Review - What's a model category?

Category $\mathcal{M}$ w/ cofibs, fibs, WEs
$\hookrightarrow$ $\twoheadrightarrow$ $\xrightarrow{\sim}$

WEs allow the definition of $\mathrm{ho}\,\mathcal{M} = \mathcal{M}[W^{-1}]$.
Cofibs + fibs allow the construction of models representing objects in $\mathrm{ho}\,\mathcal{M}$.
Analogs of homotopy extension/lifting properties, + construction of mapping cylinders + mapping path spaces

Cofibrant generation

$\mathcal{M}$ is cof. gen. if there are sets
I of generating cofibrations
J of generating acyclic cofibrations
such that

$$W \cap \mathcal{F} = I^{\boxslash}$$
$$\mathcal{F} = J^{\boxslash}$$
$$W \cap \mathcal{C} = {}^{\boxslash}(J^{\boxslash})$$ = closure of J under coproducts, pushouts along arbitrary morphisms, transfinite composition, + retracts.

relative J-cell complexes

$$\mathcal{C} = {}^{\boxslash}(I^{\boxslash})$$ = retracts of relative I-cell complexes.

$$W = (W \cap \mathcal{F}) \circ (W \cap \mathcal{C})$$

Recognition Thm $\mathcal{M}$ = bicomplete category, $W, I, J \subseteq \mathcal{M}$

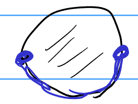
- W closed under composition, 2-out-of-3, contain all identities.
- Domains of I are I-small, likewise J
- J-cell $\subseteq W \cap {}^{\boxslash}(I^{\boxslash})$
- $I^{\boxslash} \subseteq W \cap J^{\boxslash}$
- Either $W \cap {}^{\boxslash}(I^{\boxslash}) \subseteq {}^{\boxslash}(J^{\boxslash})$ or $W \cap J^{\boxslash} \subseteq I^{\boxslash}$

Then M is a cof. gen. MC with generating cofs I, generating ACs J, and weak equivalences W.

ex. $M = \mathrm{Top}$ W = weak homotopy equivalences

$I = \{ S^{n-1} \hookrightarrow D^n \}$

$J = \{ D^{n-1} \xrightarrow{\sim} D^n \}$



- W satisfies 2-out-of-3 because it's the maps become isomorphisms under π_*.

- A relative I-cell complex is a relative cell complex, i.e., a composition of maps

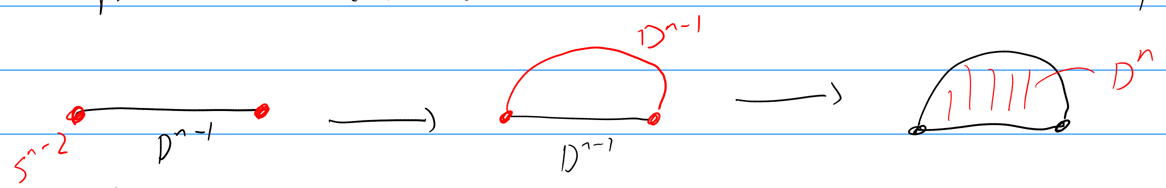
$$\begin{tikzcd} S^{n-1} \arrow[r, hook] \arrow[d] & D^n \arrow[d] \\ X \arrow[r] & X \cup e_n \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

In particular, these maps are closed inclusions, and S^{n-1}, D^{n-1} are small with respect to them Consequence of compactness.

- $J\text{-cell} \subseteq W \cap {}^{\boxslash}(I^{\boxslash})$

Any relative J-cell complex is a weak homotopy equivalence.

Maps in J can be built as relative I-cell complex



- $I^{\boxslash} = W \cap J^{\boxslash}$

A map $f \in I^{\boxslash} \iff f \in (I\text{-cell})^{\boxslash}$

$\implies f \in J^{\boxslash}$

Suppose $f : X \to Y \in I^{\boxslash}$

$$\begin{tikzcd} S^{n-1} \arrow[r] \arrow[d] & X \arrow[d] \\ D^n \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

so $\pi_{n-1} X \hookrightarrow \pi_{n-1} Y$.

$* \longrightarrow S^{n-1}$ is in I-cell.

$$\begin{tikzcd} * \arrow[r] \arrow[d] & X \arrow[d] \\ S^{n-1} \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

so $\pi_{n-1} X \twoheadrightarrow \pi_{n-1} Y$.

$\therefore f \in W$, so $I^{\boxslash} \subseteq W \cap J^{\boxslash}$.

Conversely, suppose $f \in W \cap J^{\boxslash}$.

$$\begin{tikzcd} S^{n-1} \arrow[r, "c"] \arrow[d] & X \arrow[d] \\ D^n \arrow[r] \arrow[ur, dashed, "?"] & Y \end{tikzcd}$$

$c \mapsto 0$ in $\pi_{n-1} Y$

$c \sim 0$ in $\pi_{n-1} X$.

$$\begin{tikzcd} S^{n-1} \arrow[rr] \arrow[dr] \arrow[dd] & & X \arrow[dd] \\ & D^n \arrow[ur] \arrow[dr, "\beta"] & \\ D^n \arrow[rr, "\alpha"'] & & Y \end{tikzcd}$$

(1) $\alpha \not\sim \beta$ in Y, but we get

$$S^n = D^n \coprod_{S^{n-1}} D^n \longrightarrow Y$$

This lifts to a class in $\pi_n X$.
We can use this class to replace $D^n \longrightarrow X$ with one that becomes homotopic to α in Y.

(2) $\alpha \sim \beta$ in Y.

$$\begin{tikzcd} & D^n \arrow[r, "\tilde\beta"] \arrow[d] & X \arrow[d] \\ & D^{n+1} \arrow[r, "H"] \arrow[ur, dashed] & Y \\ D^n \arrow[ur] \arrow[urr, bend right, "\alpha"'] & & \end{tikzcd}$$

homotopy from $\tilde\beta$ to a lift of α.

All this proves:

Thm. There is a cofibrantly generated model structure on Top_* where

W = weak HEs

$I = \{ S^{n-1}_+ \longrightarrow D^n_+ \}$

$J = \{ D^{n-1}_+ \longrightarrow D^n_+ \}$.

Exercise. Do the same for sSets, where

$W = \{ f : |f| \text{ is a WE in } Top \}$

$I = \{ \partial \Delta^n \longrightarrow \Delta^n \}$

$J = \{ \Lambda^n_k \longrightarrow \Delta^n \}$

$\Lambda^n_k = \partial \Delta^n$ minus a nondegenerate $(n-1)$-simplex.

$$F_d : Top_* \rightleftarrows Sp : ev_d$$

$$F_d(A) = \{ *, \dots, *, \underset{d}{A}, \underset{d+1}{\Sigma A}, \underset{d+2}{\Sigma^2 A}, \dots \}$$

$$ev_d(X) = X_d.$$

Thm. There's a cofibrantly generated model structure on Sp where

$W = \{ X \to Y : X_d \xrightarrow{\sim} Y_d \text{ in } Top_* \}$

$I = \{ F_d S^{n-1}_+ \longrightarrow F_d D^n_+ : d, n \in \mathbb{N} \}$

$J = \{ F_d D^{n-1}_+ \longrightarrow F_d D^n_+ : d, n \in \mathbb{N} \}$

Observations.

1. $ho(Sp^{level})$ is not the stable homotopy category.

(2) If $X \to Y$ is in $J^{\boxslash}$

$$\begin{tikzcd} F_d D^{n-1}_+ \arrow[r] \arrow[d] & X \arrow[d] \\ F_d D^n_+ \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} D^{n-1}_+ \arrow[r] \arrow[d] & X_d \arrow[d] \\ D^n_+ \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

So $X \to Y$ is a <u>levelwise Serre fibration</u>.

Likewise, $I^{\boxslash} = \{$<u>levelwise acyclic Serre fibrations</u>$\}$

(3) If you attach a cell to X using a map in I, this means: attach an n-cell to X_d, its suspension to X_{d+1}, etc.

Any relative I-cell complex is a levelwise relative cell complex.

<u>Proof</u>. <u>Properties of W</u>. clear.

<u>Smallness</u>. Given a sequential colimit diagram of maps in I-cell, $\{X_\alpha\}$, we want

$$\mathrm{Maps}(F_d S^{n-1}_+, \operatorname{colim} X_\alpha) = \operatorname{colim} \mathrm{Maps}(F_d S^{n-1}_+, X_\alpha).$$

$$\begin{aligned} \mathrm{Maps}(F_d S^{n-1}_+, \operatorname{colim} X_\alpha) &= \mathrm{Maps}_{Top_*}(S^{n-1}_+, ev_d(\operatorname{colim} X_\alpha)) \\ &= \mathrm{Maps}_{Top_*}(S^{n-1}_+, \operatorname{colim} ev_d(X_\alpha)) \end{aligned}$$

$$\operatorname{colim} \mathrm{Maps}(F_d S^{n-1}_+, X_\alpha) = \operatorname{colim} \mathrm{Maps}(S^{n-1}_+, ev_d(X_\alpha)).$$

<u>$J\text{-cell} \subseteq W \cap {}^{\boxslash}(I^{\boxslash})$</u>: Same arguments as before.

<u>$I^{\boxslash} = W \cap J^{\boxslash}$</u>: Checked levelwise on spectra, and true in Top_*.

$\blacksquare$

$f : X \longrightarrow Y$ is a cofibration iff:

- $f_0 : X_0 \longrightarrow Y_0$ is a cofibration in Top_*.

- $$\begin{tikzcd}
\Sigma X_n \arrow[r] \arrow[d, "\Sigma f_n"'] & X_{n+1} \arrow[d] \arrow[ddr, bend left, "f_{n+1}"] & \\
\Sigma Y_n \arrow[r] \arrow[drr, bend right] & \Sigma Y_n \coprod_{\Sigma X_n} X_{n+1} \arrow[dr, "\sigma_n"] & \\
 & & Y_{n+1}
\end{tikzcd}$$

  σ_n is a cofibration in Top_*.

In particular, let $X = *$.

$* \longrightarrow Y$ is a cofibration

$\Longleftrightarrow$ Y_0 is cofibrant and $\Sigma Y_n \hookrightarrow Y_{n+1}$ is a cofibration.

We defined, for $X \in Sp$, $K \in Top_*$,

$X \wedge K$, X^K.

$$\cdot \wedge K : Sp \rightleftarrows Sp : (\cdot)^K.$$

$$Sp(X, Y) \subseteq \prod_{n=0}^{\mathbb{N}} Top_*(X_n, Y_n).$$

View $Sp(X, Y)$ as a space by giving it the subspace topology.

$$X \wedge \cdot : Top \rightleftarrows Sp : Sp(X, \cdot).$$

Def. Suppose M is a model category which is tensored, cotensored, and enriched over Top.
M is a <u>topological model category</u> if:
For any $i: A \hookrightarrow B$ in Top,
$j: X \hookrightarrow Y$ in M,

$$i \square j : B \otimes X \coprod_{A \otimes X} A \otimes Y \longrightarrow B \otimes Y \text{ is a cofibration in } M$$

which is acyclic if i or j is.

Enriched over Top:
$$Sp(\cdot, \cdot): Sp^{op} \times Sp \longrightarrow Top$$

$$Sp(X, Y) \times Sp(Y, Z) \longrightarrow Sp(X, Z)$$
maps of spaces, etc.

$$(X \wedge K)_d = X_d \wedge K$$

$$\Sigma (X \wedge K)_d = \Sigma X_d \wedge K \longrightarrow X_{d+1} \wedge K = (X \wedge K)_{d+1}$$

$$(X^K)_d = \mathrm{Maps}_{Top_*}(K, X_d)$$

$$(X^K)_d = \mathrm{Maps}(K, X_d) \longrightarrow \mathrm{Maps}(K, \Omega X_{d+1}) = \Omega\, \mathrm{Maps}(K, X_{d+1})$$
$$\Omega\, \mathrm{Maps}(K, X_{d+1}) = \Omega (X^K)_{d+1}.$$

$$\mathrm{Maps}(K, \Omega X_{d+1}) = \mathrm{Maps}(K, \mathrm{Maps}(S^1, X_{d+1})) = \mathrm{Maps}(K \wedge S^1, X_{d+1}) = \mathrm{Maps}(S^1, \mathrm{Maps}(K, X_{d+1})).$$