overbrace{H^*A, H^*B}^{\text{known singular cohomology}}) \Rightarrow \underbrace{[A, \overbrace{\hat{B}_p}^{\text{p-completion}}]_{p-q}}_{\text{Get info}}$$

(Adams Spectral Sequence)

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## Building blocks ☆

- Eilenberg - MacLane Spaces $\qquad \pi_* K(G,n) = \begin{cases} G, & * = n \\ 0, & \text{else} \end{cases}$

- Moore Spaces $\qquad H_* M(G,n) = \begin{cases} G, & * = n \\ 0, & \text{else} \end{cases}$

# Goals For Today

Major goal: Define & compute $\pi_* X$ for $X \in \mathrm{Spectra}$

Minor goals:

1) Define $\pi_* X$
2) Find LES's (2.2.10)
3) $\pi_*$-isos (2.2.11)

# Conventions & Things to Recall

## For Spaces

- $\mathrm{Top} := \mathrm{Top}_*^{CW}$, pointed CGWH ("Spaces")

- $A \wedge B := \dfrac{A \times B}{A \vee B} \in \mathrm{Top}$ (Note: Associative up to homeo in CGWH)

- $\Sigma A := S^1 \wedge A \in \mathrm{Top}_*$ , $\Sigma^k A := (S^1)^{\wedge k} \wedge A \cong S^k \wedge A$

k-fold Smash

- $\Omega A := \mathrm{Top}(S^1, A)$ w/ compact-open topology

- $[A,B] := \mathrm{Top}(A,B)/\text{homotopy}$

- $\pi_k A := [S^k, A] \cong [\Sigma^k S^0, A]$

$\cong [S^0, \Omega^k A]$

$= \pi_0 \Omega^k A$

- $\underset{\in \mathrm{Top}}{[A,B]^{st}} := \mathrm{colim}\left( [A,B] \to [\Sigma A, \Sigma B] \to \underbrace{[\Sigma^2 A, \Sigma^2 B] \to \cdots}_{\text{Stabilizes by Freudenthal.}} \right)$

$= \mathrm{colim}_j \, [\Sigma^j A, \Sigma^j B]$

- $\underset{\in \mathrm{Top}}{\pi_k^{st} A} := [S^0, A]^{st}$ using $\Sigma S^k \cong S^{k+1}$

- Also recall

  - $\Sigma A \xrightarrow{\text{Homeo}} S^1 \wedge A$
  - $S^1 \wedge S^n \xrightarrow{\text{Homeo}} S^{n+1}$

# Context: A Category of Spectra

- **Def** ( $Sp^{\mathbb{N}}$, <u>sequential spectra</u> )

**Notation:** $Sp := Sp^{\mathbb{N}}$

- **Objects:**

$\boxed{X_n \to \Sigma X_n}$ nullhomotopic!

Suspension spectrum

$\Omega$-spectrum

$$\begin{tikzcd}
\Sigma X_0 \arrow[dr, "\sigma_0"] & \Sigma X_1 \arrow[dr, "\sigma_1"] & \Sigma X_2 \arrow[dr, "\sigma_2"] & \cdots \\
(X_0, \arrow[dr, "\omega_0"', "\simeq"] & X_1, \arrow[d] \arrow[dr, "\omega_1"', "\simeq"] & X_2, \arrow[d] \arrow[dr, "\omega_2"', "\simeq"] & \cdots\cdots) \\
& \Omega X_1 & \Omega X_2 & \cdots
\end{tikzcd}$$

$\Sigma \uparrow$

$\Omega \downarrow$

$\sigma_n \in Top_*(\Sigma X_n, X_{n+1})$

$\omega_n \in Top_*^{W}(X_n, \Omega X_{n+1})$

- **Morphisms:**

$$\begin{tikzcd}
\Sigma X_n \arrow[r, "\Sigma f_n"] \arrow[d, "\sigma_n^X"'] \arrow[dr, phantom, "\circlearrowright"] & \Sigma Y_n \arrow[d, "\sigma_n^Y"] \\
X_{n+1} \arrow[r, "f_{n+1}"] & Y_{n+1}
\end{tikzcd}$$

Commuting up to homotopy

**Note:** Not all spectra are in $Sp^{\mathbb{N}}$!

**Eg:** K-Theory

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- **Rmk:** $Sp^{\mathbb{N}} \rightsquigarrow Sp^{O} \rightsquigarrow Sp^{\Sigma}$

(Orthogonal) (Symmetric)

Also $\infty$-cat versions
Ssets, diagram spectra, ...

Problem: $A \wedge B \neq B \wedge A$, not functorial!

- **Slogan:** $\Sigma^K \approx [-K]$, left-shift

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# Examples of Spectra to Keep in Mind

- $$\Sigma^\infty K := \begin{tikzcd} \Sigma K \arrow[dr, "\sigma_0", "\cong"'] & \Sigma(S^1\wedge K) \arrow[dr, "\sigma_1", "\cong"'] & \Sigma^3 K \arrow[dr, dashed, "\sigma_2", "\cong"'] & \\ S^0\wedge K \arrow[u, "\Sigma"] , & S^1\wedge K , & S^2\wedge K , & \dots \end{tikzcd}$$
  $(K \in \mathrm{Top})$

  $S^1\wedge K \overset{\text{Homeo}}{\cong} \Sigma K$, $\quad S^1\wedge S^1\wedge K \cong S^2\wedge K$

  $\sigma_n \in \underbrace{\mathrm{Top}^{\cong}}_{\text{Homeos}}(\Sigma^n K, S^n\wedge K)$

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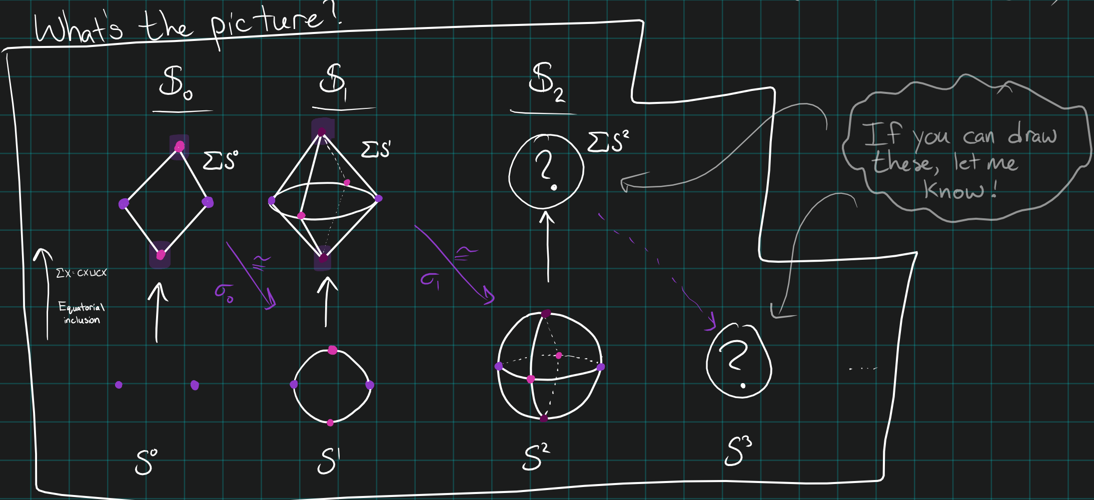
- $$\mathbb{S} := \Sigma^\infty S^0 \quad \begin{tikzcd} \Sigma S^0 \arrow[dr, "\sigma_0", "\cong"'] & \Sigma S^1 \arrow[dr, "\sigma_1", "\cong"'] & \Sigma S^2 \arrow[dr, "\sigma_2", "\cong"'] & \\ S^0 , & S^1 \arrow[u] , & S^2 \arrow[u] , & \dots \end{tikzcd}$$

  $S^1\wedge S^1 \cong \Sigma S^1$, $\quad S^1\wedge S^2 \cong \Sigma S^2$

  $\sigma_n \in \mathrm{Top}^{\cong}(\Sigma S^n, S^{n+1})$

  (Recall $S^1\wedge S^k \overset{\text{Homeo}}{\cong} S^{k+1}$)

What's the picture?




(Sphere Spectrum)

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- $$HG := \begin{tikzcd} K(G,0) \arrow[dr, "\sigma_0", "\simeq"'] , & K(G,1) \arrow[dr, "\sigma_1", "\simeq"'] , & K(G,2) \arrow[dr, dashed, "\sigma_2"] , & \dots \\ & \Omega K(G,1) & \Omega K(G,2) & \end{tikzcd}$$

  $\sigma_n \in \mathrm{Top}^{\overset{\text{weak htpy equiv.}}{W}}(K(G,n), \Omega K(G,n+1))$

  Using $\Omega K(G,n+1) \simeq K(G,n)$

(Eilenberg MacLane Spectrum)

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- $MG :=$ Complicated! But have $H_* MG = 0$ for $* \neq 0$

  $M\mathbb{Z}/n \simeq \mathrm{hocofib}\left(\mathbb{S}^{cof} \xrightarrow{\cdot n} \mathbb{S}^{fib}\right)$

  (Moore Spectrum)

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## Def (CW Spectra)

$X \in Sp^{\mathbb{N}}$ s.t. $X_n \in \mathrm{Top}^{CW}$ & $\sigma_n : S^1\wedge X_n \to X_{n+1}$ is a cellular isomorphism onto a subcomplex.

# Adjoints

**Rmk**: There is an adjunction

$$\begin{tikzcd} \mathrm{Top} \arrow[r, shift left, "\Sigma^\infty"] \arrow[r, phantom, "\perp"] & \mathrm{Sp} \arrow[l, shift left, "\Omega^\infty"] \end{tikzcd} \qquad \mathrm{Sp}(\Sigma^\infty A, B) \cong \mathrm{Top}(A, \Omega^\infty B)$$

Right adjoint

where $\Omega^\infty(X) = X_0$

**Rmk**:

Can use the unit to make an endofunctor

$$\left.\begin{aligned} Q : \mathrm{Top} &\to \mathrm{Top} \\ K &\mapsto \Omega^\infty \Sigma^\infty K \end{aligned}\right]$$

Image: Infinite loop spaces

(Actually $E_\infty$ spaces)

Then $\pi_* QK \xrightarrow[\eta_*]{} \pi_*^{st} K$ (in Top)

Can compute as $QK = \operatorname{colim}_{\overrightarrow{j}} \Omega^n \Sigma^n K$

# More Examples

Reassociate: $S^1 \wedge (S^0 \wedge K) \underset{\text{Homeo}}{\cong} \Sigma(S^0 \wedge K) \xrightarrow{\sigma_j} S^1 \wedge K$

Reassociate: $S^1 \wedge (S^1 \wedge K) \underset{\text{Homeo}}{\cong} \Sigma(S^1 \wedge K) \xrightarrow{\sigma_{j+1}} S^2 \wedge K$

- $$F_j^{\mathbb{N}} K = \text{"}\Sigma^j \Sigma^\infty K\text{"} = \Big[ *,\ *,\ \dots,\ *,\ \underset{j}{S^0 \wedge K},\ \underset{j+1}{S^1 \wedge K},\ S^2 \wedge K \Big]$$

(Right-shift!
Mnemonic: $F \to$ "Forward")

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$\Sigma * \simeq *$, $\Sigma S^1$, $\Sigma S^2$

- $$\boxed{\mathbb{S}^{(1)} := F_1^{\mathbb{N}} S^1} = \left[ \begin{matrix} (*,\ \mathbb{S}_1,\ \mathbb{S}_2,\ \dots) \\ \text{and } \sigma_j^{\mathbb{S}^{(1)}} \approx \sigma_{j+1}^{\mathbb{S}} \end{matrix} \right] = \Big( *,\ S^1,\ S^2,\ \dots \Big)$$

with $\Sigma * \simeq * \to S^1$, $\Sigma S^1 \xrightarrow{\sigma_1^{\mathbb{S}}} S^2$, $\Sigma S^2 \xrightarrow{\sigma_2^{\mathbb{S}}} \dots$

---

**Rmk:** There is an adjunction

$$\mathrm{Top} \underset{\mathrm{Ev}_j^{\mathbb{N}}}{\overset{F_j^{\mathbb{N}}}{\rightleftarrows}} Sp^{\mathbb{N}}$$

Mnemonic: $\mathrm{Ev} \to$ "Evaluation"

where $\mathrm{Ev}_j^{\mathbb{N}}(X) := X_j$

## Def (Homotopy groups of spectra, 3.2.1)

Let $X \in Sp^{\mathbb{N}}$, so $X := \{(X_n, \sigma_n : \Sigma X_n \to X_{n+1})\}_{n \in \mathbb{Z}^{\geq 0}}$. Then

$$\pi_n(X) := \operatorname{colim}_k \pi_{n+k}(X_k) \qquad (n \in \mathbb{Z})$$

What is the directed system...?

Freudenthal Suspension

$$= \operatorname{colim}\left( [S^n, X_0] \xrightarrow{?_0} [S^{n+1}, X_1] \xrightarrow{?_1} [S^{n+2}, X_2] \cdots \xrightarrow{\sim} \cdots \right)$$

Where

$$\begin{tikzcd}
{[S^{n+k}, X_k]} \arrow[rr, "\text{``}\sigma_k \Sigma\text{''}"] \arrow[d, "\cong"', "\text{Homeo}"] & & {[S^{n+k+1}, X_{k+1}]} \\
{[\Sigma^k S^n, X_k]} \arrow[r, "\Sigma^*"] & {[\Sigma^{k+1} S^n, \Sigma X_k]} \arrow[r, "\sigma_k^*"] & {[\Sigma^{k+1} S^n, X_{k+1}]} \arrow[u, "?"', "\text{Homeo}"]
\end{tikzcd}$$

$f^*$ are maps functorially induced on homs

$\sigma_k : \Sigma X_k \to X_{k+1}$

Apply $F(\cdot) = [\ ?\ , \cdot]$

$\sigma_k^* : [\ ?\ , \Sigma X_k] \to [\ ?\ , \Sigma X_{k+1}]$

**Note:** Star notation immediately dropped!

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**Rmk** This yields a monoidal functor

$$\pi_0(\cdot) : (Sp, \wedge, \mathbb{S}) \to (Ab, \otimes_{\mathbb{Z}}, \mathbb{Z})$$

$$\pi_*(\cdot) : (Sp, \wedge, \mathbb{S}) \to (\underbrace{Gr^{\mathbb{Z}}Ab}_{\mathbb{Z}\text{-graded abelian gps}}, \hat{\otimes}_{\mathbb{Z}}, \hat{\mathbb{1}})$$

where $(A \hat{\otimes}_{\mathbb{Z}} B)^n := \bigoplus_{j+k=n} A_j \otimes_{\mathbb{Z}} B_k$

$(\hat{\mathbb{1}})_n = \mathbb{Z}$ if $n=0$, $0$ else

Just put the monoidal unit in degree zero

# Computing Some Homotopy Groups

**Prop**

$$\pi_n HG = \begin{cases} G, & n = 0 \\ 0, & \text{else} \end{cases}$$

**Proof**

- $\pi_0 HG = G$

$$\begin{aligned} \pi_0 HG &:= \underset{\overrightarrow{j}}{\operatorname{colim}} \ \pi_j (HG)_j \\ &= \underset{\overrightarrow{j}}{\operatorname{colim}} \ \pi_j K(G, j) \\ &= \underset{\overrightarrow{j}}{\operatorname{colim}} \left( G \to G \to G \to \cdots \right) \\ &\cong G. \end{aligned}$$

- $\pi_n HG = 0$ for $\underline{n \geq 1}$:

$$\begin{aligned} \pi_n HG &:= \underset{\overrightarrow{j}}{\operatorname{colim}} \ \pi_{n+j} (HG)_j \\ &:= \underset{\overrightarrow{j}}{\operatorname{colim}} \ \pi_{n+j} K(G, j) \\ &= \underset{\overrightarrow{j}}{\operatorname{colim}} \left( 0 \to 0 \to \cdots \right) \end{aligned}$$

(since $n \geq 1 \Rightarrow \underline{n+j \geq 1+j > j}$

and $\pi_m K(G, n) = \delta_{m,n}$)

$$= 0.$$

$\blacksquare$

# Computing Some Homotopy Groups

## Prop

$$\pi_n F_d^{\mathbb{N}} K \cong \overbrace{\pi_{n+d} \Sigma^\infty K}^{Sp} \cong \overbrace{\pi^{st}_{n+d} K}^{Top}$$

## Proof

Recall: $\Sigma^\infty K := [K, S^1\wedge K, S^2 \wedge K, \ldots]$

$F_d^{\mathbb{N}} K := [\underset{}{*}, *, \ldots, \underset{d-1}{*}, \underset{d}{K}, \underset{d+1}{S^1\wedge K}, \ldots]$

$j = d$: $\quad (F_d^{\mathbb{N}})_d = K = (\Sigma^\infty K)_0$

$j = d+1$: $\quad (F_d^{\mathbb{N}})_{d+1} = S^1\wedge K = (\Sigma^\infty K)_1$

$\vdots$

$j = d+\ell$: $\quad (F_d^{\mathbb{N}})_{d+\ell} \quad S^\ell \wedge K = (\Sigma^\infty K)_\ell$

$$\Rightarrow (F_d^{\mathbb{N}} K)_j = \begin{cases} * & , j \le d-1 \\ (\Sigma^\infty K)_{j-d} & , j \ge d \end{cases}.$$

$$= \begin{cases} * & , j \le d-1 \\ S^{j-d} \wedge K & , j \ge d \end{cases}. \qquad (1)$$

$$\pi_n F_d^{\mathbb{N}} K := \underset{\underset{j}{\longrightarrow}}{\operatorname{colim}}\, \pi_{n+j} (F_d^{\mathbb{N}} K)_j$$

$$:= \underset{\underset{j}{\longrightarrow}}{\operatorname{colim}}\, \pi_{n+j} (S^{j-d} \wedge K) \qquad \text{by (1)}$$

$$= \underset{\underset{k}{\longrightarrow}}{\operatorname{colim}}\, \pi_{n+k+d} (\Sigma^k K) \qquad \text{Setting } k := j-d \Rightarrow j = k+d$$

$$:= \underset{\underset{k}{\longrightarrow}}{\operatorname{colim}}\, \pi_{(n+d)+k} (\Sigma^\infty K)_k \qquad \text{defn of } \Sigma^\infty K$$

1st equality $\rightsquigarrow$ $:= \pi_{n+d} (\Sigma^\infty K) \qquad$ defn of $\pi_* \Sigma^\infty K$

2nd equality $\rightsquigarrow$ $= \pi^{st}_{n+d} K$. ∎

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- ## Cor

$$\pi_* \Sigma^\infty K \cong \pi^{st}_* K$$

- ## Proof

Set $d=0$ in prop:

$$\pi_n F_0^{\mathbb{N}} K = \pi^{st}_{n+0} K \qquad \text{by prop}$$

$$= \pi^{st}_n K.$$

# Examples

**Ex** $MO = (MO(1) \to MO(2) \to \cdots)$, Thom spectrum of orthog. gp.

$$\pi_* MO = \{\text{Unoriented } C^\infty(\mathbb{R})\text{-mfds}\}/\text{Cobordism} \qquad \left(\pi_n MO = \Omega^O_n\right)$$

---

**Ex**

$\pi_* MU$ = Universal Lazard ring (related to formal gp. laws, elliptic curves, NT, ...)

$$\cong \{\text{Almost-complex } C^\infty(\mathbb{R})\text{-mfds}\}/\text{Cobordism}$$

---

· **Rmk**:

$F_1^{\mathbb{N}} \mathbb{S} \xrightarrow{\mathbb{I}} \mathbb{S}$ should be a <u>weak equivalence</u>, since $\pi_* \mathbb{S}$ is a colim – finite shifts shouldn't matter!

But this is not a weak equiv in the levelwise stable model structure.

## Def ($\pi_*$-isomorphisms, 3.2.1)

A morphism $f \in Sp(A, B)$ is a **$\pi_*$-isomorphism** iff the induced morphisms $\pi_n f \in Ab^{\cong}(\pi_n A, \pi_n B)$ $\forall n \in \mathbb{Z}$

Call $A, B$ **$\pi_*$-isomorphic** iff $\exists f$ a $\pi_*$-isomorphism.

---

## Rmk

- Motivation: Whitehead's theorem

$$f \in Mor_{CW}(X, Y) \text{ a } \pi_*\text{-iso} \Rightarrow f \in \overbrace{hoTop^{\cong}(X, Y)}^{\text{Homotopy equivalences.}}$$

(Warning: having $\pi_n X \cong \pi_n Y$ $\forall n$ isn't enough!)

- These will be our weak equivalences $W$!

- Later: A similar thm for CW-Spectra:

$$f \in Sp_{CW}^{W}(A, B) \Rightarrow [f] \in hoSp^{\cong}(A, B)$$

I.e. a weak equivalence of CW-Spectra will be a homotopy equivalence.

**Ex**

$\lambda_d : F_{d+1}^{\mathbb{N}} S^1 \xrightarrow{\sim} F_d^{\mathbb{N}} S^0$ is a $\pi_*$-isomorphism.

$$\begin{array}{ccccccccc}
 & & & & & d & d+1 & & \\
F_{d+1}^{\mathbb{N}} S^1 & = & [\,* , & * , & \cdots , & * , & S^1 , & S^2 , & \cdots\,] \\
\downarrow \lambda & & \downarrow 1 & \downarrow 1 & & \downarrow i & \downarrow 1 & \downarrow 1 & \\
F_d^{\mathbb{N}} S^0 & = & [\,* , & * , & \cdots , & S^0 , & S^1 , & S^2 , & \cdots\,]
\end{array}$$

These are the maps $\lambda_d$

Eventually isomorphic

---

**Ex**

$$\begin{array}{ccccccccc}
X^{(n)} & := & [\,* , & * , & \cdots , & * , & X_n , & X_{n+1} , & \cdots\,] \\
f \downarrow & & \downarrow f_0 & \downarrow f_1 & & \downarrow f_{n-1} & \downarrow f_n = 1 \,\wr & \downarrow f_{n+1} = 1 \,\wr & \\
X & := & [\,X_1 , & X_2 , & \cdots , & X_{n-1} , & X_n , & X_{n+1} , & \cdots\,]
\end{array}$$

Eventually isomorphic

**Moral:** Can contract finitely many terms

---

**Rmk**

Can cause homotopy in negative grading!

**Ex**

$\pi_{-3} F_4^{\mathbb{N}} S^0 \cong \pi_1 \mathbb{S}$

$$\hookrightarrow \begin{array}{c}
\begin{array}{cccccccc} & 0 & 1 & 2 & 3 & 4 & 5 & \\ ( & * , & * , & * , & * , & S^0 , & S^1 , & \cdots\,) \end{array}
\end{array}$$

$$\begin{aligned}
\pi_{-3} &= \operatorname*{colim}_{\xrightarrow[j]{}} \pi_{-3+j} (F_4^{\mathbb{N}} \mathbb{S})_j \\
&= \operatorname*{colim}_{\xrightarrow[j]{}} \pi_{j-3} \mathbb{S}^{j-4} \qquad k = j-4 \Rightarrow j-3 = k+1 \\
&= \operatorname*{colim}_{\xrightarrow[k]{}} \pi_{k+1} S^k \\
&= \operatorname*{colim}_{\xrightarrow[k]{}} \pi_{k+1} (\mathbb{S}_k) \\
&= \pi_1 \mathbb{S} \\
&= \pi_1^{st} S^0 .
\end{aligned}$$

---

- **Def**

$X \in Sp$ is **connective** $\iff$ for $n \leq 0$, $\pi_n X = 0$.

## Lemma ( $\pi_*$-isos are preserved by $\Sigma, \Omega$ )

$f \in \mathrm{Mor}^{W}_{Sp}(X,Y)$ is a $\pi_*$-isomorphism $\Longleftrightarrow$

1) $\Sigma^k f \in \mathrm{Mor}^{W}_{Sp}(\Sigma^k X, \Sigma^k Y)$ is a $\pi_*$-isomorphism $\forall\, k \in \mathbb{Z}^{\geq 0}$

$\Longleftrightarrow$

2) $\Omega^k f \in \mathrm{Mor}^{W}_{Sp}(\Omega^k X, \Omega^k Y)$ is a $\pi_*$-isomorphism $\forall\, k \in \mathbb{Z}^{\geq 0}$

# Fiber Sequences

## Def (Homotopy Fibers/Cofibers in $Sp$, $\approx 3.2.9$)

$$\left.\begin{tikzcd} \mathrm{hofib}(f) = Ff \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[r, "i_A"] \arrow[d, "f"] & A \wedge I \arrow[d] \arrow[dl, phantom, "\ulcorner", very near end] \\ B^I \arrow[r, "p_0"'] & B \arrow[r] & Cf = \mathrm{hocofib}(f) \end{tikzcd}\right\} \in \mathrm{Top}_*$$

Define levelwise for $Sp$:

$$\mathrm{hofib}^{Sp}(A \xrightarrow{f} B)_n := \mathrm{hofib}^{Top_*}(A_n \xrightarrow{f_n} B_n)$$

$$\mathrm{hocofib}^{Sp}(A \xrightarrow{f} B)_n := \mathrm{hocofib}^{Top_*}(A_n \xrightarrow{f_n} B_n)$$

## Def ((co)fiber sequences in $Sp$)

- A homotopy fiber sequence is

$$A \to B \to C \quad \text{st.} \quad \exists f \in Sp^W(A, \mathrm{hofib}(B \to C))$$

($\approx$ kernel)

- A homotopy cofiber sequence is

$$A \to B \to C \quad \text{st.} \quad \exists f \in Sp^W(C, \mathrm{hocofib}(A \to B))$$

($\approx$ cokernel)

# Some Facts

## Prop (3.2.10)

$f \in Sp(X, Y)$ induces 2 homotopy LES's

$$\cdots \to \pi_n Ff \to \pi_n X \xrightarrow{f_n^*} \pi_n Y \to \pi_{n-1} Ff \to \pi_{n-1} X \xrightarrow{f_{n-1}^*} \cdots$$

$$\cdots \to \pi_n X \xrightarrow{f_n^*} \pi_n Y \to \pi_n Cf \to \pi_{n-1} X \xrightarrow{f_{n-1}^*} \pi_{n-1} Y \to \cdots$$

## Rmk

Later we'll see that fiber sequences = cofiber sequences (maybe after passing to $hoSp$?)

# Some Facts

## Prop (3.2.11)

TFAE:

1) $f$ is a $\pi_*$-iso
2) $Cf \to *$ is a $\pi_*$-iso — Cofiber is weakly contractible
3) $Ff \to *$ is a $\pi_*$-iso — Fiber is weakly contractible

---

## Cor (3.2.12)

For any map of spectra $A \xrightarrow{f} B \in Sp$, the map $Ff \xrightarrow{g} \Omega Cf$ is a $\pi_*$-iso.

↳ Constructed in B&R

**Idea**

$$\begin{tikzcd}
 & \pi_{n+1} Cf \arrow[dr] \arrow[d, "\pi_{n+1}(g)"] & & & \pi_n(Cf) \arrow[dr] \arrow[d, "\pi_n(g)", "\wr"'] & & & \pi_{n-1} Cf \arrow[dr] \arrow[d, "\pi_{n-1}(g)"] & \\
\cdots \arrow[ur] \arrow[dr] & \pi_n Ff \arrow[r] & \pi_n A \arrow[r, "\pi_n(f)"] & \pi_n B \arrow[ur] \arrow[dr] & \pi_{n-1} Ff \arrow[r] & \pi_{n-1} A \arrow[r, "\pi_{n-1}(f)"] & \pi_{n-1} B \arrow[ur] \arrow[dr] & \pi_{n-2} Ff \arrow[r] & \cdots
\end{tikzcd}$$

**Slogan:** "Fiber seqs = cofib seqs"

# Some Facts

- Lemma (B&R, 3.2.13: preserving W)

1) If $g \in Sp^W(X,Y)$ is a levelwise h-cofib, then the pushout of $g$ along any $g' \in Sp(X,Z)$ is in W.

2) levelwise h-cofib

$$\begin{tikzcd}
 & & & P_A \arrow[dr, "\wr W"] & \\
 & & & & P_{A'} \\
B \arrow[d, "\wr"] & A \arrow[l, "i"'] \arrow[r] \arrow[d, "\wr"] & C \arrow[d, "\wr"] & & \\
B' & A' \arrow[l, "i'"'] \arrow[r] & C' & &
\end{tikzcd}$$

$\pi_*$-iso between pushouts

$\in W$

3) If $(X^i \xrightarrow[f_i]{\sim} X^{i+1} \xrightarrow[f_{i+1}]{\sim} \cdots)$ with all $f_i \in W$ & levelwise cofib, then

$$X_0 \xrightarrow[i]{} \operatorname{colim} X^i \in W \text{ \& is a levelwise h-cofib.}$$

- Prop (B&R, 3.2.14: Smash with CW preserves W)

If $f \in Sp^W(X,Y)$ and $A \in CW_*$, then

$$f \wedge 1\!\!1_A \in Sp^W(X \wedge A, Y \wedge A).$$

(Fin!) ∎