

\mathcal{C} is enriched over Top_* if:

for every $X, Y \in \mathcal{C}$ there's $\mathcal{C}(X, Y) \in \text{Top}_*$

for every $X, Y, Z \in \mathcal{C}$ there's $\mathcal{C}(Y, Z) \wedge \mathcal{C}(X, Y) \longrightarrow \mathcal{C}(X, Z)$

for every $X \in \mathcal{C}$ there's $S^0 \xrightarrow{1_X} \mathcal{C}(X, X)$
 associative + unital.

- Can do this over any symmetric monoidal category.
- There's a forgetful functor

$$\text{Top}_* - \text{Cat} \longrightarrow \text{Cat}.$$

A Top_* -enrichment on \mathcal{C} is a lift of \mathcal{C} along this functor.

• Every $\mathcal{C}(X, Y) \ni x$ which is a zero morphism.

A tensoring of $\mathcal{C} \in \text{Top}_* - \text{Cat}$ over Top_* is

$$\begin{aligned} \text{Top}_* \times \mathcal{C} &\longrightarrow \mathcal{C} \\ (K, X) &\longmapsto K \otimes X. \\ \text{Top}_*(K, \mathcal{C}(X, Y)) &\cong \mathcal{C}(K \otimes X, Y). \end{aligned}$$

A cotensoring of $\mathcal{C} \in \text{Top}_* - \text{Cat}$ over Top_* is

$$\begin{aligned} \text{Top}_*^{\text{op}} \times \mathcal{C} &\longrightarrow \mathcal{C} \\ (K, X) &\longmapsto X^K \\ \text{Top}_*(K, \mathcal{C}(X, Y)) &\cong \mathcal{C}(X^K, Y). \end{aligned}$$

$I_n S_p$ $S_p(X, Y) = \{ \text{compatible families of maps } X_n \rightarrow Y_n \}$
 $= \prod_{n=0}^{\infty} \text{Top}_*(X_n, Y_n)$

$$(K \wedge X)_n = K \wedge X_n$$

$$\Sigma(K \wedge X_n) = K \wedge \Sigma X_n \longrightarrow K \wedge X_{n+1}$$

$$(X^K)_n = X_n^K$$

$$X_n^K \longrightarrow (\Omega X_{n+1})^K = \Omega(X_{n+1}^K)$$

Def. M is a model category which is enriched, tensored,
 & cotensored over Top_* .

M is a topological model category if:

$$\text{for all } \begin{array}{l} i: A \hookrightarrow B \text{ in } \text{Top}_* \\ j: X \hookrightarrow Y \text{ in } M, \end{array}$$

$$i \circ j: \begin{array}{c} B \otimes X \\ \parallel \\ A \otimes X \end{array} \xrightarrow{A \otimes Y} B \otimes Y$$

is a cofibration in M which is acyclic if either
 i or j is.

Exercise: This is equivalent to:

$$\text{Given } \begin{array}{l} i: A \hookrightarrow B \text{ in } \text{Top}_*, \\ p: X \twoheadrightarrow Y \text{ in } M, \end{array}$$

$$\text{Hom}_{\square}(i, p): \begin{array}{c} X^B \\ \xrightarrow{\quad} \\ Y^B \end{array} \begin{array}{c} X \\ \times \\ Y^A \end{array} \begin{array}{c} X^A \\ \xrightarrow{\quad} \\ Y^A \end{array}$$

is a fibration, which is acyclic if i or p is.

Also equivalent to:

$$\text{Given } \begin{array}{l} i: X \hookrightarrow Y \text{ in } M \\ p: X' \twoheadrightarrow Y' \text{ in } M, \end{array}$$

$$M(Y, X') \xrightarrow{\quad} M(X, X') \times_{M(X, Y')} M(Y, Y')$$

is a fibration (in Top_*) which is acyclic if i or p
 is.

Other motivation:

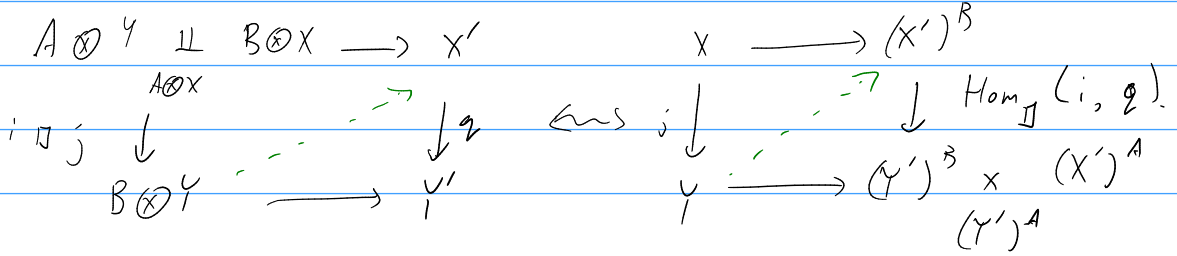
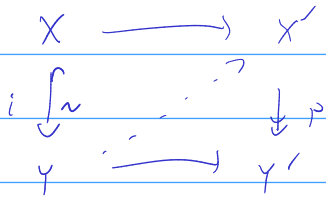
- Model structure should be "compatible" with enrichment.
- Any model category M has a space of maps between objects (Dwyer-Kan or hammock localization)

These conditions imply

DK localization spaces \cong mapping spaces coming from enrichment.

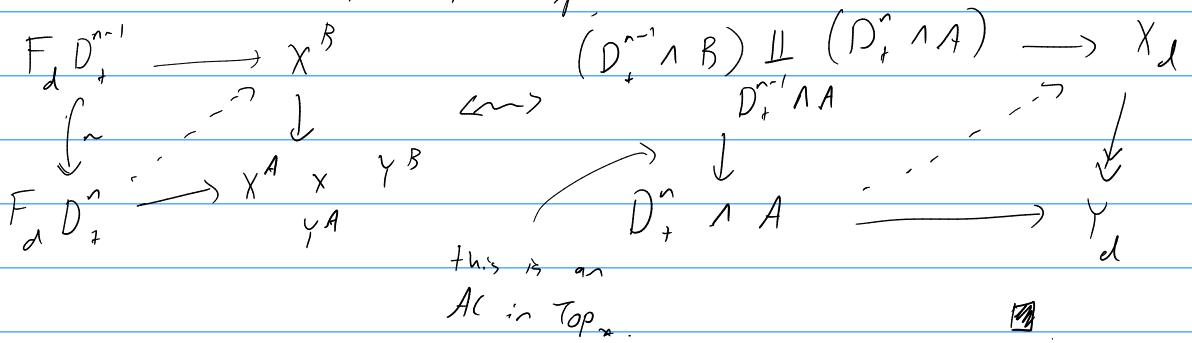
$\text{Ho}(M)$ is enriched over $\text{Ho}(\text{Top}_*)$.

$$\underline{M(Y, X')} \longrightarrow M(X, X') \quad \begin{matrix} X' \\ \bar{M}(X, Y') \end{matrix} \quad M(Y, Y')$$



Prop. S_p^{level} is a topological model category.

PF. $A \hookrightarrow B$ in Top_*
 $X \twoheadrightarrow Y$ in S_p



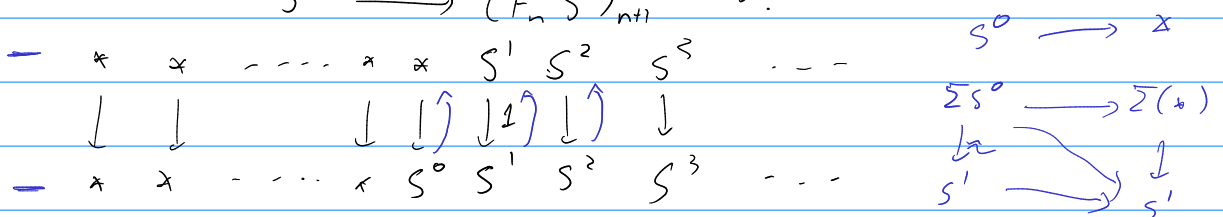
Idea of stable model structure

Add ACs to the generators of S_p^{level} .

- Same cof.
- More WEs (the π_* -isos).
- Fewer fibrations.

Def. $\lambda_n: F_{n+1} S^1 \longrightarrow F_n S^0$ adjoint to

$$S^1 \longrightarrow (F_n S^0)_{n+1} = S^1.$$



λ_n is a π_* -iso, but has no homotopy inverse.

λ_n is not a cofibration.

$$\begin{array}{ccc}
 F_{n+1} S^1 & \xrightarrow{\lambda_n} & F_n S^0 \\
 \downarrow i_1 & & \downarrow \Gamma \\
 F_{n+1} S^1 \wedge [0, 1]_+ & \longrightarrow & \text{Cyl}(X_n) \\
 \uparrow i_0 & \nearrow & \\
 F_{n+1} S^1 & & \xrightarrow{k_n}
 \end{array}$$

Thm. There's a cofibrantly generated model structure on S_p with $W = \pi_x$ -isos

$$\begin{aligned}
 I &= I_{\text{level}} = \{ F_d S_+^{n-1} \rightarrow F_d D_+^n \} \\
 J &= J_{\text{level}} \cup \{ k_d \square (S_+^{n-1} \rightarrow D_+^n) \}
 \end{aligned}$$

Prop. $f: X \rightarrow Y \in J^{\square}$ iff f is a levelwise fibration, and

$$\forall d, X_d \xrightarrow{\sim} Y_d \quad X \xrightarrow{\sim} \Omega X_{d+1}, \quad \Omega Y_{d+1}$$

Cor. ($Y = *$). X is fibrant in S_p^{stable} iff

$$X_d \xrightarrow{\sim} \Omega X_{d+1}, \text{ i.e., } X \text{ is an } \underline{\Omega\text{-spectrum}}$$

Applications

$$X \rightarrow \Omega \Sigma X \text{ is a } \pi_x\text{-iso.}$$

$$\pi_k X = \text{colim } \pi_{k+n} X_n.$$

$$\begin{array}{ccc}
 \pi_{k+n} X_n & \longrightarrow & \pi_{k+n+1} \Sigma X_n = \pi_{k+n} \Omega \Sigma X_n \\
 & \searrow & \downarrow \\
 & & \pi_{k+n+1} X_{n+1}
 \end{array}$$

$\{ \pi_{k+n} \Omega \Sigma X_n \}_n$ is cofinal in $\{ \pi_{k+n} X_n \}_n$.

So they have the same colimit.

$$\bullet \pi_n \Sigma X = \pi_{n+1} X$$

$$\pi_n \Omega X = \pi_{n+1} X \implies \Sigma, \Omega \text{ preserve } \pi_x\text{-isos.}$$

- Σ preserves cofibrations.

$$X \xrightarrow{\sim} \Omega \Sigma X \quad \Sigma : S_p^{\text{stable}} \rightleftarrows S_p^{\text{stable}} : \Omega$$

This is a Quillen equivalence,

Σ, Ω are inverse equivalences on $\text{Ho}(S_p)$.

- $\Sigma^\infty = F_0 : \text{Top}_* \rightarrow S_p$

preserves cof, & sends WEs to levelwise WEs.

(which are stable WEs, too.)

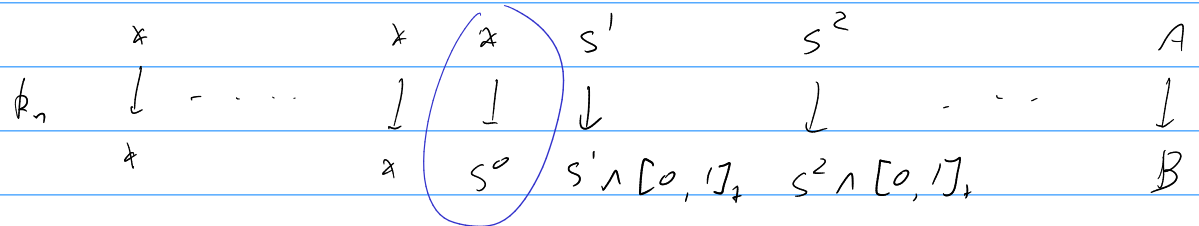
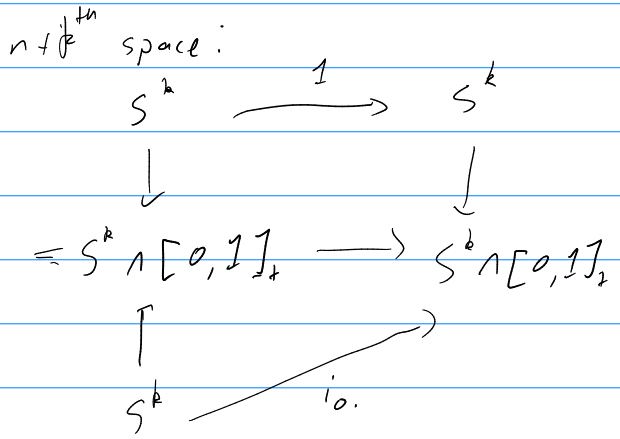
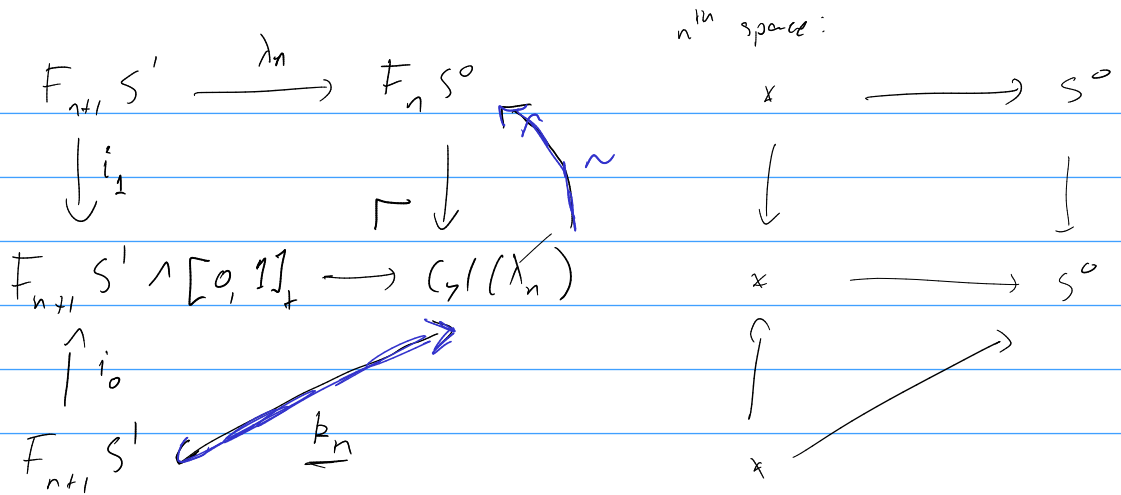
$$\Sigma^\infty : \text{Top}_* \rightleftarrows S_p : E_{v_0}$$

The homotopy-invariant concept is

$$\begin{aligned} R(E_{v_0})(X) &:= E_{v_0}(X^{\text{fib}}) \\ &= \Omega^\infty X. \end{aligned}$$

This is an infinite loop space.

$$(X^{\text{fib}})_n = \text{hocolim } \Omega^k X_{n+k}$$



$$k_n \square (S_+^{n-1} \rightarrow D_+^n) = A \cap D_+^n \coprod_{A \cap S_+^{n-1}} B \cap S_+^{n-1} \rightarrow B \cap D_+^n$$

n^{th} stage: $S_+^{n-1} \rightarrow D_+^n$

$(n+k)^{\text{th}}$ stage: $S^k \cap D_+^n \coprod_{S^k \cap S_+^{n-1}} S^k \cap [0,1]_+ \cap S_+^{n-1} \rightarrow S^k \cap [0,1]_+ \cap S_+^{n-1}$